Andrew M. W. Glass

A directed d-group that is not a group of divisibility

Czechoslovak Mathematical Journal, Vol. 34 (1984), No. 3, 475–476

Persistent URL: http://dml.cz/dmlcz/101971

Czechoslovak Mathematical Journal, 34 (109) 1984, Praha

A DIRECTED d-GROUP THAT IS NOT A GROUP OF DIVISIBILITY

A. M. W. GLASS, Bowling Green

(Received September 19, 1983)

In [1], J. Močkoř asked if there exists a directed group which admits the structure of a d-group but is not a group of divisibility. We provide such an example.

Recall that an abelian group G is a *group of divisibility* if there is an integral domain A with field of quotients K such that G is isomorphic to the multiplicative group $K^*/U(A)$ where $K^* = K \setminus \{0\}$ and $U(A)$ is the multiplicative group of units of A. Note that any group of divisibility is torsion-free (if $x^m y = 1$, then $x(x^{m-1}y) = 1$).

Recall that an abelian group $(G, +)$ with a partial order $\leqq$ defined on it is a *directed group* if for all $\alpha, \beta, \gamma \in G$ (i) there exist $\lambda, \mu \in G$ such that $\lambda \leqq \alpha, \beta \leqq \mu$ (i.e., $\lambda \leqq \alpha \leqq \mu \;\&\; \lambda \leqq \beta \leqq \mu$), and (ii) if $\alpha \leqq \beta$, then $\alpha + \gamma \leqq \beta + \gamma$. A directed abelian group $(G, +, \leqq)$ is said to be a *d-group* if there is a multivalued addition $\oplus$ on G, such that, for all $\alpha, \beta, \gamma, \delta \in G$

(1) $\alpha \oplus \beta = \beta \oplus \alpha$

(2) $\alpha \oplus (\beta \oplus \gamma) = (\alpha \oplus \beta) \oplus \gamma$

(3) $\alpha \in \beta \oplus \gamma$ implies $\beta \in \alpha \oplus \gamma$.

(4) $\alpha + (\beta \oplus \gamma) = (\alpha + \beta) \oplus (\alpha + \gamma)$

(5) $\infty \in \alpha \oplus \beta$ if and only if $\alpha = \beta$

(6) if $\alpha, \beta \geqq \gamma \;\&\; \delta \in \alpha \oplus \beta$, then $\delta \geqq \gamma$,

and (7) $\alpha \oplus \beta \neq \emptyset$

where $\beta \oplus \gamma \subseteq G \cup \{\infty\}$ and $\alpha + (\beta \oplus \gamma) = \{\alpha + \delta \colon \delta \in \beta \oplus \gamma\}$ for all $\alpha, \beta, \gamma \in G$, and the usual properties of ∞ hold (see [1]).

As noted above, to construct an example of a directed abelian d-group that is not a group of divisibility, it is enough to construct a directed abelian d-group that is not torsion-free. We now do this.

We will write $\alpha \parallel \beta$ if $\alpha \nleqq \beta \;\&\; \beta \nleqq \alpha$.

Let $G = \boldsymbol{Q} \oplus C_3$ where $\boldsymbol{Q}$ is the additive group of rationals with the usual ordering and C_3 is a cyclic group of order 3, say $C_3 = \{0, a, 2a\}$. Partially order G by: $(q, na) < (q', n'a)$ if and only if $q < q'$ in $\boldsymbol{Q}$, where $n, n' \in \{0, 1, 2\}$. Clearly, G is a directed group; indeed, it is a tight Riesz group (i.e., if $\alpha, \beta < \gamma, \delta$, there is $\lambda \in G$ such that $\alpha, \beta < \lambda < \gamma, \delta$). Note that

$$(*) \qquad (q, na) \parallel (q', n'a) \quad \text{if and only if} \quad q = q' \;\&\; n \neq n' .$$

Hence if $\alpha \parallel \beta$ in G, there is a unique $\gamma \in G$ such that $\gamma \parallel \alpha \,\&\, \gamma \parallel \beta$; we write $\langle \alpha \parallel \beta \rangle$ for this γ. Also

$$(\dagger) \qquad \text{if} \quad \alpha > \beta \,\&\, \beta \parallel \gamma\,, \quad \text{then} \quad \alpha > \gamma\,.$$

Let

$$\alpha \oplus \beta = \begin{cases} \{\alpha\} & \text{if} \quad \alpha < \beta \\ \{\beta\} & \text{if} \quad \alpha > \beta \\ \{\langle \alpha \parallel \beta \rangle\} & \text{if} \quad \alpha \parallel \beta \\ \{\delta\colon \delta > \alpha\} \cup \{\infty\} & \text{if} \quad \alpha = \beta\,. \end{cases}$$

Clearly $\oplus$ satisfies (1), (5) & (7), and a tedious check by cases establishes that it also satisfies (2), (3), (4) & (6). Hence G is a d-group as required. We show three of the more interesting cases used in establishing (2).

(i) $\alpha > \beta \,\&\, \beta \parallel \gamma$. By ($\dagger$), $\alpha > \gamma \,\&\, \alpha > \langle \beta \parallel \gamma \rangle$. Hence $\alpha \oplus \beta = \{\beta\}$, so $(\alpha \oplus \beta) \oplus \gamma = \langle \beta \parallel \gamma \rangle$; $\beta \oplus \gamma = \langle \beta \parallel \gamma \rangle$, so $\alpha \oplus (\beta \oplus \gamma) = \langle \beta \parallel \gamma \rangle$. Thus $(\alpha \oplus \beta) \oplus \gamma = \alpha \oplus (\beta \oplus \gamma)$.

(ii) $\alpha \parallel \beta \,\&\, \beta \parallel \gamma \,\&\, \gamma \parallel \alpha$. Hence $\alpha \oplus \beta = \{\gamma\}$, so $(\alpha \oplus \beta) \oplus \gamma = \{\infty\} \cup \{\delta\colon \delta > \gamma\}$. $\beta \oplus \gamma = \{\alpha\}$, so $\alpha \oplus (\beta \oplus \gamma) = \{\infty\} \cup \{\delta\colon \delta > \alpha\}$. By ($\dagger$), $\{\delta\colon \delta > \alpha\} = \{\delta\colon \delta > \gamma\}$. Thus $(\alpha \oplus \beta) \oplus \gamma = \alpha \oplus (\beta \oplus \gamma)$.

(iii) $\alpha = \beta < \gamma$. Hence $\alpha \oplus \beta = \{\infty\} \cup \{\delta\colon \delta > \alpha\}$, so $(\alpha \oplus \beta) \oplus \gamma = \{\delta \oplus \gamma\colon \delta > \alpha\}$. But if $\delta > \gamma$, $\delta \oplus \gamma = \{\gamma\}$; if $\delta < \gamma$, $\delta \oplus \gamma = \{\delta\}$; if $\delta \parallel \gamma$, $\delta \oplus \gamma = \langle \delta \parallel \gamma \rangle$; if $\delta = \gamma$, $\delta \oplus \gamma = \{\lambda\colon \lambda > \gamma\} \cup \{\infty\}$. Hence $(\alpha \oplus \beta) \oplus \gamma = \alpha \oplus \beta$. Now $\beta \oplus \gamma = \{\beta\}$ so $\alpha \oplus (\beta \oplus \gamma) = \alpha \oplus \beta = (\alpha \oplus \beta) \oplus \gamma$.

Of course, $\boldsymbol{Q}$ could be replaced by any abelian linearly ordered group.

We note in closing that J. Močkoř has kindly pointed out to us that our observation that a group of divisibility must be torsion-free is a special case of a result of J. Ohm [2]; viz: If $K \neq \{0\}$ is a partially ordered abelian group, H is a linearly ordered group and $\{0\} \to K \to^{\phi} J \to^{\psi} H \to \{0\}$ is lex-exact (i.e., if $j \in J$, then $j > 0$ if and only if: $j\psi > 0$ or $j = k\phi$ for some $k > 0$), then J is not a group of divisibility. In our case, $K = C_3$ with the trivial order, $J = G$ and $H = \boldsymbol{Q}$, with ϕ & ψ the natural maps.

References

[1] *Močkoř, J.:* Semi-valuations and d-groups, Czech. Math. J., *32 (107)*, 1982, 77—89.

[2] *Ohm, J.:* Semi-valuations and groups of divisibility, Canad. J. Math., *21 (1969)*, 576—591.

Author's address: Bowling Green State University, Bowling Green, Ohio 43403, U.S.A.