Czechoslovak Mathematical Journal

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Czechoslovak Mathematical Journal, Vol. 34 (1984), No. 4, 528–540

Persistent URL: `http://dml.cz/dmlcz/101979`

Czechoslovak Mathematical Journal, 34 (109) 1984, Praha

LARGE SYSTEMS OF INDEPENDENT OBJECTS IN CONCRETE CATEGORIES II

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(Received September 2, 1982)

In combinatorics two points a, b of a graph are called *adjacent* if they lie on an edge – and they are called *independent* in the opposite case. Many results of classical combinatorics concern the problem of independent points. There is one typical question of infinite combinatorics which has no finite parallel: when a graph is self-independent – meaning that there is a set of independent points with the same cardinality as the whole underlying set of the graph. These questions were generalized for general categories, see [K_3]. Roughly speaking, two subcategories $\mathscr{K}_1, \mathscr{K}_2$ of a category $\mathscr{K}$ are called *independent* if there is no morphism between an object in $\mathscr{K}_1$ and an object in $\mathscr{K}_2$. These problems were studied for special types of categories in [HS]. An important generalization of this question for concrete categories was studied in [K_3]. We recall the basic definitions:

Let $(\mathscr{K}, U)$, $(\mathscr{L}, V)$ be concrete categories (i.e. $U: \mathscr{K} \to \text{Set}$, $V: \mathscr{L} \to \text{Set}$ are faithful functors). Then functors $\Psi, \Phi: \mathscr{K} \to \mathscr{L}$ are independent if for every pair a, b of objects of $\mathscr{K}$ there is no morphism between Φa and Ψb.

A functor $\Phi: \mathscr{K} \to \mathscr{L}$ is called *strong* if there is a functor $F: \text{Set} \to \text{Set}$, called the *underlying functor of* Φ, with $F \circ U = V \circ \Phi$.

A functor $\Phi: \mathscr{K} \to \mathscr{L}$ is called a *full embedding* if it is faithful and full (i.e. for $a, b \in \mathscr{K}$, $\Phi(\text{Hom}(a, b)) = \text{Hom}(\Phi a, \Phi b)$).

Let $(\mathscr{K}, U)$, $(\mathscr{L}, V)$ be concrete categories. Then $(\mathscr{K}, U)$ is (*strongly*) *$\mathscr{L}$-spanned from α upwards*, where α is a cardinal, if for every cardinal $\beta \geqq \alpha$ there is a class $\mathscr{F}_\beta$ of (strong) full embeddings from $\mathscr{L}$ to $\mathscr{K}$ such that

a) if $\Psi, \Phi \in \mathscr{F}_\beta$, $\Psi \neq \Phi$ then Ψ and Φ are independent;
b) if a is an object of $\mathscr{L}$ then for every $\Psi \in \mathscr{F}_\beta$, $\text{card } U(\Psi a) \leqq \max \{\text{card } Va, \beta\}$;
c) $\text{card } \mathscr{F}_\beta = \text{card } \{a;\ a \text{ is an object of } \mathscr{K},\ \text{card } Ua \leqq \beta\}/\sim$ where $a \sim b$ iff a, b are isomorphic.

We say that $(\mathscr{K}, U)$ is (*strongly*) *ultimately $\mathscr{L}$-spanned* if $(\mathscr{K}, U)$ is (strongly) $\mathscr{L}$-spanned from α upwards for a cardinal α. If $\mathscr{L}$ has exactly one morphism (i.e. $\mathscr{L}$

is a one-object discrete category) then we say that $(\mathscr{K}, U)$ is *ultimately discretely spanned* or *discretely spanned from α upwards.*

In the foregoing definitions we assume that all proper classes have the same power — denote it by c — which is bigger than any set cardinal α. This is correct in the Bernays-Gödel model of the set theory with the strong axiom of choice. In this paper we shall work in this model of sets.

An important role among concrete categories is played by $S(F)$ categories (defined in [HPT]) where F is a set-valued functor: If $F: \mathscr{K} \to \text{Set}$ is a functor then objects of $S(F)$ are the pairs (A, α) where A is an object of $\mathscr{K}$ and $\alpha \subset FA$, morphisms from (A, α) to (B, β) are the morphisms $f: A \to B$ of $\mathscr{K}$ such that $Ff(\alpha) \subset \beta$ if F is covariant, $Ff(\beta) \subset \alpha$ for F contravariant.

Many papers have been devoted to the investigation of $S(F)$, particularly for F: Set $\to$ Set. Every current category is a reflective or coreflective subcategory of $S(F)$ for some functor F. Categories which have full embeddings to $S(F)$ were investigated in [HP_1], [P_1] etc. The most important result was proved by Hedrlín and Kučera. They proved that under the set axiom (M)*) every concrete category can be fully embedded to $S(Q_2)$ where $Q_2 = \text{Hom}(2, -)$: Set $\to$ Set, see [H], [H K], [K], and Kučera proved that every concrete category can be fully embedded to $S(P_2)$ where $P_2 = \text{Hom}(-, 2)$: Set $\to$ Set, see [K]. Moreover, every day-life concrete category can be strongly fully embedded to $S(F)$ for a functor F such that the underlying functor is an identity, [KuP]. In the paper [AHS] it was proved that $S(F)$ is a "universal" initially complete fibre-small concrete category. This is only a brief list of interesting properties of $S(F)$. It leads us to a detailed investigation of $S(F)$.

The aim of this paper is to characterize the categories $S(F)$, with F: Set $\to$ Set contravariant, such that $S(F)$ is ultimately $\mathscr{D}$-spanned, where $\mathscr{D}$ is the category of directed graphs (clearly $S(Q_2) = \mathscr{D}$). It continues the paper [K_3] where the same question was studied for covariant functors F: Set $\to$ Set and is based on the papers [K_1, K_2] where the categories $S(F)$ into which $\mathscr{D}$ is fully embeddable (i.e. the binding $S(F)$ categories) were characterized.

We say that F: Set $\to$ Set is nearly faithful if there is a cardinal β such that if $f, g: X \to Y$ are mappings with $Ff = Fg$ then $f = g$ or card Im f, card Im $g < \beta$. Then we prove:

Main Theorem. *For a contravariant set functor* F: Set $\to$ Set *the following are equivalent*:

a) *$S(F)$ is strongly ultimately $S(Q_2)$-spanned;*
b) *$S(F)$ is strongly ultimately $S(P_2)$-spanned;*
c) *$S(F)$ is ultimately discretely spanned;*

*) Axiom (M) — there is a cardinal γ such that each ultrafilter closed under intersections of γ sets is trivial.

d) *there is a full embedding of* $S(Q_2)$ *to* $S(F)$;
e) F *is nearly faithful.*

To compare the covariant and the contravariant case we formulate the Main Theorem from [K$_3$].

Theorem. *For a covariant set functor* F: Set $\to$ Set *the following are equivalent*:

a) $S(F)$ *is strongly ultimately* $S(Q_2)$*-spanned*;
b) $S(F)$ *is ultimately discretely spanned*;
c) *there is a full embedding of* $S(Q_2)$ *to* $S(F)$;
d) *there are sets* A, B *such that* A *is finite and* $F(A \cup B) \neq \operatorname{Im} Fi \cup \operatorname{Im} Fj$ *where* $i: A \to A \cup B$, $j: B \to A \cup B$ *are inclusions.*

The proofs in this paper are based on [K$_2$, K$_4$ and KP]. These techniques belong to infinitary combinatorics, while the results are formulated in the language of the theory of categories. The proof of Main Theorem is divided into several parts. First we prove that $S(P_2)$ is strongly $S(P_2)$-spanned from $\aleph_0$ upwards. Then we construct a strong full embedding from $S(Q_2)$ to $S(P_2)$ such that for every infinite set X the underlying functor F fulfils card FX = card X. Further, if we use [K$_2$] we get that if there is an embedding from $S(P_2)$ to $S(F)$ then $S(F)$ is strongly ultimately $S(P_2)$-spanned, hence by the results in [K$_2$] we get Main Theorem. In the end we show some testing categories for the decision whether $S(F)$ is binding and we formulate a big characterization theorem.

First we prove an auxiliary lemma:

Lemma 1. *For every infinite set* X, *every subset* $U \subset X$ *and every non-empty graph* (X, R) (*i.e.* $\emptyset \neq R \subset X \times X$) *there is a set* $\mathfrak{A} \subset \exp(\exp X)$ (*where* $\exp X = \{Z; Z \subset X\}$) *such that*

(i) card $\mathfrak{A} = 2^{2^\alpha}$ *where* α = card X;
(ii) $\mathscr{K}_1, \mathscr{K}_2 \in \mathfrak{A}$, $\mathscr{K}_1 \neq \mathscr{K}_2$ *implies* $\mathscr{K}_1 - \mathscr{K}_2 \neq \emptyset \neq \mathscr{K}_2 - \mathscr{K}_1$;
(iii) *every* $\mathscr{K} \in \mathfrak{A}$ *fulfils*: a) *if* $K \in \mathscr{K}$ *then* K *is infinite*;
b) *if* $K_1, K_2 \in \mathscr{K}$ *then either* $K_1 = K_2$ *or* card $(K_2 - K_1)$, card $(K_1 - K_2) > 1$;
c) $U, X \notin \mathscr{K}$ *and there are* $(x, y) \in R$, $K \in \mathscr{K}$ *with* $x, y \in K$.

Proof. Choose a decomposition $\{T, X_1, X_2, X_3, X_4, X_5, X_6, X_7, X_8\}$ of X such that

1) $U \neq T$ and either $U \subset X_1$ or $U \cap X_i \neq \emptyset$ foi $i = 1, 2, \dots 8$;
2) card X_1 = card X_2 = card X_3 = card X_4 = card X_5 = card X_6 = card X_7 = card X_8 = card X;
3) there is $(x, y) \in R$ with $x, y \in T$.

Choose bijections $\varphi_i: X_1 \to X_i$, $i = 2, 3, \dots, 8$. Then card $(\exp(\exp X_1)) = 2^{2^\alpha}$.

Define a mapping ψ: $\exp(\exp X_1) \to \exp(\exp X)$ as follows: for $\mathscr{S} \in \exp(\exp X_1)$ put $\psi(\mathscr{S}) = \{W \cup \varphi_2(W) \cup (X_3 - \varphi_3(W)) \cup (X_4 - \varphi_4(W));\ W \in \mathscr{S}\} \cup \{\varphi_5(V) \cup \cup \varphi_6(V) \cup (X_7 - \varphi_7(V)) \cup (X_8 - \varphi_8(V));\ V \in \exp X_1 - \mathscr{S}\} \cup \{T\}$. If $\mathscr{S}_1 \mathscr{S}_2 \in \in \exp(\exp X_1)$ and $V \in \mathscr{S}_1 - \mathscr{S}_2$ then $V \cup \varphi_2(V) \cup (X_3 - \varphi_3(V)) \cup (X_4 - \varphi_4(V)) \in \in \psi(\mathscr{S}_1) - \psi(\mathscr{S}_2)$ and $\varphi_5(V) \cup \varphi_6(V) \cup (X_7 - \varphi_7(V)) \cup (X_8 - \varphi_8(V)) \in \psi(\mathscr{S}_2) - \psi(\mathscr{S}_1)$. Hence $\psi(\mathscr{S}_1) \neq \psi(\mathscr{S}_2)$ and so ψ is one-to-one. Thus we get that if we define $\mathfrak{A} = \operatorname{Im} \psi = \{\psi(\mathscr{S});\ \mathscr{S} \in \exp(\exp X_1)\}$ then $\mathfrak{A}$ fulfils (i) and (ii). Since either $U \subset X_1$ or $U \cap X_i \neq \emptyset$ for $i = 1, 2, \ldots, 8$ we have that $U \notin \psi(\mathscr{S})$ for each $\mathscr{S} \in \in \exp(\exp X_1)$. It is easy to see that $X \notin \psi(\mathscr{S})$ and since $T \in \psi(\mathscr{S})$ and there are $(x, y) \in R$ with $x, y \in T$, we conclude that (iii) c) is fulfilled. Since φ_i $(i = 2, \ldots, 8)$ are bijections we get that every set in $\psi(\mathscr{S})$ is infinite and (iii) a) is fulfilled. Let $a \in V - W$. If $V, W \in \mathscr{S}$ then a, $\varphi_2(a) \in [V \cup \varphi_2(V) \cup (X_3 - \varphi_3(V)) \cup (X_4 - \varphi_4(V))] - [W \cup \varphi_2(W) \cup (X_3 - \varphi_3(W)) \cup (X_4 - \varphi_4(W))]$ and $\varphi_3(a), \varphi_4(a) \in \in [W \cup \varphi_2(W) \cup (X_3 - \varphi_3(W)) \cup (X_4 - \varphi_4(W))] - [V \cup \varphi_2(V) \cup (X_3 - \varphi_3(V)) \cup (X_4 - \varphi_4(V))]$. If $V, W \notin \mathscr{S}$ then $\varphi_5(a), \varphi_6(a) \in [\varphi_5(V) \cup \varphi_6(V) \cup \cup (X_7 - \varphi_7(V)) \cup (X_8 - \varphi_8(V))] - [\varphi_5(W) \cup \varphi_6(W) \cup (X_7 - \varphi_7(W)) \cup (X_8 - \varphi_8(W))]$, $\varphi_7(a), \varphi_8(a) \in [\varphi_5(W) \cup \varphi_6(W) \cup (X_7 - \varphi_7(W)) \cup (X_8 - \varphi_8(W))] - [\varphi_5(V) \cup \varphi_6(V) \cup (X_7 - \varphi_7(V)) \cup (X_8 - \varphi_8(V))]$. Moreover, if $V \in \mathscr{S}$, $W \notin \mathscr{S}$ then $[V \cup \varphi_2(V) \cup (X_3 - \varphi_3(V)) \cup (X_4 - \varphi_4(V))] \cap [\varphi_5(W) \cup \varphi_6(W) \cup (X_7 - \varphi_7(W)) \cup (X_8 - \varphi_8(W))] = \emptyset$ and thus (iii) b) is fulfilled, too. The lemma is proved.

Proposition 2. *$S(P_2)$ is strongly $S(P_2)$-spanned from $\aleph_0$ upwards.*

Proof. In [VPH] it is proved that for every infinite cardinal α there is a rigid graph $(\bar{X}, \bar{R})$, i.e. a graph without nonidentical compatible mappings into itself, with $\operatorname{card} \bar{X} = \alpha$. If we use the šíp-construction described in [M] or $[K_5, K_6]$, with the particular šíp defined in [HN] or $[K_6]$ we get that there is a symmetric, rigid, connected graph (X, R) such that every edge lies in a triangle and $\operatorname{card} X = \alpha$. In what follows we shall assume that an infinite cardinal α is given and that $G_\alpha = (X, R)$ is a graph with these properties. Further, choose a set $U \subset X$ with $2 < \operatorname{card} U < \operatorname{card} X$.

Now for every set $\mathscr{K} \subset \exp X$ such that

a) $K \in \mathscr{K}$ implies that K is infinite;
b) $K_1, K_2 \in \mathscr{K}$ implies either $K_1 = K_2$ or $\operatorname{card} K_1 - K_2$, $\operatorname{card} K_2 - K_1 > 1$;
c) $U, X \notin \mathscr{K}$ and there are $(x, y) \in R$ and $K \in \mathscr{K}$ with $x, y \in K$,

and for every object (Y, S) of $S(P_2)$ we shall construct an object $(Y \vee X, \Phi_{\mathscr{K}}(S))$ ($\vee$ denotes disjoint union) such that

1) if $\varphi: (Y, S) \to (Z, V)$ is a morphism of $S(P_2)$ then $\varphi \vee 1_X: (Y \vee X, \Phi_{\mathscr{K}}(S)) \to (Z \vee X, \Phi_{\mathscr{K}}(V))$ is a morphism of $S(P_2)$ as well;
2) If $\varphi: (Y \vee X, \Phi_{\mathscr{K}_1}(S)) \to (Z \vee X, \Phi_{\mathscr{K}_2}(V))$ is a morphism of $S(P_2)$ then $\mathscr{K}_2 \subset \mathscr{K}_1$ and there is a morphism $\psi: (Y, S) \to (Z, V)$ of $S(P_2)$ with $\varphi = \psi \vee 1_X$.

In this case $\Phi_{\mathscr{K}}: S(P_2) \to S(P_2)$ is a strong full embedding such that the underlying functor is non-increasing from α and if $\mathscr{K}_1 - \mathscr{K}_2 \neq \emptyset \neq \mathscr{K}_2 - \mathscr{K}_1$ then $\Phi_{\mathscr{K}_2}$ and $\Phi_{\mathscr{K}_1}$ are independent and Lemma 1 concludes the proof.

Define $\Phi_{\mathscr{K}}(S)$: A mapping $f: X \vee Y \to \{0, 1\}$ is in $\Phi_{\mathscr{K}}(S)$ if either $f/Y \in S$ and $f^{-1}(1) \cap X = U$,

or there is $(x, y) \in R$ with $f^{-1}(1) = \{x, y\}$,
or $f^{-1}(1) \in \mathscr{K}$,
or $f^{-}(1) = X$.

We prove 1). If $f \in \Phi_{\mathscr{K}}(V)$ we have to prove that $P_2(\varphi \vee 1_X)(f) = (\varphi \vee 1_X) \circ f \in \Phi_{\mathscr{K}}(S)$. If $f/Z \in V$ and $f^{-1}(1) \cap X = U$ then $(\varphi \vee 1_X) \circ f/Y = \varphi \circ (f/Y) = P_2(\varphi) \circ (f/Y) \in S$ and $((\varphi \vee 1_X) \circ f)^{-1}(1) \cap X = (\varphi \vee 1_X)^{-1}(f^{-1}(1)) \cap X = 1_X^{-1}(f^{-1}(1) \cap X) = (f^{-1}(1) \cap X) = U$ and so $(\varphi \vee 1_X) \circ f \in \Phi_{\mathscr{K}}(S)$. Further we have $((\varphi \vee 1_X) \circ f)^{-1}(1) = (\varphi \vee 1_X)^{-1}(f^{-1}(1))$ and hence if $f^{-1}(1) \subset X$ then $((\varphi \vee 1_X) \circ f)^{-1}(1) = f^{-1}(1)$, which yields 1).

The proof of 2) will be divided into auxiliary statements:

(i) For every $(x_1, x_2) \in R$, $\operatorname{card}(\varphi^{-1}(\{x_1, x_2\}) \cap X) > 1$. Indeed, there is $f \in \Phi_{\mathscr{K}_2}(V)$ with $f^{-1}(1) = \{x_1, x_2\}$ and since $P_2\, \varphi(f) = \varphi \circ f \in \Phi_{\mathscr{K}_1}(S)$ we have $\operatorname{card}((\varphi \circ f)^{-1}(1) \cap X) = \operatorname{card}(\varphi^{-1}(\{x_1, x_2\}) \cap X) > 1$.

(ii) For every $y \in Y$, $\varphi(y) \in Z$. Assume the contrary, i.e. there is a point $y \in Y$ with $\varphi(y) \in X$. Choose $x_1, x_2 \in X$ with $(\varphi(y), x_1)$, $(\varphi(y), x_2)$, $(x_1, x_2) \in R$ (this is possible because X is infinite, G_α is connected and each edge lies in a triangle), and choose $f_1, f_2, f_3 \in \Phi_{\mathscr{K}_2}(V)$ with $f_1^{-1}(1) = \{\varphi(y), x_1\}$, $f_2^{-1}(1) = \{\varphi(y), x_2\}$, $f_3^{-1}(1) = \{x_1, x_2\}$. Then $\varphi \circ f_i \in \Phi_{\mathscr{K}_1}(S)$, $i = 1, 2, 3$ and $(\varphi \circ f_i)^{-1}(1) \cap Y \neq \emptyset$ for $i = 1, 2$. Hence $(\varphi \circ f_i)^{-1}(1) \cap X = U$ for $i = 1, 2$ and so $\varphi^{-1}(\{\varphi(y), x_i\}) \cap X = U$ for $i = 1, 2$. Therefore $\varphi^{-1}(\varphi(y)) \cap X = U$ and $\varphi^{-1}(x_1) \cap X = \emptyset = \varphi^{-1}(x_2) \cap X$, which implies $(\varphi \circ f_3)^{-1}(1) \cap X = \emptyset$ – a contradiction with (i).

(iii) If $x_1 \neq x_2$, $x_1, x_2 \in X$, $\varphi(x_1) \in X$ then $\varphi(x_1) \neq \varphi(x_2)$. Assume the contrary, i.e. for some points $x_1, x_2 \in X$, $x_1 \neq x_2$ we have $\varphi(x_1) = \varphi(x_2) \in X$. Denote $x = \varphi(x_1)$. Choose $\bar{x}_1, \bar{x}_2$ so that $x, \bar{x}_1, \bar{x}_2$ form a triangle in (X, R). We prove that $\operatorname{card} \varphi^{-1}(\bar{x}_1)$, $\operatorname{card} \varphi^{-1}(\bar{x}_2) > 1$. If $\varphi^{-1}(\bar{x}_1) = \emptyset$ then by (i), $\operatorname{card} \varphi^{-1}(\bar{x}_2) > 1$ and by (ii), $\varphi^{-1}(\{x, \bar{x}_2\}) \subset X$. Since $\operatorname{card} \varphi^{-1}(\{x, \bar{x}_2\}) > 3$ we get by the definition of $\Phi_{\mathscr{K}_1}(S)$ (there is $f \in \Phi_{\mathscr{K}_2}(V)$, $f^{-1}(1) = \{x, \bar{x}_2\}$) that $\varphi^{-1}(\{x, \bar{x}_2\})$ is infinite. Hence either $\varphi^{-1}(x)$ or $\varphi^{-1}(\bar{x}_2)$ are infinite. Assume that $\varphi^{-1}(x)$ is infinite. Then $\varphi^{-1}(x) \in \mathscr{K}$ (there is $f \in \Phi_{\mathscr{K}_2}(V)$ with $f^{-1}(1) = \{x, \bar{x}_1\}$) and $\varphi^{-1}(\{x, \bar{x}_2\}) \supsetneqq \varphi^{-1}(x)$. By the property b) of $\mathscr{K}_1$, $\varphi^{-1}(\{x, \bar{x}_2\}) \notin \mathscr{K}_1$ and therefore $\varphi^{-1}(\{x, \bar{x}_2\}) = X$ by the definition of $\Phi_{\mathscr{K}_1}(S)$ – a contradiction with (i) (obviously there exists $(u, v) \in R$ with $\{u, v\} \cap \cap \{x, \bar{x}_2\} = \emptyset$). Thus $\varphi^{-1}(\bar{x}_1) \neq \emptyset$, analogously $\varphi^{-1}(\bar{x}_2) \neq \emptyset$. Since there are $f_1, f_2 \in \Phi_{\mathscr{K}_2}(V)$ with $f_1^{-1}(1) = \{x, \bar{x}_1\}$, $f_2^{-1}(1) = \{x, \bar{x}_2\}$ and $\operatorname{card} \varphi^{-1}(\{x, \bar{x}_1\})$, $\operatorname{card} \varphi^{-1}(\{x, \bar{x}_2\}) \geqq 3$ we get that $\varphi^{-1}\{x, \bar{x}_1\}$, $\varphi^{-1}(\{x, \bar{x}_2\})$ are infinite (because

$\varphi \circ f_1, \varphi \circ f_2 \in \Phi_{\mathscr{K}_1}(S))$. Since $\varphi^{-1}(\{x, \bar{x}_1\}) \neq X \neq \varphi^{-1}\{x, \bar{x}_2\}$ by (i), we necessarily get $\varphi^{-1}(\{x, \bar{x}_1\})$, $\varphi^{-1}(\{x, \bar{x}_2\}) \in \mathscr{K}_1$ and therefore card $(\varphi^{-1}(\{x, \bar{x}_1\}) - \varphi^{-1}(\{x, \bar{x}_2\})) = \text{card}\, \varphi^{-1}(\bar{x}_1) > 1$ and card $(\varphi^{-1}(\{x, \bar{x}_2\}) - \varphi^{-1}(\{x, \bar{x}_1\})) = \text{card}\, \varphi^{-1}(\bar{x}_2) > 1$.

Now we get that if card $\varphi^{-1}(\bar{x}) > 1$ and $(\bar{x}, \tilde{x}) \in R$ then card $\varphi^{-1}(\tilde{x}) > 1$ since $(\bar{x}, \tilde{x})$ is in a triangle and so for every $z \in X$, card $\varphi^{-1}(z) > 1$. By the property c) of $\mathscr{K}_2$ there are $g_1, g_2, g_3 \in \Phi_{\mathscr{K}_2}(V)$ with $g_1^{-1}(1) \subsetneqq g_2^{-1}(1) \subsetneqq g_3^{-1}(1) = X$, hence $(\varphi \circ g_1)^{-1}(1) \subsetneqq (\varphi \circ g_2)^{-1}(1) \subsetneqq (\varphi \circ g_3)^{-1}(1) \subseteq X$ (by the above property and (ii)) and since card $\varphi^{-1}(z) > 1$ we get that $(\varphi \circ g_1)^{-1}(1) \in \mathscr{K}_1$ and therefore $(\varphi \circ g_2)^{-1}(1) = X$, which contradicts the definition of $\Phi_{\mathscr{K}_1}(S)$. Hence (iii) is proved.

(iv) $\varphi(X) \supset X$. Indeed, if $\varphi^{-1}(x) = \emptyset$ for some $x \in X$ then there is $\bar{x}$ with $(x, \bar{x}) \in R$ and therefore card $\varphi^{-1}(\{x, \bar{x}\}) \cap X > 1$ by (i) but card $\varphi^{-1}(\{x, \bar{x}\}) \leqq 1$ by (iii) – a contradiction.

(v) $\varphi(X) = X$. Indeed, there is $f \in \Phi_{\mathscr{K}_2}(V)$ with $f^{-1}(1) = X$, hence $\varphi \circ f \in \Phi_{\mathscr{K}_1}(S)$ and by (iv), $(\varphi \circ f)^{-1}(1) \subset X$. Consequently, either $(\varphi \circ f)^{-1}(1) = X$, then $\varphi^{-1}(X) = X$ (and $\varphi(X) = X$ by (iv)), or $(\varphi \circ f)^{-1}(1) \in \mathscr{K}_1$ (and then $\varphi^{-1}(X) \in \mathscr{K}_1$). But there are $g_1, g_2 \in \Phi_{\mathscr{K}_2}(V)$ with $g_1^{-1}(1) \subsetneqq g_2^{-1}(1) \subsetneqq f^{-1}(1)$. Then (iv) yields $(\varphi \circ g_1)^{-1}(1) \subsetneqq (\varphi \circ g_2)^{-1}(1) \subsetneqq (\varphi \circ f)^{-1}(1)$. Since $(\varphi \circ f)^{-1}(1) \in \mathscr{K}_1$ then the property b) of $\mathscr{K}_1$ implies $(\varphi \circ g_2)^{-1}(1) \notin \mathscr{K}_1$ and card $(\varphi \circ g_2)^{-1}(1) = 2$ and card $(\varphi \circ g_1)^{-1}(1) < 2$ – a contradiction.

(vi) $\varphi/X = 1_X$. Since φ/X is a bijection from X to itself (see (iii) and (v)) we get that φ/X is a compatible mapping from G_α to itself and so $\varphi/X = 1_X$ because G_α is rigid.

(vii) $\mathscr{K}_2 \subset \mathscr{K}_1$. For every $K \in \mathscr{K}_2$ there is $f \in \Phi_{\mathscr{K}_2}(V)$ with $f^{-1}(1) = K$, hence $(\varphi \circ f)^{-1}(1) = K \in \mathscr{K}_1$.

(viii) $\varphi/Y: (Y, S) \to (Z, V)$ is a morphism of $S(P_2)$. If $f \in V$ then there is $g \in \Phi_{\mathscr{K}_2}(V)$ with $g^{-1}(1) \cap X = U$ and $g/Z = f$. Since $\varphi \circ g \in \Phi_{\mathscr{K}_1}(V)$ and $(\varphi \circ g)^{-1}(1) \cap X = g^{-1}(1) \cap X = U$ we get that $(\varphi \circ g)/Y \in S$ but $(\varphi \circ g)/Y = \varphi/Y \circ g = \varphi/Y \circ g/Z = \varphi/Y \circ f$ and therefore φ/Y is a morphism of $S(P_2)$.

(ix) $\varphi = \varphi/Y \vee 1_X$.

This follows from (ii) and (vi). The proof is complete.

We say that a functor F: Set $\to$ Set is non-increasing from α upwards if for every set X with card $X \geqq \alpha$ we have card $FX = \text{card}\, X$.

Proposition 3. *There is a strong full embedding from $S(Q_2)$ to $S(P_2)$ such that the underlying functor is non-increasing from $\aleph_0$ upwards.*

Proof. The proof is similar to that of the foregoing proposition but simpler. Let $G = (X, R)$ be a symmetric, rigid, connected graph such that every edge lies in a triangle and card $X = \aleph_0$ (there is such a graph – see the foregoing proof). Choose a decomposition $\{X_1, X_2, X_3, X_4\}$ on X such that every set is infinite. We shall define a strong full embedding Φ: $S(Q_2) \to S(P_2)$: if $(Y, V) \in S(Q_2)$ then put $\Phi(Y, V) = (Q_2 Y \vee X, \Phi(V))$, and f: $Q_2 Y \vee X \to \{0, 1\}$ is in $\Phi(V)$ if either

$f^{-1}(1) \cap X = X_1$ and $f^{-1}(1) \cap V = \emptyset$ or
$f^{-1}(1) = X$ or
$f^{-1}(1) = \{x_1, x_2\}$ for some $(x_1, x_2) \in R$ or
$f^{-1}(1) = X_2 \cup \{(y_1, y_2);\ y_2 \in Y, y_1 \in A\}$ for some $A \subset Y$ or
$f^{-1}(1) = X_3 \cup \{(y_1, y_2);\ y_1 \in Y, y_2 \in A\}$ for some $A \subset Y$ or
$f^{-1}(1) = X_4 \cup \bigcup\{Q_2 Y_i;\ i \in I\}$ for some decomposition $\{Y_i;\ i \in I\}$ of Y

(notice that $Q_2 Y = \{(y_1, y_2);\ y_1, y_2 \in Y\} = Y \times Y$) and for $\varphi: (Y, S) \to (Z, T)$ define $\Phi\varphi = Q_2\varphi \vee 1_X$. The proof that Φ is a strong full embedding will be divided into several parts – analogously as in the foregoing proof.

(i) *If* $\varphi: (Y, S) \to (Z, T)$ *is compatible then* $Q_2\varphi \vee 1_X$: $\Phi(Y, S) \to \Phi(Z, T)$ *is a morphism of* $S(P_2)$.

Proof. If $f \in \Phi(T)$ then

either a) $f^{-1}(1) \cap X = X_1$ and $f^{-1}(1) \cap T = \emptyset$, then $((Q_2\varphi \vee 1_X) \circ f)^{-1}(1) = (Q_2\varphi)^{-1} \circ f^{-1}(1) \cup 1_X^{-1} \circ f^{-1}(1) = (Q_2\varphi)^{-1} \circ (f^{-1}(1)) \cup X_1$ and because $Q_2\, \varphi(S) \subset T$ we obtain $(Q_2\varphi)^{-1} \circ f^{-1}(1) \cap S = ((Q_2\varphi \vee 1_X) \circ f)^{-1}(1) \cap T = \emptyset$ and so $(Q_2\varphi \vee 1_X) \circ f \in \Phi(S)$,

or b) $f^{-1}(1) = \{x_1, x_2\}$ for some $(x_1, x_2) \in R$, then $((Q_2\varphi \vee 1_X) \circ f)^{-1}(1) = \{x_1, x_2\}$ – and so $(Q_2\varphi \vee 1_X) \circ f \in \Phi(S)$,

or c) $f^{-}(1) = X_2 \cup \{(y_1, y_2);\ y_2 \in Z, y_1 \in A\}$ for some $A \subset Z$, then $((Q_2\varphi \vee 1_X) \circ f)^{-1}(1) = X_2 \cup (\{(y_1, y_2);\ y_2 \in Y, y_1 \in \varphi^{-1}(A)\})$ and so $((Q_2\varphi \vee 1_X) \circ f) \in \Phi(S)$,

or d) $f^{-1}(1) = X_3 \cup \{(y_1, y_2);\ y_1 \in Z, y_2 \in A\}$ for some $A \subset Z$ – the proof is the same as that of c),

or e) $f^{-1}(1) = X_4 \cup \bigcup\{Q_2 Z_i;\ i \in I\}$ for a decomposition $\{Z_i;\ i \in I\}$ of Z, then $((Q_2\varphi \vee 1_X) \circ f)^{-1}(1) = X_4 \cup \{Q_2(\varphi^{-1}(Z_i));\ i \in I\}$ and so $(Q_2\varphi \vee 1_X) \circ f \in \Phi(S)$,

or f) $f^{-1}(1) = X$, then $((Q_2\varphi \vee 1_X) \circ f)^{-1}(1) = X$, too, and thus $(Q_2\varphi \vee 1_X) \circ f \in \Phi(S)$,

and (i) is proved. Hence we immediately get that Φ is a functor.

Assume that $\varphi: (Q_2 Y \vee X, \Phi(S)) \to (Q_2 Z \vee X, \Phi(T))$ is a morphism of $S(P_2)$.

(ii) *For every* $(x_1, x_2) \in R$, card $\varphi^{-1}(\{x_1, x_2\}) \cap X \geqq 2$.

Proof is analogous to the foregoing one.

(iii) *For every* $y \in Q_2 Y$, $\varphi(y) \in Q_2 Z$.

Proof. Assume the contrary, which means that $\varphi(y) \in X$ for some $y \in Q_2 Y$. Let $x = \varphi(y)$. There are x_1, x_2 such that x, x_1, x_2 form a triangle in (X, R). Then there are $f_1, f_2 \in \Phi(T)$ with $f_1^{-1}(1) = \{x, x_1\}$, $f_2^{-1}(1) = \{x, x_2\}$ and $(\varphi \circ f_i)^{-1}(1) \cap \cap Q_2 Y \neq \emptyset$ for $i = 1, 2$. Since $\varphi \circ f_i \in \Phi(S)$, $i = 1, 2$, we have $(\varphi \circ f_i)^{-1}(1) \cap X = X_j$ for some $j = 1, 2, 3, 4$. Thus either $(\varphi \circ f_1)^{-1}(1) \cap X = (\varphi \circ f_2)^{-1}(1) \cap \cap X$, then $(\varphi \circ f_1)^{-1}(1) \cap X = \varphi^{-1}(x)$ and $\varphi^{-1}(x_1) \cap X = \emptyset = \varphi^{-1}(x_2) \cap X$ – a contradiction with (ii), or $(\varphi \circ f_1)^{-1}(1) \cap X \cap (\varphi \circ f_2)^{-1}(1) \cap X = \emptyset$, then $\varphi^{-1}(x) \cap X = \emptyset$ and $\varphi^{-1}\{x_1, x_2\} \cap X = ((\varphi \circ f_1)^{-1}(1) \cap X) \cup ((\varphi \circ f_2)^{-1}(1) \cap \cap X) = X_j \cup X_k$ where $j, k = 1, 2, 3, 4$, $j \neq k$. On the other hand, there is $f_3 \in \Phi(T)$ with $f_3^{-1}(1) = \{x_1, x_2\}$ and hence $\varphi \circ f_3 \notin \Phi(S)$ – a contradiction. Therefore (iii) holds.

(iv) *If* $x_1 \neq x_2$, $x_1, x_2 \in X$ *then* $\varphi(x_1), \varphi(x_2) \in X$ *implies* $\varphi(x_1) \neq \varphi(x_2)$.

Proof is the same as that of (iii) in the foregoing proof. Now by the same argument as in the foregoing proof we get that $\varphi(X) \supset X$, hence $\varphi(X) = X$ and

(v) $\varphi / X = 1_X$.

(vi) *There is* $\psi_1: Y \to Z$ *such that for every* $(y_1, y_2) \in Q_2 Y$, $\varphi(y_1, y_2) = (\psi_1(y_1), z)$ *for some* $z \in Z$.

Proof. For every $z \in Z$ there is $f_z \in \Phi(T)$ with $f_z^{-1}(1) = X_2 \cup \{(z, z_1);\ z_1 \in Z\}$. Hence $\varphi \circ f_z \in \Phi(S)$ and $(\varphi \circ f_z)^{-1}(1) \cap X = X_2$ – therefore $(\varphi \circ f_z)^{-1}(1) = X_2 \cup \{(y_1, y_2);\ y_2 \in Y,\ y_1 \in A_z\}$. Then for $z \neq \bar{z}$ we have $A_z \cap A_{\bar{z}} = \emptyset$ and since $\varphi(Q_2 Y) \subset Q_2 Z$ (by (iii)) we get that $\{A_z;\ z \in Z,\ A_z \neq \emptyset\}$ is a decomposition of Y. Define $\psi_1(y) = z$ iff $y \in A_z$. Then ψ_1 evidently has the required property.

(vii) $\varphi / Q_2 Y = \psi_1 \times \psi_2$, $\varphi = (\psi_1 \times \psi_2) \vee 1_X$.

Proof. Analogously as in (vi) we find $\psi_2: Y \to Z$ with $\varphi(y_1, y_2) = (z, \psi_2(y_2))$ for every $(y_1, y_2) \in Q_2 Y$ – this means that $\varphi / Q_2 Y = \psi_1 \times \psi_2$. The rest follows from (iii) and (v).

(viii) $\psi_1 = \psi_2$.

Proof. Assume that $\psi_1(x) \neq \psi_2(x)$. There is $f \in \Phi(T)$ with $f^{-1}(1) = \{(z, z);\ z \in Z\} \cup X_4$. Then $(\varphi \circ f)^{-1}(1) \cap X = X_4$ and so $(\varphi \circ f)^{-1}(1) = X_4 \cup \{Q_2 Y_i;\ i \in I\}$, where $\{Y_i;\ i \in I\}$ is a decomposition of Y, but $(x, x) \notin (\varphi \circ f)^{-1}(1)$ – a contradiction. Thus $\psi_1 = \psi_2 = \psi$, $\varphi = Q_2 \psi \vee 1_X$.

(ix) $\psi: (Y, S) \to (Z, T)$ *is a morphism of* $S(Q_2)$.

Proof. Since there is $f \in \Phi(T)$ with $f^{-1}(1) = X_1 \cup (Q_2 Z - T)$, hence $\varphi \circ f \in \Phi(S)$ and $(\varphi \circ f)^{-1}(1) \cap X = X_1$; therefore $(\varphi \circ f)^{-1}(1) \cap S = \emptyset$, which implies $Q_2\, \psi(S) \subset \subset T$.

(x) Φ *is a strong full embedding with an underlying functor non-increasing from* $\aleph_0$ *upwards.*

Proof. By (i), (iii), (v), (vii), (viii), (ix) we get that Φ is a strong full embedding. By Theorem 1,4 of [K3] we get the rest of the assertion.

The proof of Proposition 3 is complete.

Corollary 4. *$S(P_2)$ is strongly $S(Q_2)$-spanned from $\aleph_0$ upwards.*

Proof follows from Propositions 2 and 3 if we use the result from [K_3] (Proposition 1.8).

Now we recall some notions from [K_2] and [K_3]. For a contravariant functor F and $x \in FX$ define a subfunctor F^x, $F^x Y = \{F f(x);\ f\colon Y \to X\}$ and set $F(x)\, Y = \{F\, g(x);\ \exists f\colon X \to Z$ onto, $g\colon Y \to X$ with $g \circ f$ is onto, $x \in F f(FZ)\}$.

Lemma 5. *For every contravariant set functor $F\colon$ Set $\to$ Set, for every point $x \in FX$ and for every infinite set Y with* card $Y \geqq$ card X *we have*:

a) card $F^x Y =$ card $F(x)\, Y$;
b) card $F(x)\, Y =$ card 2^Y.

Proof of a). Denote by $\mu\colon Y \to X \vee Y$ the sum injection.

Then for each $f\colon Y \to X$ there is a mapping $h\colon X \vee Y \to X$ onto with $f = \mu \circ h$. Thus $F\mu(F(x)\,(X \vee Y)) = F^x Y$ and since card $Y =$ card $X \vee Y$ we get card $F(x)\, Y =$ card $F(x)\,(X \vee Y) \geqq$ card $F^x Y$. The converse inequality is obvious.

Proof of b) – see [K_4].

Hence we get

Proposition 6. *If there is a full embedding of $S(P_2)$ to $S(F)$ then $S(F)$ is strongly ultimately $S(P_2)$-spanned.*

Proof. In [K_2] it is proved that if there is a full embedding of $S(P_2)$ to $S(F)$ then there is $x \in FX$ such that there is a strong full embedding of $S(P_2)$ to $S(F^x)$ such that the underlying functor is non-increasing from β upwards, where β is a suitable cardinal. By Lemma 5 and Proposition 2 (and Propositions 1,8, 1,9 in [K_3]), $S(F^x)$ is strongly $S(P_2)$-spanned from $\max\{\beta, \aleph_0, \text{card}\, X\}$ upwards. We prove that $S(F)$ is strongly $S(P_2)$-spanned from $\max\{\beta, \aleph_0, \text{card}\, X\}$ upwards. Choose a cardinal $\alpha \geqq \max\{\beta, \aleph_0, \text{card}\, X\}$. Then card $\{(Z, \gamma);\ (Z, \gamma)$ is an object of $S(F)$, card $Z \leqq \alpha\}/\sim\ =$ card $2^{F\alpha}$ (where $(Z, \gamma) \sim (V, \delta)$ if they are isomorphic, and α is the set ordinals smaller than α), hence we have to construct a family $\{G_i;\ i \in I\}$ of independent strong full embeddings from $S(P_2)$ to $S(F)$ such that the underlying functors are non-increasing from α upwards and card $I =$ card $2^{F\alpha}$. We can assume that there is a family $\{H_j;\ j \in J\}$ of independent strong full embeddings from $S(P_2)$ to $S(F^x)$ such that the underlying functors are non-increasing from α upwards and, moreover, the functor C_α is their subfunctor and card $J =$ card $2^{2^\alpha} =$ card $2^{F^x}\alpha$. Now we use the idea from [K_3] to complete the proof. We choose a bijection $\varphi\colon \exp F_x\alpha \to J$ and for a subset $U \subset F\alpha$ define a functor $G_U\colon S(P_2) \to S(F)$ as follows: Put $j = \varphi(U \cap F^x\alpha)$. Then for an object (Y, T) of $S(P_2)$ define $G_U(Y, T) = (Z, V)$ where

$H_j(Y, T) = (Z, W)$ and $V = W \cup Fk_Y(U - F^x\alpha)$ where $k_Y: Z \to \alpha$ is a mapping such that $k_Y/\alpha = 1_\alpha$ and $k_Y(Z - \alpha) = \{0\}$ ($\alpha \subset Z$ since C_α is a subfunctor of the underlying functor of H_j). If $f: (Y, T) \to (Y', T')$ is a morphism then put $G_U f = H_j f$. Since $H_j f/\alpha = 1_\alpha$ we get $k_Y = k_{Y'} \circ f$ and so $Fk_Y(U - F^x\alpha) = Ff(Fk_{Y'}(U - F^x\alpha)$, and because H_j is a functor we get that G_U is a functor, too. If $f: G_U(Y, T) \to$ $\to G_{U'}(Y', T')$ is a morphism of $S(F)$ then we get that $U' \supset U$, $U' \cap F^x\alpha = U \cap F^x\alpha$ and there is a morphism $g: (Y, T) \to (Y', T')$ of $S(P_2)$ with $f = H_{\varphi(U \cap F^x\alpha)} g$. Indeed, since F^x is a subfunctor of F we get: if $G_U(Y, T) = (Z, V)$, $G_U'(Y', T') = (Z', V')$ then f is a morphism from $(Z, V \cap F^x Z) = H_{\varphi(U \cap F^x\alpha)}(Y, T)$ into $(Z', V' \cap F^x Z') =$ $= H_{\varphi(U' \cap F^x\alpha)}(Y', T')$ and from the properties of H_j we get the second and the third assertion. Hence $f/\alpha = 1_\alpha$ and so we get the first one, too. To conclude the proof it suffices to construct a set $\mathfrak{A} \subset \exp F\alpha$ such that a) $V_1, V_2 \in \mathfrak{A}$, $V_1 \neq V_2$ implies $V_1 - (F^x\alpha \cup V_2) \neq \emptyset \neq V_2 - (F^x\alpha \cup V_1)$ and b) $\operatorname{card} \mathfrak{A} = \operatorname{card} \exp F\alpha$. Since $\operatorname{card} F\alpha \geqq \operatorname{card} F^x\alpha = \operatorname{card} 2^\alpha$, we easily construct the required set (for example, we can use the simplified way from Lemma 1) and the proof is complete.

In the end we give some conditions equivalent with the universality of $S(F)$ (by which we mean that every concrete category can be fully embedded to $S(F)$ without set theoretical assumptions; e.g., $S(P_2)$ is universal [K]). To this purpose we recall some notions from [K_2].

For a contravariant functor F and for $x \in FX$ put $\mathscr{F}_x = \{\sim;\ \sim$ is an equivalence on X, $x \in Ff(F(X/\sim))$ where $f: X \to X/\sim$ is the canonical quotient mapping$\}$ and denote by e_x the cointersection of all equivalences $\sim$ in $\mathscr{F}_x$ (i.e., e_x is the finest equivalence which is coarser than every equivalence $\sim$ in $\mathscr{F}_x$).

An equivalence e is called a *finite decomposition* if all its classes are finite and it has only a finite number of non-singleton classes.

Lemma 7. *If* F: Set $\to$ Set *is a contravariant functor and* $x \in FX$ *then every finite decomposition* e *which is finer than* e_x *lies in* $\mathscr{F}_x$.

Proof. see [K_2].

The following result is easy to obtain:

Lemma 8. *If* F: Set $\to$ Set *is a contravariant functor and* $x \in FX$, *then any mappings* $f, g: Y \to X$ *for which there is* $e \in \mathscr{F}_x$ *such that* $(f(y), g(y)) \in e$ *for each* $y \in Y$, *fulfil* $Ff(x) = Fg(x)$.

Proof. Let $h: X \to X/e$ be the canonical mapping. Then for some $x_1 \in F(X/\sim)$, $Fh(x_1) = x$ and $f \circ h = g \circ h$. Thus $Ff(x) = F(f \circ h)(x_1) = F(g \circ h)(x_1) =$ $= Fg(x)$.

We recall that $t: X \to X$ is a transposition if there are $x, y \in X$ such that $t(x) = y$, $t(y) = x$ and for $z \in X$, $z \neq x, y$, $t(z) = z$.

If $f: X \to Y$, $g: X \to Z$ are mappings onto, $h: X \to V$ is their counion (i.e. there are $f_1: V \to Y$, $g_1: V \to Z$ with $f = f_1 \circ h$, $g = g_1 \circ h$ and whenever $f = k_1 \circ k$, $g = k_2 \circ k$ for some k then there is a unique r with $h = r \circ k$), then a contravariant functor F dualizes this counion if $Ff(FY) \cup Fg(FZ) = Fh(FV)$. We say that F *dualizes* counions if it dualizes all counions, and F dualizes counions with finite decompositions when it dualizes all counions of $f: X \to Y$, $g: X \to Z$ such that the equivalence $\sim$ on X is a finite decomposition and $x \sim y$ iff $f(x) = f(y)$.

We recall a notion from [K3]:

Let $(\mathscr{K}, U)$, $(\mathscr{L}, V)$ be concrete categories. Then a family $\{\Phi_i: (\mathscr{K}, U) \to (\mathscr{L}, V); i \in I\}$ is called a $(\mathscr{K}, \alpha)$-span in $\mathscr{L}$ if

a) Φ_i are independent full embeddings;
b) for each object K of $\mathscr{K}$, card $V(\Phi_i(K)) \leqq \max\{\alpha, \text{card } U(K)\}$ for each $i \in I$;
c) card I = card $\{L$ is an object of $\mathscr{L}$; card $VL \leqq \alpha\}/\sim$, where $L \sim L_1$ iff L, L_1 are isomorphic.

In particular, if $(\mathscr{K}, U)$ is discrete then we say that $(\mathscr{L}, U)$ has a discrete α-span.

Now we formulate a big characterization theorem (compare it with the analogous theorem for covariant set functors in [K3]).

Theorem 9. *For a contravariant set functor* F: Set $\to$ Set *the following are equivalent*:

1) $S(F)$ *is universal*;
2) $S(F)$ *is binding*;
3) *there is a strong full embedding from* $S(P_2)$ *to* $S(F)$;
4) *there is a strong full embedding from* $S(Q_2)$ *to* $S(F)$;
5) $S(F)$ *has more than* card $2^{F\emptyset}$ + card 2^{F1} *non-isomorphic rigid objects*;
6) *there is a rigid object* (X, V) *in* $S(F)$ *with* card $X > 1$;
7) *there is an object* (X, V) *in* $S(F)$ *such that its monoid of endomorphisms is finite and different from the monoid of all mappings of* X;
8) $S(F)$ *is strongly ultimately* $S(P_2)$*-spanned*;
9) $S(F)$ *is strongly ultimately* $S(Q_2)$*-spanned*;
10) $S(F)$ *is strongly ultimately* $\mathscr{M}$*-spanned for every finite single-object category*;
11) $S(F)$ *is strongly ultimately* $\mathscr{M}$*-spanned for some finite single-object category*;
12) $S(F)$ *is ultimately discretely spanned*;
13) $S(F)$ *has an* $(\mathscr{M}, \alpha)$*-span for a finite single-object category and an infinite cardinal* $\alpha >$ card $2^{F\emptyset}$ + card 2^{F1};
14) F *is nearly faithful*;
15) F *does not dualize counions with a finite decomposition*;
16) *there is a set* X *and* $x \in FX$ *such that* e_x *is nontrivial* (*i.e. there are* x, y *such that* $(x, y) \notin e_x$);

17) *there is a set X and a transposition $t: X \to X$ with $Ft \neq F1_X$;*
18) *there is a set X and a mapping $f: X \to X$ such that there is only a finite number of points x of X with $f(x) \neq x$ and $Ff \neq F1_X$;*
19) *there is a cardinal α such that for every set X with* card $X \geqq \alpha$ *and every transposition $t: X \to X$, $Ft \neq F1_X$;*
20) *there is a cardinal α such that for every set X with* card $X \geqq \alpha$ *and every mapping $f: X \to X$ such that $f(x) \neq x$ only for a finite number of points x of X we have $Ff \neq F1_X$.*

Note. We recall that a concrete category is binding if there is a full embedding of the category of graphs into it (or there is a full embedding of any category of algebras into it), and is universal if every concrete category has a full embedding into it. It is easy to see that every universal concrete category is binding and as was proved e.g. in [K], the converse implication is equivalent to the set axiom that there is only a set of measurable cardinals. Nonetheless, if F: Set $\to$ Set is a contravariant functor then the two properties are equivalent without any set assumptions. This is not true for covariant set functors, see [K], e.g. $S(Q_2)$ is only binding. On the other hand, for a covariant set power functor P (i.e. $PX = \exp X$, and $Pf(Z) = f(Z)$), $S(P)$ is universal, see [K].

If we compare the results in [K_1 and K_2] with those in [K_3] and Theorem 9 we find that they are similar. Let us call the reader's attention to a difference. The role of $S(Q_2)$ is played, for a contravariant functor, by the category $S(P_2)$, but the role of $S(P)$ where P is a covariant power-set functor is not matched by any contravariant functor because every $S(F)$ can be fully embedded to $S(P_2)$ without any set assumptions. The second difference consists in the fact that there is a faithful covariant functor F such that $S(F)$ is not binding. While for covariant functors the properties – binding and faithful – are independent, this is not true for contravariant functors. As was proved in [P_1], if F is a faithful contravariant set functor than there is a strong full embedding of $S(P_2)$ to $S(F)$ such that the underlying functor is the identity functor and therefore $S(F)$ is universal.

Proof of Theorem 9. In [K_2] the following implications were proved: 1) ⇒ 3) ⇒ ⇒ 5) ⇒ 6) ⇒ 14) ⇒ 15) ⇒ 16) ⇒ 17) ⇒ 19) ⇒ 1). It is evident that 1) ⇒ 2), 4) ⇒ 2), 1) ⇒ 7), 14) ⇒ 20) ⇒ 19), 20) ⇒ 18). From [VPH] it follows that 2) ⇒ 5). By Proposition 3 we get 3) ⇒ 4). By Proposition 6 we get 3) ⇒ 8) and by Proposition 3, 8) ⇒ 9). By [HP_1] (see also [K_3]), 9) ⇒ 10) ⇒ 12) ⇒ 5) and 10) ⇒ 11) ⇒ 13). By Lemmas 7 and 8 we get that ¬16) ⇒ ¬18) and so 18) ⇒ 16). For the same reasons ¬16) ⇒ ¬7), thus 7) ⇒ 16). To complete the proof we have to show 13) ⇒ 7) & 16). Assume that $S(F)$ has an $(\mathcal{M}, \alpha)$-span, i.e. there is a family $\{(X_i, V_i); i \in I\}$ of independent objects of $S(F)$ such that their endomorphism monoid is isomorphic with $\mathcal{M}$ and card $I = \alpha$. Since $\mathcal{M}$ is finite it suffices to show that $\mathcal{M}$ is not a monoid of all mappings of X_i. Assume the contrary, then card $X_i^{X_i} =$ card $\mathcal{M}$ for each i. Further, $\alpha >$ card $2^{F\emptyset}$ + card 2^{F1} and so card $X_i > 1$ and there are

$i \in I$ and $x \in V_i$ with $x \notin Ff(F1)$ for $f: X_i \to \{0\} = 1$. Since card $X_i < \aleph_0$ we get by Lemma 7 that either e_x is non-trivial or $x \in Ff(F1)$ — this completes the proof.

It is evident that Main Theorem follows from Theorem 9. The fact that $S(P_2)$ is discretely spanned from $\aleph_0$ upwards follows from the results of Kučera [K] and was proved independently by Pultr, whose proof has not been published.

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