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Czechoslovak Mathematical Journal, Vol. 34 (1984), No. 3, 439–474

Persistent URL: http://dml.cz/dmlcz/101970

Czechoslovak Mathematical Journal, 34 (109) 1984, Praha

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(Received July 20, 1983)

INTRODUCTION

In 1965, R. C. O'Neil published a paper [17] the main theorem of which asserts that for every Abelian group G and for every two spaces X and Y such that the product $X \times Y$ is paracompact and regular there exist isomorphisms

$$(1) \qquad \check{H}^n(X \times Y; G) \approx \bigoplus_{i+j=n} \check{H}^i(X; \check{H}^j(Y; G)), \quad n = 0, 1, 2, \ldots .$$

Unfortunately, O'Neil's proof of this theorem is correct only under the additional assumption that either the space Y is compact oı the space X is homotopy equivalent to a CW complex K such that the product $K \times Y$ is compactly generated. Moreover, the examples of G. E. Bredon [5] and E. G. Skljarenko [1] have shown that without this or some other additional assumption isomorphisms (1) really need not exist.

The problem of general sufficient conditions for the existence of isomorphisms (1) was then studied in [1] and [3]. The sufficient conditions given in Theorem C of [1] have shown that in the case of a non-compact space Y the existence of isomorphisms (1) may depend on the local properties of the space X. In [3] the problem was studied in connection with similar problems for the normal Čech cohomology (see Definition 1.2) and for Grothendieck cohomology with constant coefficients, and more general results were obtained than those of [1]. It was proved, for example, that for every Abelian group G, for all paracompact regular spaces X and Y, and for all closed subspaces $A \subset X$ and $B \subset Y$ there are homomorphisms

$$(2) \qquad \bigoplus_{i+j=n} \check{H}^i(X, A; \check{H}^j(Y, B; G)) \to \check{H}^n(X \times Y, X \times B \cup A \times Y; G), \quad n = 0, 1, 2, \ldots .$$

which are natural with respect to the argument (X, A) and bijective for every pair (X, A) such that for all integers k

$$\varinjlim \{\check{H}^k(U \times Y, U \times B; G) \mid U \text{ is a neighborhood of } x\} \approx \check{H}^k(Y, B; G)$$

if $x \in X - A$, and

$$\varinjlim \{\hat{H}^k(U \times Y, U \times B \cup (U \cap A) \times Y; G) | \ U \text{ is a neighborhood of } x\} = 0$$

if $x \in A$. Similar results were also obtained for the normal Čech cohomology and for Grothendieck cohomology.

In this paper the results of [3] are further generalized and improved. It is proved that for a module G over a principal ideal domain Λ there are Λ-homomorphisms (2) with the properties described above, and sufficient conditions are given for their commuting with the connecting homomorphisms of appropriate cohomology exact sequences and for their commuting with the homomorphisms induced by continuous maps $g : (Y, B) \to (Y', B')$.

As a matter of fact, none of the results just mentioned can be found in this paper in an explicit form. With the exception of Theorem 2.7, all our main results are formulated only for the normal Čech cohomology, and it is left to the reader to state explicitly the correspodnding results for Čech or Grothendieck cohomology, which can be easily derived from them.

The paper is organized as follows. In Section 1 we recall the definition and some properties of the normal Čech cohomology groups needed in the sequel, and in Section 2 we state our main results. Section 3 contains preliminary remarks on semi-simplicial Λ-modules and their geometric realizations, which play an important role in Section 4, and the remaining two Sections 4 and 5 are devoted to proofs of two of our main theorems.

Our results do not depend on which sign conventions are used, but some arguments in our proofs do. For this reson we remark that everywhere in this paper only the natural sign conventions of [19] are used.

1. PRELIMINARIES ON THE NORMAL ČECH COHOMOLOGY

The normal Čech cohomology groups were introduced and studied in [3] and [4] (see also Remark 1.3). In this section we recall their definition and some properties needed in the sequel. More details can be found in [3] and [4].

1.1. Let X be a topological space and $A \subset X$ a subspace. By a covering of the pair (X, A) we mean a pair $\mathscr{U} = (\mathscr{U}_X, \mathscr{U}_A)$ consisting of a covering $\mathscr{U}_X = \{U_i | \ i \in I_X\}$ of the space X and of a covering $\mathscr{U}_A = \{U_i \cap A | \ i \in I_A\}$ of A, where $I_A \subset I_X$.

Given two coverings

$$\mathscr{U} = (\mathscr{U}_X, \mathscr{U}_A) : \mathscr{U}_X = \{U_i | \ i \in I_X\}, \ \mathscr{U}_A = \{U_i \cap A | \ i \in I_A\},$$

$$\mathscr{V} = (\mathscr{V}_X, \mathscr{V}_A) : \mathscr{V}_X = \{V_j | \ j \in J_X\}, \ \mathscr{V}_A = \{V_j \cap A | \ j \in J_A\}$$

of a pair (X, A), we say that $\mathscr{V}$ refines $\mathscr{U}$ and call $\mathscr{V}$ a refinement of $\mathscr{U}$ if there is a map $\varphi : (J_X, J_A) \to (I_X, I_A)$ such that $V_j \subset U_{\varphi(j)}$ for all $j \in J_X$. We also write $\mathcal{U} < \mathcal{V}$ in this case.

A covering $\mathcal{U} = (\mathcal{U}_X, \mathcal{U}_A)$ of a pair (X, A) is said to be *normal* if $\mathcal{U}_X$ and $\mathcal{U}_A$ are normal (= uniformizable = numerable) coverings of the spaces X and A, respectively. The class of all normal coverings of the pair (X, A) is clearly quasiordered and directed by the relation $<$ and contains cofinal subsets.

1.2. Definition. For a pair (X, A) of topological spaces, for an Abelian group G, and for an integer n let

$$(1.1) \qquad \hat{H}^n(X, A; G) = \varinjlim \{H^n(\mathcal{U}_X, \mathcal{U}_A; G) \mid \mathcal{U} = (\mathcal{U}_X, \mathcal{U}_A) \in \operatorname{cov}(X, A)\},$$

where $\operatorname{cov}(X, A)$ is a cofinal subset of the class of all normal coverings of the pair (X, A) and $H^n(\mathcal{U}_X, \mathcal{U}_A; G)$ denotes the n-th cohomology group with coefficients in G of the simplicial pair $(N(\mathcal{U}_X), N(\mathcal{U}_A))$ consisting of the nerves of the coverings $\mathcal{U}_X$ and $\mathcal{U}_A$, and let

$$(1.2) \qquad \hat{h}^n(X, A; G) = \varinjlim \{H^n(\mathcal{U}, \mathcal{U} \cap A; G) \mid \mathcal{U} \in \operatorname{cov}(X)\},$$

$$(1.3) \qquad \hat{h}^n_{\mathfrak{m}}(X, A; G) = \varinjlim \{H^n(\mathcal{U}, \mathcal{U} \cap A; G) \mid \mathcal{U} \in \operatorname{cov}_{\mathfrak{m}}(X)\}$$

where $\operatorname{cov}(X)$ is a cofinal subset of the class of all normal coverings of X, $\mathfrak{m}$ is an infinite cardinal, and $\operatorname{cov}_{\mathfrak{m}}(X)$ is a cofinal subset of the class of all normal coverings $\mathcal{U} = \{U_i \mid i \in I\}$ of X with $\operatorname{card} I \leqq \mathfrak{m}$.

Defining induced homomorphisms

$$f^* = \hat{H}^n(f; G) : \hat{H}^n(Y, B; G) \to \hat{H}^n(X, A; G)$$

where f is a continuous map from (X, A) into (Y, B), and connecting homomorphisms

$$(1.4) \qquad \hat{\delta}^* = \hat{\delta}^n(X, A; G) : \hat{H}^n(A; G) \to \hat{H}^{n+1}(X, A; G)$$

in the usual way, we obtain a cohomology theory $(\hat{H}^*(-; G), \hat{\delta}^*)$ on the category Top_2 of all topological pairs and their continuous maps, which is called the normal Čech cohomology theory (with coefficients in G). Similarly (1.2) and (1.3) can be regarded as cofunctors on the category Top_2.

Clearly, if G is a left (right) module over a ring Λ then (1.1)–(1.3) can be regarded as cofunctors from Top_2 into the category of left (right) modules over Λ in a canonical way, and (1.4) are Λ-homomorphisms.

1.3. Remark. The groups (1.2) and (1.3) play an auxiliary role in the theory of the normal Čech cohomology. The groups (1.2) were introduced and studied also by K. Morita in [15].

1.4. Theorem. *The normal Čech cohomology theory* $(\hat{H}^*(-; G), \hat{\delta}^*)$ *on* Top_2 *satisfies the axioms of homotopy, exactness and dimension, and the following functional excision axiom: If* $U \subset A \subset X$ *and* A *is a functional neighborhood* *of U then the inclusion map $(X - U, A - U) \hookrightarrow (X, A)$ induces an isomorphism $\hat{H}^*(X, A; G) \approx \hat{H}^*(X - U, A - U; G)$.*

Moreover, it is also additive in the following sense: If X is the topological sum of subspaces X_i $(i \in I)$ then the inclusion maps $X_i \hookrightarrow X$ $(i \in I)$ induce an isomorphism $\hat{H}^n(X; G) \approx \prod_{i \in I} \hat{H}^n(X_i; G)$ for all $n = 0, 1, 2, \ldots$.

The cofunctors $\hat{h}^n(-; G)$ and $\hat{h}^n_{\mathfrak{m}}(-; G)$ satisfy the axiom of homotopy, and there are canonical identifications $\hat{h}^n_{\mathfrak{m}}(X, A; G) = \hat{h}^n(X, A; G) = \hat{H}^n(X, A; G) = \check{H}^n(X, A; G)$ for every closed pair (X, A) where X is a paracompact regular space containing a dense subset of cardinality $\leqq \mathfrak{m}$.

1.5. We call a pair (X, A) $\mathfrak{m}$-*normal*, where $\mathfrak{m}$ is an infinite cardinal, if for every normal covering $\mathscr{U}$ of A of cardinality $\leqq \mathfrak{m}$ there is a normal covering $\mathscr{V}$ of X with $\mathscr{U} < \mathscr{V} \cap A$. This notion is clearly equivalent to the notion of the $P^{\mathfrak{m}}$-embedded subspace A of X in the sense of [16]. For properties of $\mathfrak{m}$-normal pairs see e.g. [3], [4], [10] and [16]. Here we only remark that every pair (X, A), where the inclusion map $A \hookrightarrow X$ is a cofibration, is normal, i.e. $\mathfrak{m}$ -normal for all $\mathfrak{m}$.

In the following theorem we use the well known fact that on the category of semisimplicial pairs (X, A) there is a canonical identification $H^*(X, A; G) = \hat{H}^*(|X|, |A|; G)$ commuting with the connecting homomorphisms, and some properties of semi-simplicial sets $K(G, n)$ and $L(G, n + 1)$. For these properties see 3.11, 3.12 and [11, pp. 226–231].

1.6. Theorem. *Let G be an Abelian group, n a non-negative integer and $\mathfrak{m}$ an infinite cardinal, $\mathfrak{m} \geqq$ card G. Let $K(G, n)$ be the Eilenberg-MacLane semi-simplicial group associated with the pair (G, n), let $|K|$ be its Milnor's geometric realization, and let $c_n \in H^n(K, *; G) = \hat{h}^n_{\mathfrak{m}}(|K|, *; G) = \hat{h}^n(|K|, *; G) = \hat{H}^n(|K|, *; G)$ be the fundamental cohomology class. Then for every topological pair (X, A) the formulae*

$$\hat{t}^n_{\mathfrak{m}}(X, A)([f]) = f^*(c_n) \in \hat{h}^n_{\mathfrak{m}}(X, A; G),$$

$$\hat{t}^n(X, A)([f]) = f^*(c_n) \in \hat{h}^n(X, A; G),$$

$$\hat{T}^n(X, A)([f]) = f^*(c_n) \in \hat{H}^n(X; A; G)$$

where f is a continuous map from (X, A) to $(|K|, *)$, define group homomorphisms

$$\hat{t}^n_{\mathfrak{m}}(X, A) : [X, A; |K|, *] \to \hat{h}^n_{\mathfrak{m}}(X, A; G), \tag{1.5}$$

$$\hat{t}^n(X, A) : [X, A; |K|, *] \to \hat{h}^n(X, A; G), \tag{1.6}$$

$$\hat{T}^n(X, A) : [X, A; |K|, *] \to \hat{H}^n(X, A; G) \tag{1.7}$$

which are natural with respect to both arguments (X, A) and G. The homomorphisms (1.5) and (1.6) are bijective for every pair (X, A), and (1.7) is bijective for all $(\aleph_0 +$ card $G)$-normal pairs (X, A). Moreover, for every $(\aleph_0 +$ card $G)$-normal *pair (X, A) we have the commutative diagrams*

$$\begin{array}{ccc} [A; |K(G,n)|] & \xrightarrow{\hat{T}^n} & \hat{H}^n(A; G) \\ \uparrow \approx & & \\ [X, A; |L(G, n+1)|, |K(G,n)|] & & \Big\downarrow \hat{\delta}^n \\ \Big\downarrow |p|_* & & \\ [X, A; |K(G, n+1)|, *] & \xrightarrow{T^{n+1}} & \hat{H}^{n+1}(X, A; G) \end{array}$$

$(n = 0, 1, 2, \ldots)$ *where* $p : L(G, n+1) \to K(G, n+1)$ *is the canonical projection. If G is a left (right) Λ-module then (1.5)–(1.7) are Λ-homomorphisms.*

1.7. Corollary. *If* $\mathfrak{m} \geqq \aleph_0 + \text{card}\, G$ *and the pair* (X, A) *is* $(\aleph_0 + \text{card}\, G)$-*normal, then there are canonical isomorphisms* $\hat{h}^*_{\mathfrak{m}}(X, A; G) \approx \hat{h}^*(X, A; G) \approx \approx \hat{H}^*(X, A; G)$.

1.8. Besides the normal Čech cohomology theory we shall also consider axiomatic cohomology theories (h^*, δ^*) on Top_2 which satisfy the Eilenberg-Steenrod axioms of homotopy, exactness and dimension, and the functional excision axiom (see Theorem 1.4). We shall call such cohomology theories normal to distinguish between them and cohomology theories in the sense of Eilenberg-Steenrod. Moreover, we shall suppose everywhere in this paper that our normal cohomology theories (h^*, δ^*) are non-negative, i.e. that $h^n = 0$ for $n < 0$. It is easy to see that almost all the results of [8] concerning axiomatic cohomology theories remain true also for more general normal cohomology theories.

1.9. Theorem. *Let (h^*, δ^*) be an additive normal cohomology theory on* Top_2, *G an Abelian group and $*$ a one-point space. For every group homomorphism $\varphi : G \to h^0(*)$ there exists a unique natural transformation of cohomology theories*

$$T_\varphi : (\hat{H}^*(-; G), \hat{\delta}^*) \to (h^*, \delta^*)$$

such that $T^0_\varphi(*)$ equals the composition of the canonical isomorphism $\hat{H}^0(*; G) \approx G$ and of φ.

If (h^*, δ^*) takes values in the category of left (right) modules over a ring Λ, G is a left (right) Λ-module and φ is a Λ-homomorphism, then all the homomorphisms

$$T^n_\varphi(X, A) : \hat{H}^n(X, A; G) \to h^n(X, A)$$

are Λ-homomorphisms.

1.10. In the following theorem, the normal Čech cohomology groups $\hat{H}^*(X; \mathscr{A})$ of a space X with coefficients in a presheaf $\mathscr{A}$ of Abelian groups on X are used. These groups are defined quite analogously to the Čech cohomology groups $\check{H}^*(X; \mathscr{A})$, the only difference consisting in replacing the class of all open coverings of X by the class of all normal open coverings of X.

1.11. Theorem. *Let (h^*, δ^*) be a (non-negative) additive normal cohomology* *theory on* Top_2. *For every topological pair* (X, A) *and every continuous map* $f: X \to Y$ *there exist a first-quadrant cohomology spectral sequence* $E = \{E_r, d_r, \iota_r\}_{r \geqq 2}$ *and a decreasing filtration* $Fh^*(X, A) = \{F^p h^*(X, A)\}_p$ *of* $h^*(X, A)$ *with the following properties*:

(a) $F^0 h^*(X, A) = h^*(X, A)$ *and* $F^{n+1} h^n(X, A) = 0$.

(b) E *converges to* $h^*(X, A)$ *with respect to the filtration* $Fh^*(X, A)$, *i.e. there exists an isomorphism* $E_\infty \approx GrFh^*(X, A)$ *of bigraded Abelian groups, where* $GrFh^*(X, A)$ *is the bigraded Abelian group associated with the filtration* $Fh^*(X, A)$.

(c) *There are isomorphisms* $E_2^{p,q} \approx \hat{H}^p(Y; Ph^q(f, f|A))$, *where* $Ph^q(f, f|A)$ *is the presheaf of Abelian groups on* Y *defined by putting* $Ph^q(f, f|A)(U) = h^q(f^{-1}(U), f^{-1}(U) \cap A)$ *for all open subsets* U *of* Y.

The spectral sequence E *and the filtration* $Fh^*(X, A)$ *are functorial in the usual sense, and the isomorphisms of* (b) *and* (c) *are natural.*

If (h^*, δ^*) *takes values in the category* $\mathcal{M}_\Lambda$ *of left (right)* Λ*-modules, then* E *becomes in a canonical way a spectral sequence in* $\mathcal{M}_\Lambda$, $Fh^*(X, A)$ *is a filtration of the left (right)* Λ*-module* $h^*(X, A)$, *and the isomorphisms of* (b) *and* (c) *are* Λ*-isomorphisms.*

1.12. Let (h^*, δ^*) be a normal cohomology theory on Top_2, s^q an oriented geometric q-simplex and s^{q-1} an oriented $(q-1)$-face of s^q. In [4, p. 37] natural isomorphisms

(1.8) $$[s^{q-1} : s^q] : h^{n-1}((s^{q-1}, \dot{s}^{q-1}) \times (X, A)) \approx h^n((s^q, \dot{s}^q) \times (X, A))$$

(1.9) $$[s^q] : h^{n-q}(X, A) \approx h^n((s^q, \dot{s}^q) \times (X, A))$$

were defined, reducing in the case X = one-point space and $A = \emptyset$ to the well-known isomorphisms of Eilenberg-Steenrod [8, Chap. III] and having the same formal properties. If (h^*, δ^*) takes values in the category of left (right) modules over a ring Λ then (1.8) and (1.9) are of course Λ-isomorphisms.

It follows from Theorem 1.6 and from the properties of normal pairs that $\hat{h}^*((K, L) \times (X, A); G) \approx \hat{h}^*((K, L) \times (X/\bar{A}, *); G) \approx \hat{H}^*((K, L) \times (X/\bar{A}, *); G)$ for every pair (X, A) and every closed pair (K, L) where the space K is compact and regular. Using this fact we can define the isomorphisms

(1.8′) $$[s^{q-1} : s^q] : \hat{h}^{n-1}((s^{q-1}, \dot{s}^{q-1}) \times (X, A); G) \approx \hat{h}^n((s^q, \dot{s}^q) \times (X, A); G)$$

(1.9′) $$[s^q] : \hat{h}^{n-q}(X, A; G) \approx \hat{h}^n((s^q, \dot{s}^q) \times (X, A); G)$$

having the same formal properties as isomorphisms (1.8) and (1.9).

1.13. We conclude this preparatory section with the remark that for a right Λ-module G and a left Λ-module H the cross products

$$\times : \hat{h}^p(X, A; G) \otimes_\Lambda \hat{h}^q(Y, B; H) \to \hat{h}^{p+q}((X, A) \times (Y, B); G \otimes_\Lambda H)$$
$$\times : \hat{h}^p(X, A; G) \otimes_\Lambda \hat{H}^q(Y, B; H) \to \hat{H}^{p+q}((X, A) \times (Y, B); G \otimes_\Lambda H)$$

can be defined, having all the usual properties. This easily follows from the properties of normal coverings.

Let Λ, Λ' and Γ be principal ideal domains and let $\omega : \Gamma \to \Lambda$, $\omega' : \Gamma \to \Lambda'$ be ring homomorphisms.

2.1. Theorem. *There exist Λ-homomorphisms*

$$\theta^n_\Lambda = \theta^n_\Lambda(X, A; Y, B; G) : \bigoplus_{i+j=n} \hat{h}^i(X, A; \hat{h}^j(Y, B; G)) \to \to \hat{h}^n(X \times Y, X \times B \cup A \times Y; G) \tag{2.1}$$

defined for all non-negative integers n, all Λ-modules G and all pairs (X, A) and (Y, B) of topological spaces, and having the following properties:

(a) *They are natural with respect to the argument (X, A).*

(b) *Let G be a Λ-module, G' a Λ'-module and (Y, B), (Y', B') topological pairs. If*

$$\mathrm{Ext}_\Gamma(\hat{h}^{i+1}(Y', B'; G'), \hat{h}^i(Y, B; G)) = 0 \quad \textit{for} \quad i = 0, 1, \ldots, n-1,$$

then the diagram

$$\begin{array}{ccc} \bigoplus\limits_{i+j=n} \hat{h}^i(X, A; \hat{h}^j(Y', B'; G')) & \xrightarrow{\theta^n_{\Lambda'}} & \hat{h}^n(X \times Y', X \times B' \cup A \times Y'; G') \\ \downarrow & & \downarrow \\ \bigoplus\limits_{i+j=n} \hat{h}^i(X, A; \hat{h}^j(Y, B; G)) & \xrightarrow{\theta^n_\Lambda} & \hat{h}^n(X \times Y, X \times B \cup A \times Y; G) \end{array}$$

where the vertical Γ-homomorphisms are induced by a continuous map $g : (Y, B) \to \to (Y', B')$ and a Γ-homomorphism $\gamma : G' \to G$, commutes for every pair (X, A) and every such g and γ. If (X, A) consists of a geometric simplex and its boundary then this diagram commutes without any assumption on G, G', (Y, B) and (Y', B').

(c) *Let G be a Λ-module and G' a Λ'-module, and let (Y, B), (Y', B') be closed topological pairs. If the inclusion maps $B \subsetneq Y$ nad $B' \subsetneq Y'$ are cofibrations and if*

$$\mathrm{Ext}_\Gamma(\hat{h}^{i+1}(Y', B'; G'), \hat{h}^i(Y, B; G)) = 0 \quad \textit{for} \quad i = 0, 1, \ldots, n, \tag{2.2}$$

then the diagram

$$\begin{array}{ccc} \bigoplus\limits_{i+j=n} \hat{h}^i(A; \hat{h}^j(Y', B'; G')) & \xrightarrow{\theta^n_{\Lambda'}} & \hat{h}^n(A \times Y', A \times B'; G') \\ \Big\downarrow \bigoplus\limits_i (g^* \circ \gamma_*) \circ \hat{\delta}^* & & \Big\downarrow (\mathrm{id} \times g)^* \circ \gamma_* \circ \hat{\delta}^* \\ \bigoplus\limits_{i+j=n} \hat{h}^{i+1}(X, A; \hat{h}^j(Y, B; G)) & \xrightarrow{\theta^{n+1}_\Lambda} & \hat{h}^{n+1}(X \times Y, X \times B \cup A \times Y; G) \end{array}$$

commutes for every closed pair (X, A) such that the inclusion map $A \subsetneq X$ is a cofibration, for every continuous map $g : (Y, B) \to (Y', B')$ and for every Γ-homomorphism $\gamma : G' \to G$. If (X, A) consists of a geometric simplex and its boundary, the condition (2.2) may be omitted.

(d) *Let G be a Λ-module and G' a Λ'-module, and let (Y, B) and (Y', B') be closed* *topological pairs. If the inclusion maps $B \hookrightarrow Y$ and $B' \hookrightarrow Y'$ are cofibrations and if*

$$\text{(2.3)} \qquad \mathrm{Ext}_\Gamma(\hat{h}^i(B'; G'), \hat{h}^i(Y, B; G)) = 0 \quad \textit{for} \quad i = 0, 1, \dots n\,,$$

then the diagram

$$\text{(2.4)} \qquad \begin{array}{ccc} \bigoplus_{i+j=n} \hat{h}^i(X, A; \hat{h}^j(B'; G')) & \xrightarrow{\theta^n_{A'}} & \hat{h}^n(X \times B', A \times B'; G') \\ \Big\downarrow \bigoplus_i (-1)^i g^* \circ \gamma_* \circ \delta^* & & \Big\downarrow (\mathrm{id} \times g)^* \circ \gamma_* \circ \delta^* \\ \bigoplus_{i+j=n} \hat{h}^i(X, A; \hat{h}^{j+1}(Y, B; G)) & \xrightarrow{\theta^{n+1}_A} & \hat{h}^{n+1}(X \times Y, X \times B \cup A \times Y; G) \end{array}$$

is commutative for every pair (X, A) such that the inclusion map $A \hookrightarrow X$ is a cofibration, for every continuous map $g : (Y, B) \to (Y', B')$ and for every Γ-homomorphism $\gamma : G' \to G$. If (X, A) consists of a geometric simplex and its boundary, the condition (2.3) may be omitted.

(e) *The diagram*

$$\begin{array}{ccc} \bigoplus_{i+j=n} \hat{h}^i(X, A; \Lambda) \otimes_\Lambda \hat{h}^j(Y, B; G) & \xrightarrow{\otimes} & \bigoplus_{i+j=n} \hat{h}^i(X, A; \hat{h}^j(Y, B; G)) \\ \times \Big\downarrow & & \Big\downarrow \theta^n_\Lambda \\ & \hat{h}^n(X \times Y, X \times B \cup A \times Y; G) & \end{array}$$

where $\times$ denotes the cross product, commutes for all Λ-modules G and for all pairs (X, A) and (Y, B).

(f) *The diagram*

$$\text{(2.5)} \qquad \begin{array}{ccc} \hat{h}^p(\nabla(p), \dot{\nabla}(p); \hat{h}^{n-p}(Y, B; G)) & \xrightarrow{\theta^n_\Lambda} & \hat{h}^n((\nabla(p), \dot{\nabla}(p)) \times (Y, B); G) \\ [\nabla(p)] \Big\uparrow \approx & & \approx \Big\uparrow [\nabla(p)] \\ & \hat{h}^{n-p}(Y, B; G) & \end{array}$$

commutes for every pair (Y, B), for every Λ-module G and for $p = 0, 1, \dots, n$.

(g) *Finally, $\theta^n_\Lambda(X, A;\ Y, B;\ G)$ is an isomorphism if either (X, A) has the homotopy type of a CW pair or the space Y is compact.*

Proof of this theorem will be given in Section 4.

2.2. Corollary. *$\theta^n_\Lambda(X, A; Y, B; G)$ is an isomorphism if and only if canonical projections $X \to |N(\mathscr{U})|$ $(\mathscr{U} \in \mathrm{cov}(X))$ induce an isomorphism*

$$\text{(2.6)} \qquad \varinjlim_{\mathscr{U} \in \mathrm{cov}(X)} \hat{h}^n(|N(\mathscr{U})| \times Y, |N(\mathscr{U})| \times B \cup |N(\mathscr{U} \cap A)| \times Y; G) \approx$$
$$\approx \hat{h}^n(X \times Y, X \times B \cup A \times Y; G)\,.$$

Proof. This follows from assertions (a) and (g) of Theorem 2.1 because canonical projections $X \to |N(\mathscr{U})|$ induce an isomorphism $\varinjlim \hat{h}^*(|\mathscr{U})|, |N(\mathscr{U} \cap A)|; G) \approx \approx \hat{h}^*(X, A; G)$ for every pair (X, A) and every Abelian group G.

2.3. Theorem. *There exist Λ-homomorphisms*

$$\Phi^n = \Phi^n(X, A; Y, B; G) : \bigoplus_{i+j=n} \hat{h}^i(X, A; \hat{H}^j(Y, B; G)) \to \tag{2.7}$$
$$\to \hat{H}^n(X \times Y, X \times B \cup A \times Y; G)$$

defined for all non-negative integers n, all Λ-modules G and all pairs (X, A) and (Y, B) of topological spaces, and having the following properties:

(a) *They are natural with respect to the argument (X, A).*

(b) *Let G be a Λ-module and G' a Λ'-module. If topological pairs (Y, B) and (Y', B') satisfy the condition*

$$\operatorname{Ext}_\Gamma(\hat{H}^{i+1}(Y', B'; G'), \hat{H}^i(Y, B; G)) = 0 \quad \textit{for} \quad i = 0, 1, \ldots, n - 1 ,$$

then the diagram

$$\begin{array}{ccc} \bigoplus_{i+j=n} \hat{h}^i(X, A; \hat{H}^j(Y', B'; G')) & \xrightarrow{\Phi^n_{\Lambda'}} & \hat{H}^n(X \times Y' \; X \times B' \cup A \times Y'; G') \\ \downarrow & & \downarrow \\ \bigoplus_{i+j=n} \hat{h}^i(X, A; \hat{H}^j(Y, B; G)) & \xrightarrow{\Phi^n_{\Lambda}} & \hat{H}\,(X \times Y, X \times B \cup A \times Y; G) \end{array}$$

where the vertical Γ-homomorphisms are induced by a continuous map $g : (Y, B) \to$ $\to (Y', B')$ and a Γ-homomorphism $\gamma : G' \to G$, commutes for every pair (X, A) and every such g and γ. If (X, A) consists of a geometric simplex and its boundary then this diagram commutes without any assumptions on G, G', (Y, B) and (Y', B').

(c) *Let G be a Λ-module and G' a Λ'-module, and let (Y, B) and (Y', B') be such topological pairs that*

$$\operatorname{Ext}_\Gamma(\hat{H}^{i+1}(Y', B'; G'), \hat{H}^i(Y, B; G)) = 0 \quad \textit{for} \quad i = 0, 1, \ldots, n . \tag{2.8}$$

Then the diagram

$$\begin{array}{ccc} \bigoplus_{i+j=n} \hat{h}^i(A; \hat{H}^j(Y', B'; G')) & \xrightarrow{\Phi^n_{\Lambda'}} & \hat{H}^n(A \times Y', A \times B'; G') \\ \Big\downarrow \bigoplus_i (g^* \circ \gamma_*) \circ \hat{\delta}^* & & \Big\downarrow (\mathrm{id} \times g)^* \circ \gamma_* \circ \hat{\delta}^* \\ \bigoplus_{i+j=n} \hat{h}^{i+1}(X, A; \hat{H}^j(Y, B; G)) & \xrightarrow{\Phi^{n+1}_{\Lambda}} & \hat{H}^{n+1}(X \times Y, X \times B \cup A \times Y; G) \end{array}$$

commutes for every pair (X, A) such that the inclusion map $A \subset X$ is a cofibration, for every continuous map $g : (Y, B) \to (Y', B')$ and for every Γ-homomorphism $\gamma : G' \to G$. If (X, A) consists of a geometric simplex nad its boundary, the condition (2.8) may be omitted.

(d) *Let G be a Λ-module and G' a Λ'-module, and let for topological pairs (Y, B) and (Y', B')*

$$\operatorname{Ext}_\Gamma(\hat{H}^i(B'; G'), \hat{H}^i(Y, B; G)) = 0 \quad \textit{for} \quad i = 0, 1, \ldots, n . \tag{2.9}$$

Then the diagram

$$\begin{array}{ccc}
\bigoplus_{i+j=n} \hat{h}^i(X, A; \hat{H}^j(B'; G')) & \xrightarrow{\Phi^n_{A'}} & \hat{H}^n(X \times B', A \times B'; G') \\
\Big\downarrow \bigoplus_i (-1)^i g^* \circ \gamma_* \circ \hat{\delta}^* & & \Big\downarrow (\mathrm{id} \times g)^* \circ \gamma_* \circ \hat{\delta}^* \\
\bigoplus_{i+j=n} \hat{h}^i(X, A; \hat{H}^{j+1}(Y, B; G)) & \xrightarrow{\Phi^{n+1}_A} & \hat{H}^{n+1}(X \times Y, X \times B \cup A \times Y; G)
\end{array}$$

is commutative for every pair (X, A) such that the inclusion map $A \hookrightarrow X$ is a cofibration, for every continuous map $g : (Y, B) \to (Y', B')$ and for every Γ-homomorphism $\gamma : G' \to G$. If (X, A) consists of a geometric simplex and its boundary, the assumption (1.9) may be omitted.

(e) *The diagram*

$$\begin{array}{ccc}
\bigoplus_{i+j=n} \hat{h}^i(X, A; \Lambda) \otimes_\Lambda \hat{H}^j(Y, B; G) & \xrightarrow{\otimes} & \bigoplus_{i+j=n} \hat{h}^i(X, A; H^j(Y, B; G)) \\
\times \Big\downarrow & & \Big\downarrow \Phi^n_A \\
& \hat{H}^n(X \times Y, X \times B \cup A \times Y; G) &
\end{array}$$

commutes for all Λ-modules G and all pairs (X, A) and (Y, B).

(f) *The diagram*

$$\begin{array}{ccc}
\hat{h}^p(\nabla(p), \dot{\nabla}(p); \hat{H}^{n-p}(Y, B; G)) & \xrightarrow{\Phi^n_A} & \hat{H}^n((\nabla(p), \dot{\nabla}(p)) \times (Y, B); G) \\
[\nabla(p)] \Big\uparrow \approx & & \approx \Big\uparrow [\nabla(p)] \\
& \hat{H}^{n-p}(Y, B; G) &
\end{array}$$

is commutative for every pair (Y, B), for every Λ-module G and $p = 0, 1, \ldots, n$.

(g) *Finally, $\Phi^n_A(X, A; Y, B; G)$ is an isomorphism if either (X, A) has the homotopy type of a CW pair or the inclusion map $A \subset X$ is a cofibration, A is closed in X and the spaces Y and B are compact.*

Proof. This theorem is in fact an easy corollary to Theorem 2.1. To show this let us consider the diagram

$$(2.10)\qquad \begin{array}{ccc}
\bigoplus_{i+j=n} \hat{h}^i(X, A; \hat{H}^j(Y, B; G)) & \xrightarrow{\Phi^n_A} & \hat{H}^n(X \times Y, X \times B \cup A \times Y; G) \\
\approx \Big\downarrow \beta_1 & & \Big\downarrow \beta_1' \\
\bigoplus_{i+j=n} \hat{h}^i(X, A; \hat{H}^j(\tilde{Y}, \tilde{B}; G)) & & \hat{H}^n(X \times \tilde{Y}, X \times \tilde{B} \cup A \times \tilde{Y}; G) \\
\approx \Big\uparrow \beta_2 & & \Big\uparrow \beta_2' \\
\bigoplus_{i+j=n} \hat{h}^i(X, A; \hat{h}^j(\tilde{Y}, \tilde{B}; G)) & \xrightarrow{\theta^n_A} & \hat{h}^n(X \times \tilde{Y}, X \times \tilde{B} \cup A \times \tilde{Y}; G)
\end{array}$$

where $\tilde{Y} = Y \times \{0\} \cup B \times [0, 1] \subset Y \times [0, 1]$, $\tilde{B} = B \times \{1\}$, $\beta_1, \beta_1', \beta_2$ and β_2' are the obvious canonical homomorphisms, and $\Phi^n_A = \Phi^n_A(X, A; Y, B; G)$ is to be defined. If (X, A) is a closed pair and the inclusion map $A \hookrightarrow X$ is a cofibration then the inclusion map $X \times \tilde{\tilde{B}} \cup A \times \tilde{Y} \hookrightarrow X \times \tilde{Y}$ is also a cofibration (see e.g. [6]) and therefore β'_2 is an isomorphism. Further, the pair $\{X \times B, A \times Y\}$ is excisive in this case, which implies that β'_1 is also bijective. We conclude that under our assumption on the pair (X, A) there is a unique homomorphism Φ^n_A making the diagram (2.10) commutative. This defines homomorphisms $\Phi^n_A(X, A; Y, B; G)$ for all CW pairs, and in the general case we apply the same procedure as that we used in 4.14 to define $\theta^n_A(X, A; Y, B; G)$ for general pairs (X, A). Verifying the properties (a)–(g) for the homomorphisms $\Phi^n_A(X, A; Y, B; G)$ defined in this way makes no trouble.

2.4. Corollary. *$\Phi^n_A(X, A; Y, B; G)$ is an isomorphism if and only if the canonical projections $X \to |N(\mathcal{U})|$ $(\mathcal{U} \in \operatorname{cov}(X))$ induce an isomorphism*

$$\varinjlim_{\mathcal{U} \in \operatorname{cov}(X)} \hat{H}^n(|N(\mathcal{U})| \times Y, |N(\mathcal{U})| \times B \cup |N(\mathcal{U} \cap A)| \times Y; G) \approx \approx \hat{H}^n(X \times Y, X \times B \cup A \times Y; G)\,. \tag{2.11}$$

2.5. If h is a cofunctor from the category Top_2 into the category of Abelian groups, $f: X \to Y$ is a continuous map and A is a subspace of X, we denote by $Ph(f, f \mid A)$ the presheaf of Abelian groups on Y defined by the formula $Ph(f, f \mid A)(U) = h(f^{-1}(U), f^{-1}(U) \cap A)$ and by $Sh(f, f \mid A)$ the sheaf of Abelian groups on Y generated by the presheaf $Ph(f, f \mid A)$. We have

$$Sh(f, f \mid A)_y = \varinjlim_{U} h(f^{-1}(U), f^{-1}(U) \cap A)$$

where U ranges over all (open) neighborhoods of the point y, and the inclusion maps $(f^{-1}(y), f^{-1}(y) \cap A) \hookrightarrow (f^{-1}(U), f^{-1}(U) \cap A)$ induce a canonical homomorphism

$$r_y : Sh(f, f \mid A)_y \to h(f^{-1}(y), f^{-1}(y) \cap A)\,.$$

2.6. If A is a subspace of a space X we denote by $rd_X A$ the relative dimension of A in X in the sense of P. S. Alexandroff. We recall that, by definition,

$$rd_X A = \sup_F \dim F$$

where F ranges over all non-empty closed subsets of X contained in A, and that $rd_X \emptyset = -\infty$.

Now let us suppose that for a fixed non-negative integer n, for a fixed Abelian group G and for a fixed topological pair (Y, B) we are given group homomorphisms

$$\Phi^n(X, A) : \bigoplus_{i+j=n} \hat{h}^i(X, A; \hat{H}^j(Y, B; G)) \to \hat{H}^n((X, A) \times (Y, B); G)\,, \tag{2.12}$$

which are defined for all pairs (X, A), natural, and bijective whenever (X, A) is homotopy equivalent to a CW pair.

2.7. Theorem. *Let (h^*, δ^*) be an additive normal cohomology theory on the category Top_2, let $G = h^0(*)$ be its coefficient group, and let $T : (\hat{H}^*(-; G), \hat{\delta}^*) \to$*

$\to (h^*, \delta^*)$ *be the unique natural transformation of cohomology theories extending the canonical isomorphism* $H^0(*; G) \approx G = h^0(*)$. *Let* (X, A) *be a topological pair,* $(Z, C) = (X, A) \times (Y, B) = (X \times Y, X \times B \cup A \times Y)$ *and let* $p_1 : Z \to X$ *be the canonical projection. Finally, let* N *be a non-negative integer, and let us put*

$$v(N) = 1 + \max_{0 \leq i < N} (rd_X M_i + i),$$

where

$$M_i = \{x \in X \mid r_x : Sh^i(p_1, p_1 \mid C) \not\approx h^i(p_1^{-1}(x), p_1^{-1}(x) \cap C)\}.$$

If $v(N) < N$ *and*

(a) *the subspace* A *is closed,*

(b) *the subspace* X-int A *is paracompact and each neighborhood in* X *of each point* $x \in X$-int A *is functional,*

(c) *the pair* $(X, X\text{-int } A)$ *is normal in the sense of* 1.5, *and*

(d) *the canonical homomorphism* $T^i(Y, B) : \hat{H}^i(Y, B; G) \to h^i(Y, B)$ *is bijective for* $i = 0, 1, \ldots, N - 1$ *and injective for* $i = N$,

then the homomorphism

$$(2.13) \qquad T^n(Z, C) \circ \Phi^n(X, A) : \bigoplus_{i+j=n} \hat{h}^i(X, A; \hat{H}^j(Y, B; G)) \to h^n((X, A) \times (Y, B))$$

is surjective in the case $n = v(N)$, *bijective in the case* $v(N) < n < N$ *and injective in the case* $n = N$.

Proof of this theorem will be given in Section 5.

2.8. Corollary. *If an additive normal cohomology theory* (h^*, δ^*) *on* Top_2 *with the coefficient group* G, *pairs* (X, A) *and* (Y, B), *and a non-negative integer* N *satisfy the conditions* (a)–(d) *of Theorem* 2.7, *and if for* $i = 0, 1, \ldots, N - 1$

$$r_x : Sh^i(p_1, p_1 \mid C)_x \approx h^i(Y, B) \quad \text{for} \quad x \in X - A,$$

$$Sh^i(p_1, p_1 \mid C)_x = 0 \quad \text{for} \quad x \in A$$

then (2.13) *is an isomorphism for* $n = 0, 1, \ldots, N - 1$ *and a monomorphism for* $n = N$.

2.9. Corollary. *If an additive normal cohomology theory* (h^*, δ^*) *on* Top_2 *with the coefficient group* G, *pairs* (X, A) *and* (Y, B), *and a non-negative integer* N *satisfy the conditions* (a)–(d) *of Theorem* 2.7, *and if* (X, A) *is locally contractible at every point* $x \in X$-int A (see 5.8), *then* (2.13) *is bijective for* $n < N$ *and injective for* $n = N$.

Proof. Apply Lemma 5.8.

2.10. Remark. Let us suppose that we are given Λ-homomorphisms (2.7), which are defined for all non-negative integers n, for all Λ-modules G and for all (X, A) and (Y, B), and which are natural with respect to (X, A). Then we can define Λ-

homomorphisms

$$(2.14) \qquad \Psi^n_\Lambda = \Psi^n_\Lambda(X, A; Y, B; h^*, \delta^*) : \bigoplus_{i+j=n} \hat{H}^i(X, A; \hat{H}^j(Y, B; G)) \to$$
$$\to h^n((X, A) \times (Y, B)),$$

where (h^*, δ^*) is an additive normal cohomology theory on Top_2 with values in the category of Λ-modules and with $h^0(*) = G$, and where (X, A) and (Y, B) are such that the pair $\{X \times B, A \times Y\}$ of subspaces of the space $X \times Y$ is excisive with respect to (h^*, δ^*), in such a way that the diagram

$$(2.15) \qquad \begin{array}{ccc} \bigoplus_{i+j=n} \hat{h}^i(X, A; \hat{H}^j(Y, B; G)) & \xrightarrow{\Phi^n_\Lambda} & \hat{H}^n((X, A) \times (Y, B); G) \\ \downarrow & & \downarrow T^n \\ \bigoplus_{i+j=n} \hat{H}^i(X, A; \hat{H}^j(Y, B; G)) & \xrightarrow{\Psi^n_\Lambda} & h^n((X, A) \times (Y, B)) \end{array}$$

where T is the unique natural transformation of cohomology theories extending the canonical isomorphism $\hat{H}^0(*; G) \approx G = h^0(*)$, will commute. Such (2.14) can be constructed e.g. with the help of the canonical isomorphism $\hat{H}^*(X, A; -) \approx \approx \hat{h}^*(\tilde{X}, \tilde{A}; -)$, where $\tilde{X} = X \times \{0\} \cup A \times [0, 1] \subset X \times [0, 1]$ and $\tilde{A} = A \times \{1\}$. Moreover, one can show without difficulties that the homomorphisms (2.14) are characterized by naturality and by commutativity of the diagram (2.15), and that if (2.7) have any of the properties listed in Theorem 2.3 then (2.14) have a similar property. Finally, a theorem can be proved similar to Theorem 2.7.

3. PRELIMINARY REMARKS ON SEMI-SIMPLICIAL Λ-MODULES

In proving the theorems of the preceding section, an important role is played by semi-simplicial methods. In this section, we recall some properties of Milnor's geometric realization functor from the category of ss. sets and ss. maps into the category of CW complexes and cell maps and generalize to ss. Λ-modules some well-known results on ss. Abelian groups. The basic references are [11], [12] and [13]. The terminology and notation we use are those of [11]. In particular, we denote by $[n]$ the set $\{0, 1, \ldots, n\}$, by $\Delta(n)$ the ss. n-simplex, i.e. the ss. set of all non-decreasing functions $[\ldots] \to [n]$, and by $\nabla(n)$ the standard geometric n-simplex spanned by the vectors of the standard basis of $\mathbb{R}^{n+1}$.

3.1. Theorem. *For all ss. sets X and Y, Milnor's geometric realization functor induces a map of homotopy sets* $[X; Y] \to [|X|; |Y|]$. *If Y is a Kan ss. set, this map is a bijection.*

A similar result holds for n-ads of ss. sets.

Proof. This follows immediately from [11, p. 47, Satz 5.7, and p. 48, Satz 5.8].

The following theorem was first proved by J. Milnor [13]. Let us remark that for topological spaces X and Y we denote by $X \times_k Y$ their cartesian product with the topology induced inductively by the compact subspaces of the ordinary cartesian product $X \times Y$.

3.2. Theorem. *Let X and Y be ss. sets, and let $pr_1 : X \times Y \to X$ and $pr_2 : X \times Y \to Y$ be the canonical projections. The maps $|pr_1|$ and $|pr_2|$ induce a continuous bijective map $i: |X \times Y| \to |X| \times |Y|$ and a homeomorphism $i_k: |X \times Y| \to |X| \times_k |Y|$. Consequently, if either both ss. sets X and Y are countable or one of them is locally finite then i is a homeomorphism. In the general case, i is only a homotopy equivalence.*

Similar assertions hold for ss. n-ads.

By a k-topological left (right) Λ-module, where Λ is a discrete ring, we mean a left (right) Λ-module X together with a topology on X such that the addition, the multiplication by elements of Λ and the map $x \mapsto -x$ of X into itself are continuous on compact subspaces.

3.3. Corollary. *If X is a ss. left (right) Λ-module then $|X|$ becomes in a canonical way a k-topological left (right) Λ-module. If 0 denotes both the trivial ss. Λ-submodule of X and the zero element of $|X|$, we have $|0| = \{0\}$.*

Since every ss. group is a Kan ss. set, see e.g. [11, p. 33 or 12, p. 67], we get from 3.1 and 3.3

3.4. Corollary. *If Y is a ss. left (right) Λ-module, then for every ss. set X, Milnor's geometric realization functor induces an isomorphism $[X; Y] \approx [|X|; |Y|]$ of left (right) Λ-modules.*

A similar assertion holds for an n-ad Y of ss. left (right) Λ-modules and for every n-ad X of ss. sets.

3.5. Remark. Let Y be a ss. left (right) Λ-module and let i be the canonical bijection $|Y \times Y| \to |Y| \times |Y|$. If X is a compactly generated topological space, $[X; |Y|]$ is a left (right) Λ-module with the addition induced by the map $|\alpha| \circ i^{-1}$, where $\alpha: Y \times Y \to Y$ is the addition in Y. It can be easily shown that the same addition is induced by any map $|\alpha| \circ j$, where j is a homotopy inverse of the map $i: (|Y \times Y|; |Y \times 0|, |0 \times Y|) \to (|Y| \times |Y|; |Y| \times \{0\}, \{0\} \times |Y|)$. (It follows from 3.2 and [14, Lemma 1] that such a j exists.) Moreover, every such map $|\alpha| \circ j$ is a commutative H-group structure on $|Y|$.

Similar assertions hold for n-ads.

3.6. Theorem. *A ss. map of connected (pointed) Kan ss. sets is a homotopy equivalence if and only if it induces an isomorphism of all homotopy groups.*

(See [11, p. 203, Folgerung 7.2] or [12, § 12].)

3.7. Corollary. *A ss. homomorphism $f: X \to Y$ of ss. groups is a homotopy equi-*

valence of (pointed) ss. sets if and only if it induces an isomorphism $f_*\colon \pi_i(X, *) \approx \approx \pi_i(Y, *)$ *for* $i = 0, 1, \ldots,$ *with* $*$ *denoting the neutral elements of* X_0 *and* Y_0.

3.8. Let M denote the Moore functor from the category of ss. Abelian groups into the category of positive chain complexes of Abelian groups (see [11, p. 198] for its definition).

If X is a ss. left (right) module over a ring Λ then it is clear from the definition of M that $M(X)$ has a canonical structure of a positive chain complex of left (right) Λ-modules. Similarly, if $f\colon X \to Y$ is a ss. homomorphism of ss. left (right) Λ-modules, $M(f)$ is a Λ-homomorphism. This means that M induces a functor from the category of ss. left (right) Λ-modules into the category of positive chain complexes of left (right) Λ-modules. We shall denote this induced functor by M_Λ.

Analogously, the Dold-Kan functor D from the category of positive chain complexes of Abelian groups into the category of ss. Abelian groups (see [11, p. 222] for its definition) induces a functor D_Λ from the category of positive chain complexes of left (right) Λ-modules into the category of ss. left (right) Λ-modules.

From [11, p. 223, Satz 1.5] we immediately get

3.9. Theorem. *For every ss. left (right) Λ-module X there is a canonical ss. isomorphism*

$$\iota_X\colon X \approx D_\Lambda \circ M_\Lambda(X)$$

of ss. left (right) Λ-modules, and for every positive chain complex Y of left (right) Λ-modules there is a canonical isomorphism

$$\iota'_Y\colon Y \approx M_\Lambda \circ D_\Lambda(Y)$$

of chain complexes over Λ. Both ι_X and ι'_Y are, in fact, determined by the underlying structure of a ss. Abelian group or a chain complex of Abelian groups, respectively, and

$$M_\Lambda(\iota_X) = \iota'_{M_\Lambda X}, \quad D_\Lambda(\iota'_Y) = \iota_{D_\Lambda Y}. \tag{3.1}$$

Finally, if $f, g\colon Y \to Y'$ are homotopic homomorphisms of positive chain complexes of left (right) Λ-modules then $D_\Lambda f, D_\Lambda g : D_\Lambda Y \to D_\Lambda Y'$ are homotopic ss. homomorphisms of ss. left (right) Λ-modules.

3.10. Proposition. *For every ss. left (right) Λ-module X, the inclusions $(M_\Lambda X)_n \subset \subset X_n$, $n = 0, 1, 2, \ldots$, induce a Λ-isomorphism $H_*(M_\Lambda X) \approx \pi_*(X, *)$, where $*$ denotes the zero element of X_0.*

(See [11, p. 198].)

3.11. Throughout this paper, $K(\pi, n)$ denotes the Eilenberg-MacLane ss. Abelian group corresponding to an Abelian group π and a natural number n. Let us recall that, by definition, $K(\pi, n) = Dk(\pi, n)$, where $k(\pi, n)$ is a chain complex with $k_n(\pi, n) = \pi$ and $k_i(\pi, n) = 0$ for $i \neq n$, that $\pi_i(K(\pi, n), *) = 0$ for $i \neq n$, and that there is a canonical isomorphism $\pi_n(K(\pi, n), *) \approx \pi$, which is defined as the composition

$$\pi_n(K(\pi, n), *) \approx H_n(MDk(\pi, n)) \xrightarrow{(\iota')_*^{-1}} H_n(k(\pi, n)) = \pi ,$$

where $\approx$ denotes the canonical isomorphism of Proposition 3.10. It is clear that for a left (right) Λ-module π, $K(\pi, n)$ is a ss. left (right) Λ-module, and that the canonical isomorphism $\pi_n(K(\pi, n), *) \approx \pi$ is a Λ-isomorphism.

Also the following convention is used: If X is a ss. Abelian group, the zero element of X_0, the trivial ss. subgroup of X and the zero element of $|X|$ are all denoted by $*$.

3.12. Remark. It follows from 3.1 that $|K(\pi, n)|$ is a CW-complex of type (π, n). Consequently, it has an H-group structure (with $*$ as the homotopy unit), which is unique up to homotopy and commutative. If π is a left (right) Λ-module, one can easily see that every such H-group structure commutes up to homotopy with operators from Λ. It follows that for every topological pair (X, A) the homotopy set $[X, A; |K(\pi, n)|, *]$ has a canonical Abelian group structure, which extends in a canonical way to a left (right) Λ-module structure if π is a left (right) Λ-module.

We shall now introduce an important auxiliary notion of weak ss. Λ-homomorphism of ss. Λ-modules, where Λ is a principal ideal domain.

3.13. Definition. Let Λ be a principal ideal domain. A ss. map $f: X \to X'$ of ss. Λ-modules will be called a *weak ss. Λ-homomorphism* if $f(*) = *$ and there is a diagram

$$\begin{array}{ccc} X & \xrightarrow{\ f\ } & X \\ \uparrow & & \uparrow \\ \varphi & & \psi \\ & D_\Lambda Y & \end{array}$$

which is homotopy commutative as a diagram of ss. maps of pointed ss. sets, and where Y is a free chain complex over Λ, φ and ψ are ss. Λ-homomorphisms, and φ is a homotopy equivalence of (pointed) ss. sets.

3.14. Proposition. *Let Λ be a principal ideal domain and let X and X' be ss. Λ-modules.*

(a) *Every weak ss. Λ-homomorphism $f: X \to X'$ commutes up to homotopy fixed on $*$ with the addition and the multiplication by elements of Λ in X and X', respectively.*

(b) *The composition of weak ss. Λ-homomorphisms is again a weak ss. Λ-homomorphism.*

(c) *For every homomorphism $h: \pi_*(X, *) \to \pi_*(X', *)$ of graded Λ-modules there is a weak ss. Λ-homomorphism $f: X \to X'$ inducing h.*

(d) *If* $\mathrm{Ext}_\Lambda\,(\pi_i(X, *)), \pi_{i+1}(X', *)) = 0$ *for all* $i = 0, 1, 2, \ldots,$ *then ss. Λ-homo-*

morphisms $f, g: X \to X'$ are homotopic as ss. maps of (pointed) ss. sets if and only if $f_* = g_*: \pi_*(X, *) \to \pi_*(X', *)$.

(e) *Let $\omega : \Gamma \to \Lambda$ be a homomorphism of principal ideal domains. Then every weak ss. Λ-homomorphism $f: X \to X'$ is also a weak ss. Γ-homomorphism from the ss. Γ-module X into the ss. Γ-module X.*

Proof. Let us write simply M and D instead of M_Λ and D_Λ, respectively.

Assertion (a) is obvious. The proof of all the other assertions is based on the properties of the functors M and D, on the Homotopy Classification Theorem [9, p. 177, Theorem 3.1] and on the notion of a free approximation, in the sense of [19, p. 225], of a chain complex over Λ.

Ad (b). Let $f: X \to X'$ and $f': X' \to X''$ be two weak ss. Λ-homomorphisms, and let us consider the diagram

$$\begin{array}{ccccc} X & \xrightarrow{f} & X & \xrightarrow{f'} & X \\ \uparrow{\scriptstyle\varphi} & & \uparrow{\scriptstyle\psi}\ \uparrow{\scriptstyle\varphi'} & & \uparrow{\scriptstyle\psi'} \\ & DY & & DY & \end{array}$$

where both triangles have the properties described in Definition 3.13, and the diagram

$$(3.2)\qquad \begin{array}{ccc} & MX' & \\ {\scriptstyle M\psi}\nearrow & & \nwarrow{\scriptstyle M\varphi'} \\ Y = MDY & \overset{\chi}{\dashrightarrow} & MDY' = Y' \end{array}$$

(To simplify the notation, we regard the natural equivalences ι and ι' of Theorem 3.9 as identities, which is possible in view of (3.1).) Since Y is free over Λ and, by Proposition 3.10, $M\varphi': Y' \to MX'$ induces an isomorphism of the homology groups, by the Homotopy Classification Theorem there is a homomorphism $\chi : Y \to Y'$ of chain complexes over Λ making the diagram (3.2) commutative up to homotopy. Using the last assertion of Theorem 3.9 we easily obtain that $\psi' \circ D\chi \sim f' \circ f \circ \varphi$, which proves that $f' \circ f$ is a weak ss. Λ-homomorphism.

Ad (c). Let $\chi : Y \to MX$ be a free approximation of the chain complex MX over Λ. Let us regard $H_*(MX)$ as a chain complex with trivial differential and choose a homomorphism $\tau : Y \to H_*(MX)$ of chain complexes over Λ such that $\mathrm{id} = \tau_* \circ (\chi_*)^{-1} : H_*(MX) \to H_*(MX)$; such a τ exists because Y is free over Λ and Λ is a principal ideal domain. In the same way let us choose $\chi' : Y' \to MX'$ and $\tau' : Y' \to H_*(MX')$. Since Y is free over Λ and τ' induces an isomorphism in homology, there is a homomorphism $\eta : Y \to Y'$ of chain complexes over Λ such that the diagram

$$(3.3)\qquad \begin{array}{ccccc} Y & \xrightarrow{\tau} & H_*(MX) & \xrightarrow{\approx} & \pi_*(X, *) \\ {\scriptstyle\eta}\downarrow & & & & \downarrow{\scriptstyle h} \\ Y' & \xrightarrow{\tau'} & H_*(MX') & \xrightarrow{\approx} & \pi_*(X', *) \end{array}$$

where $\approx$ denotes the canonical Λ-isomorphisms of Proposition 3.10, is homotopy commutative. Similarly, using the fact that $D\chi : DY \to X$ is a ss. homotopy equivalence of pointed ss. sets by 3.10 and 3.7, we obtain a ss. map $f: X \to X'$ satisfying $f(*) = *$ and making the diagram

(3.4)
$$\begin{array}{ccc} DY & \xrightarrow{D\chi} & X \\ {\scriptstyle D\eta}\downarrow & & \downarrow{\scriptstyle f} \\ DY' & \xrightarrow{D\chi'} & X' \end{array}$$

of ss. maps of pointed ss. sets homotopy commutative. Since $\tau_* = \chi_*$ and $\tau'_* = \chi'_*$, applying the homology functor to (3.3) and the functors $H_* \circ M$ and π_* to (3.4) yields the commutative diagram

$$\begin{array}{ccccccc} \pi_*(X, *) & & \xrightarrow{\qquad\qquad h \qquad\qquad} & & & & \pi_*(X', *) \\ \approx\uparrow & & & & & & \uparrow\approx \\ H_*(MX) & \xrightarrow{(\tau_*)^{-1}} & H_*(Y) & \xrightarrow{\eta_*} & H_*(Y') & \xrightarrow{\tau'_*} & H_*(MX') \\ \approx\downarrow & & & & & & \downarrow\approx \\ \pi_*(X, *) & & \xrightarrow{\qquad\qquad f_* \qquad\qquad} & & & & \pi_*(X', *) \end{array}$$

where $\approx$ again denotes the canonical isomorphisms of Proposition 3.10. This diagram shows that $f_* = h_*$.

Ad (d). There is a diagram

(3.5)
$$\begin{array}{ccccc} X & \xrightarrow{f} & X' & \xleftarrow{g} & X \\ {\scriptstyle \varphi}\uparrow & {\scriptstyle \psi}\nearrow & & \nwarrow{\scriptstyle \psi'} & \uparrow{\scriptstyle \varphi'} \\ DY & & & & DY' \end{array}$$

whose both triangles have the properties described in Definition 3.13. Since Y is free over Λ and $M\varphi'$ induces an isomorphism of the homology groups by Proposition 3.10, there is a homomorphism $\eta : Y \to Y'$ of chain complexes over Λ such that $M\varphi' \circ \eta \sim M\varphi$, and therefore also $\varphi' \circ D\eta \sim \varphi$ by the last assertion of Theorem 3.9. Using this, the assumption $f_* = g_*$ and the diagram (3.5), it is easy to show that $\psi_* = \psi'_* \circ (D\eta)_* : \pi_*(DY, *) \to \pi_*(X', *)$. Therefore also $(M\psi)_* = (M\psi')_* \circ \circ\, \eta_* : H_*(Y) \to H_*(MX')$ by Proposition 3.10. Since Y is free over Λ and the assumption of the assertion (d) is by Proposition 3.10 equivalent to the assumption $\mathrm{Ext}_\Lambda\,(H_i(Y), H_{i+1}(MX')) = 0$ for $i = 0, 1, 2, \ldots,$ the Homotopy Classification Theorem yields $M\psi \sim (M\psi') \circ \eta$, and therefore $\psi \sim \psi' \circ (D\eta)$ in view of Theorem 3.9. This, the relation $\varphi' \circ D\eta \sim \varphi$ and the homotopy commutativity of the diagram (3.5) immediately imply $f \sim g$, which was to prove.

The last assertion (e) follows immediately from [19, p. 225, Lemma 12], which asserts that any chain complex over a principal ideal domain has a free approximation.

3.15. Lemma. *Let Λ be a principal ideal domain and let B be a Λ-subcomplex of an acyclic Λ-complex Y. Then for every free approximation $\beta : \tilde{B} \to B$ of the Λ-complex B there is a commutative diagram*

$$\begin{array}{ccc} \tilde{B} & \xrightarrow{\subset} & \tilde{Y} \\ {\scriptstyle\beta}\downarrow & & \downarrow{\scriptstyle\gamma} \\ B & \xrightarrow{\subset} & Y \end{array}$$

where $\gamma : \tilde{Y} \to Y$ is a free approximation of Y and $\tilde{B}_n$ is a direct summand of $\tilde{Y}_n$ for all n.

Proof. Let $\tilde{Y}$ be the direct sum of the cone over $\tilde{B}$ and of a free approximation of Y. Then $\tilde{B}_n$ is a direct summand of $\tilde{Y}_n$ for all n and β can be extended to a Λ-homomorphism $\gamma : \tilde{Y} \to Y$, which is clearly a free approximation of Y.

3.16. Lemma. *Let G be a Λ-module and let $l(G, n + 1)$, $n \geqq 0$, be the following chain complex over Λ:*

$$\begin{array}{ccccccccc} \ldots \to 0 \to & l_{n+1}(G, n+1) & \to & l_n(G, n+1) & \to 0 \to \ldots \\ & \| & & \| & \\ & G & = & G & \end{array}$$

Let Y be a chain complex over Λ and let B be a Λ-subcomplex of Y. If B_n is a direct summand of Y_n then for every homomorphism $\beta : B \to k(G, n)$ of Λ-complexes there is a homomorphism $\gamma : Y \to l(G, n + 1)$ of Λ-complexes such that the diagram

$$\begin{array}{ccc} B & \xrightarrow{\subset} & Y \\ {\scriptstyle\beta}\downarrow & & \downarrow{\scriptstyle\gamma} \\ k(G, n) & \xrightarrow{\subset} & l(G, n+1) \end{array}$$

commutes.

The proof is trivial.

3.17. Proposition. *Let Λ be a principal ideal domain and let A be a ss. Λ-submodule of a homotopically trivial ss. Λ-module X. Then for every weak ss. Λ-homomorphism $f : A \to \prod_{n=0}^{\infty} K(G_n, n)$ there is a commutative diagram of ss. Λ-homomorphisms*

$$
\begin{array}{ccccc}
A & \xleftarrow{\ \alpha\ } & D_A B & \xrightarrow{\ \alpha'\ } & \prod_{n=0}^{\infty} K(G_n, n) \\
\downarrow \subset & & \downarrow \subset & & \downarrow \subset \\
X & \xleftarrow{\ \beta\ } & D_A Y & \longrightarrow & \prod_{n=0}^{\infty} L(G_n, n+1) = \prod_{n=0}^{\infty} D_A l(G_n, n+1) \\
\downarrow & & \downarrow & & \downarrow \\
X/A & \longleftarrow & D_A Y/D_A B & \longrightarrow & \prod_{n=0}^{\infty} K(G_n, n+1) \\
 & & \| & & \\
 & & D_A(Y/B) & &
\end{array}
$$

where α and β are homotopy equivalences of pointed ss. sets, B, Y and Y/B are free Λ-complexes, and $\alpha' \sim f \circ \alpha$.

Proof. This easily follows from Lemmas 3.15 and 3.16 and from the properties of the functors D_A and M_A.

4. TWO SPECIAL CLASSES OF MAPS AND THE PROOF OF THEOREM 2.1

4.1. Definition. Let $f : (X, A) \to (Y, B)$ be a continuous map. We shall say that f is an *LCR-map* or that f is *locally a map with compact range* if each point $x \in X$ has a neighborhood U_x whose image $f(U_x)$ is contained in a compact subspace of Y, and each point $x \in A$ has a neighborhood V_x in A with the image $f(V_x)$ contained in a compact subspace of B.

We shall say that a continuous homotopy $h_t : (X, A) \to (Y, B)$, $0 \leqq t \leqq 1$, is an *LCR-homotopy* if the map $H : (X, A) \times [0, 1] \to (Y, B)$ defined by the relation $H(x, t) = h_t(x)$ is an LCR-map.

4.2. If $f, g : (X, A) \to (Y, B)$ are LCR-maps, let us write $f \sim_{LCR} g$ if and only if there is an LCR-homotopy $h_t : (X, A) \to (Y, B)$, $0 \leqq t \leqq 1$, with $f = h_0$ and $g = h_1$. Clearly, the relation $\sim_{LCR}$ is an equivalence on the set of all LCR-maps of (X, A) into (Y, B). Let us denote by $[X, A; Y, B]_{LCR}$ the corresponding set of equivalence classes and by $[f]_{LCR}$ the equivalence class of an LCR-map f. Then $[-; -]_{LCR}$ is a bifunctor on the category $\mathrm{Top}_2^{op} \times \mathrm{Top}_2$, and the the map $[X, A; Y, B]_{LCR} \to$ $\to [X', A'; Y', B']_{LCR}$ induced by continuous maps $f : (X', A') \to (X, A)$ and $g : (Y, B) \to (Y', B')$ depends only on the homotopy classes $[f]$ and $[g]$ of maps f and g, respectively.

4.3. Lemma. *If (X, A) has the homotopy type of a CW pair then there are a continuous homotopy $h_t : (X, A) \to (X, A)$, $0 \leqq t \leqq 1$, and an open covering $\{U_i \mid i \in I\}$ of X such that $h_0 = \mathrm{id}$ and each set $h_1(U_i)$ $(h_1(U_i \cap A))$, $i \in I$, is contained in a compact separable subspace of X (resp. A).*

Proof. According to the general bridge mapping theorem, see [2] or [7, Appendix], there is a locally finite normal covering $\mathscr{V} = \{V_j | j \in J\}$ of X and a continuous map $f: (|N(\mathscr{V})|, |N(\mathscr{V} \cap A)|) \to (X, A)$ such that $f \circ p_{\mathscr{V}} \sim \mathrm{id}_{(X,A)}$ for each canonical map $p_{\mathscr{V}}: (X, A) \to (|N(\mathscr{V})|, |N(\mathscr{V} \cap A)|)$. If $\mathscr{U} = \{U_i | i \in I\}$ is an open covering of X such that the set $\{j \in J | V_j \cap U_i \neq \emptyset\}$ is finite for all $i \in I$, and if h_t: $(X, A) \to (X, A)$, $0 \leqq t \leqq 1$, is a continuous homotopy from $\mathrm{id}_{(X,A)}$ to $f \circ p_{\mathscr{V}}$, then $\mathscr{U}$ and h_t, $0 \leqq t \leqq 1$, are easily verified to have the desired properties.

4.4. Proposition. *If (X, A) or (Y, B) has the homotopy type of a CW pair then the canonical map*

$$[X, A; Y, B]_{LCR} \to [X, A; Y, B] \tag{4.1}$$

is bijective.

Proof. Let us suppose that (X, A) has the homotopy type of a CW pair, and let h_t, $0 \leqq t \leqq 1$, be the homotopy from Lemma 4.3. If $f: (X, A) \to (Y, B)$ is a continuous map then $f \sim f \circ h_1$ and $f \circ h_1$ is clearly an LCR-map. This proves that (4.1) is surjective. If $f_1, f_2: (X, A) \to (Y, B)$ are LCR-maps and $f_1 \sim f_2$, then clearly $f_i \sim_{LCR} f_i \circ h_1$ for $i = 1, 2$ and $f_1 \circ h_1 \sim_{LCR} f_2 \circ h_1$, which implies $f_1 \sim_{LCR} f_2$ and proves the injectivity of (4.1).

The case of (Y, B) having the CW homotopy type is treated similarly.

4.5. Definition. Let $f: (X, A) \times (Y, B) \to (Z, C)$ be a map (not necessarily continuous). We shall say that f is an *LCR^*-map* if for each pair $(K, L) \subset (X, A)$ consisting of compact spaces the restriction of f onto $(K, L) \times (Y, B)$ is an LCR-map into (Z, C).

We shall say that a homotopy $h_t: (X, A) \times (Y, B) \to (Z, C)$, $0 \leqq t \leqq 1$, (not necessarily continuous) is an *LCR^*-homotopy* if for each compact pair $(K, L) \subset$ $\subset (X, A)$ the restriction of this homotopy to $(K, L) \times (Y, B)$ is an LCR-homotopy into (Z, C).

4.6. If $f, g: (X, A) \times (Y, B) \to (Z, C)$ are LCR^*-maps, let us write $f \sim_{LCR^*} g$ if and only if there is an LCR^*-homotopy between f and g. This defines an equivalence on the set of all LCR^*-maps of $(X, A) \times (Y, B)$ into (Z, C), and if we denote the corresponding set of equivalence classes by $[(X, A) \times (Y, B); Z, C]_{LCR^*}$, then $[- \times -; -]_{LCR^*}$ may be considered as a functor on $\mathrm{Top}_2^{op} \times \mathrm{Top}_2^{op} \times \mathrm{Top}_2$. Just as in 4.2 it is clear that the map $[(X, A) \times (Y, B); Z, C]_{LCR^*} \to [(X', A') \times$ $\times (Y', B'); Z', C']_{LCR^*}$ induced by continuous maps $f: (X', A') \to (X, A)$, $g:$ $: (Y', B') \to (Y, B)$ and $h: (Z, C) \to (Z', C')$ depends only on the homotopy classes $[f]$, $[g]$ and $[h]$.

4.7. Proposition. *If (X, A) is homotopy equivalent to a CW pair then the canonical map*

$$[(X, A) \times (Y, B); Z, C]_{LCR} \to [(X, A) \times (Y, B); Z, C]_{LCR^*} \tag{4.2}$$

is bijective. If Z is a k-topological Λ-module and C a submodule then (4.2) *is a Λ-isomorphism.*

Proof. This follows from Lemma 4.3 in a similar way as Proposition 4.4.

4.8. For every two pairs (Y, B) and (Z, C) let $S(Y, B; Z, C)$ denote the following ss. set: n-simplexes of $S(Y, B; Z, C)$ are LCR-maps $s: \nabla(n) \times Y \to Z$ such that $s(\nabla(n) \times B) \subset C$ and the restriction $\nabla(n) \times B \to C$ is also an LCR-map, and if $\alpha : [\ldots] \to [n]$ is a non-decreasing function then the corresponding face operator α^* is defined by $\alpha^*(s) = s \circ (|\alpha| \times \mathrm{id}_Y)$.

Clearly, $S(Y, B; Z, C)$ is always a Kan ss. set, and if Z is a k-topological Λ-module and C a submodule then it is a ss. Λ-module.

4.9. Proposition. *For every ss. pair (X, A) and every two topological pairs (Y, B) and (Z, C) there is a canonical bijection*

$$[X, A; S(Y, B; Z, C), S(Y, Y; Z, C)] \approx [(|X|, |A|) \times (Y, B); Z, C]_{LCR^*} \tag{4.3}$$

natural with respect to all three arguments (X, A), (Y, B) and (Z, C). If Z is a k-topological Λ-module and C is a submodule of Z then (4.3) is a Λ-isomorphism.

Proof. Let $\chi_x : \nabla(\dim x) \to |X|$ be the characteristic map corresponding to a simplex $x \in X$. It is easy to see that the formula

$$x \in X\,, \quad t \in \nabla(\dim x)\,, \quad y \in Y \Rightarrow f(x)(t, y) = \tilde{f}(\chi_x(t), y)$$

defines a one-to-one correspondence between ss. maps

$$f : (X, A) \to (S(Y, B; Z, C), S(Y, Y; Z, C))$$

and LCR^*-maps

$$\tilde{f} : (|X|, |A|) \times (Y, B) \to (Z, C)\,,$$

which is natural with respect to all three arguments. Moreover, using 3.2 and the functoriality of the geometric realization one verifies without difficulties that this correspondence preserves the relation of homotopy. Consequently, passing from maps to their homotopy classes gives the desired bijection (4.3).

Combining 1.6, 3.5, 4.4 and 4.9 we obtain

4.10. Proposition. *Let n be a non-negative integer and G an Abelian group. For every ss. pair (X, A) and every topological pair (Y, B) there is a canonical group isomorphism*

$$\hat{h}^n(|X| \times Y, |X| \times B \cup |A| \times Y; G) \approx [X, A; S(Y, B; |K(G, n)|, *), *]\,, \tag{4.4}$$

which is natural with respect to all the arguments (X, A), (Y, B) and G. If G is a Λ-module then (4.4) is a Λ-isomorphism.

4.11. Corollary. *For every topological pair (Y, B), for every Abelian group G and for every non-negative integers n and q there is an isomorphism*

$$\alpha_{n,q} : \pi_q(S(Y, B; |K(G, n)|, *), *) \approx \hat{h}^{n-q}(Y, B; G)\,, \tag{4.5}$$

which is natural with respect to both arguments (Y, B) *and* G. *If* G *is a* Λ*-module then* (4.5) *is a* Λ*-isomorphism.*

Proof. We have

$$(4.6)\qquad \pi_q(S(Y, B; |K(G, n)|, *), *) = [\Delta(q), \dot{\Delta}(q); S(Y, B; |K(G, n)|, *), *]$$

by the definition of homotopy groups,

$$(4.7)\qquad [\Delta(q)\,\dot{\Delta}(q); S(Y, B; |K(G, n)|, *), *] \approx \hat{h}^n((\nabla(q), \dot{\nabla}(q)) \times (Y, B); G)$$

by Proposition 4.10 and the canonical identification $(\nabla(q), \dot{\nabla}(q)) = (|\Delta(q)|, |\dot{\Delta}(q)|)$, and finally (see 1.12)

$$(4.8)\qquad [\nabla(q)]^{-1} : \hat{h}^n((\nabla(q), \dot{\nabla}(q)) \times (Y, B); G) \approx \hat{h}^{n-q}(Y, B; G)\,.$$

The isomorphism (4.5) is now defined as the composition of the isomorphisms (4.6) to (4.8).

4.12. Construction of isomorphisms $\theta^n_{\Lambda,ss}(X, A; Y, B; G)$. Let Λ be a principal ideal domain. For $n = 0, 1, 2, \ldots$ we shall now construct Λ-isomorphisms $\theta^n_{\Lambda,ss} = \theta^n_{\Lambda,ss}(X, A; Y, B; G)$,

$$(4.9)\qquad \theta^n_{\Lambda,ss} : \bigoplus_{i+j=n} \hat{h}^i(|X|, |A|; \hat{h}^j(Y, B; G)) \approx \hat{h}^n(|X| \times Y, |X| \times B \cup |A| \times Y; G)\,.$$

where (X, A) is a ss. pair, (Y, B) a topological pair and G a module over Λ.

By Theorem 1.6 we have a canonical Λ-isomorphism

$$(4.10)\qquad \bigoplus_{i+j=n} \hat{h}^i(|X|, |A|; \hat{h}^j(Y, B; G)) \approx \bigoplus_{i+j=n} [|X|, |A|; |K(\hat{h}^j(Y, B; G), i)|, *]\,,$$

and by Corollary 3.4 we have a canonical Λ-isomorphism

$$(4.11)\qquad \bigoplus_{i+j=n} [|X|, |A|; |K(\hat{h}^j(Y, B; G), i)|, *] \approx \bigoplus_{i+j=n} [X, A; K(\hat{h}^j(Y, B; G), i), *]\,.$$

On the other hand, we have the canonical Λ-isomorphism

$$(4.12)\qquad [X, A; S(Y, B; |K(G, n)|, *), *] \approx \hat{h}^n(|X| \times Y, |X| \times B \cup |A| \times Y; G)$$

of Proposition 4.10. Consequently, it suffices to construct a Λ-isomorphism

$$(4.13)\qquad \bigoplus_{i+j=n} [X, A; K(\hat{h}^j(Y, B; G), i), *] \approx [X, A; S(Y, B; |K(G, n)|, *), *]$$

and then to define (4.9) as the composition of the isomorphisms (4.10)–(4.13) taken in an appropriate order.

Our construction of the Λ-isomorphism (4.13) is based on Proposition 3.14. By the assertion (c) of this proposition, there is a weak ss. Λ-homomorphism

$$\vartheta^n_\Lambda = \vartheta^n_\Lambda(Y, B; G) : S(Y, B; |K(G, n)|, *) \to \prod_{i=0}^{n} K(\hat{h}^{n-i}(Y, B; G), i)$$

such that the diagram

$$\begin{array}{ccc}\pi_q(S(Y,B;|K(G,n)|,*),*) & \xrightarrow{(\vartheta^n_A)_*} & \pi_q(\prod_{i=0}^{n} K(\hat{h}^{n-i}(Y,B;G),i),*)\\ \alpha_{n,q}\downarrow & & \downarrow (pr_q)_*\\ \hat{h}^{n-q}(Y,B;G) & \xleftarrow{\approx} & \pi_q(K(\hat{h}^{n-q}(Y,B;G),q),*)\end{array}$$

where $\approx$ in the lower row denotes the canonical isomorphism of 3.11, commutes for $q = 0, 1, 2, \ldots, n$. By Corollary 3.7, ϑ^n_A is a homotopy equivalence of pointed ss. sets, and we define (4.13) to be the A-isomorphism induced by ϑ^n_A.

It is clear from the construction, from the homotopy invariance of both sides in (4.9) and from Theorem 3.1 that (4.9) is natural with respect to continuous maps $f:(|X|,|A|)\to(|X'|,|A'|)$. In general case, however, it is not natural with respect to the arguments (Y, B) and G because the isomorphism (4.13) depends on the choice of the weak ss. A-homomorphism ϑ^n_A and this choice cannot be made in a canonical way. Finally, (4.9) also depends on the basic ring A : if $\omega : \Gamma \to A$ is a homomorphism of principal ideal domains then for a A-module G it may happen that $\theta^n_{\Gamma,ss} \neq \theta^n_{A,ss}$.

4.13. Definition of isomorphisms $\theta^n_{A,CW}(X, A; Y, B; G)$. Let A be a principal ideal domain. For $n = 0, 1, 2, \ldots, (X, A)$ a topological pair of CW homotopy type, (Y, B) an arbitrary topological pair and G a A-module, we define a A-isomorphism $\theta^n_{A,CW} = \theta^n_{A,CW}(X, A; Y, B; G)$,

$$\theta^n_{A,CW} : \bigoplus_{i+j=n} \hat{h}^i(X, A; \hat{h}^j(Y, B; G)) \approx \hat{h}^n(X\times Y, X\times B\cup A\times Y; G)\,, \tag{4.14}$$

in the following way.

It is well-known that the canonical projection $p:(|S(X)|,|S(A)|)\to(X,A)$, where $S(X)$ and $S(A)$ are the singular ss. sets of spaces X and A, respectively, is a homotopy equivalence if the pair (X, A) has the CW homotopy type. Using this fact we define (4.14) as the unique map making the diagram

$$\begin{array}{ccc}\bigoplus_{i+j=n}\hat{h}^i(|S(X)|,|S(A)|;\hat{h}^j(Y,B;G)) & \xrightarrow{\theta^n_{A,ss}} & \hat{h}^n((|S(X)|,|S(A)|)\times(Y,B);G)\\ \approx\uparrow \bigoplus_{i+j=n} p^* & & \approx\uparrow (p\times \mathrm{id})^*\\ \bigoplus_{i+j=n}\hat{h}^i(X,A;\hat{h}^j(Y,B;G)) & \xrightarrow{\theta^n_{A,CW}} & \hat{h}^n((X,A)\times(Y,B);G)\end{array}$$

commutative.

It is easy to see that (4.14) is natural with respect to the argument (X, A) and that $\theta^n_{A,ss}(X, A; Y, B; G) = \theta^n_{A,CW}(|X|, |A|; Y, B; G)$ for every ss. pair (X, A).

4.14. Proof of Theorem 2.1. Let us consider the diagram

$$\begin{array}{ccc}\varinjlim_{\mathscr{U}} \bigoplus_{i+j=n}\hat{h}^i(X_{\mathscr{U}},A_{\mathscr{U}};\hat{h}^j(Y,B;G)) & \xrightarrow{\varinjlim \theta^n_{A,CW}} & \varinjlim_{\mathscr{U}} \hat{h}^n((X_{\mathscr{U}},A_{\mathscr{U}})\times(Y,B);G)\\ \approx\downarrow & & \downarrow\\ \bigoplus_{i+j=n}\hat{h}^i(X,A;\hat{h}^j(Y,B;G)) & \xrightarrow{\theta^n_{A}} & \hat{h}^n((X,A)\times(Y,B);G)\end{array}\tag{4.15}$$

where $\mathscr{U}$ runs over the set $\operatorname{cov}(X)$, $(X_{\mathscr{U}}, A_{\mathscr{U}}) = (|N(\mathscr{U})|, |N(\mathscr{U} \cap A)|)$ and both vertical Λ-homomorphisms are induced by canonical projections $p_{\mathscr{U}} : (X, A) \to (X_{\mathscr{U}}, A_{\mathscr{U}})$. It is easy to see that the left vertical homomorphism is an isomorphism. Using this fact we define $\theta^n_\Lambda = \theta^n_\Lambda(X, A; Y, B; G)$ as the unique Λ-homomorphism making this diagram commutative. Clearly we have $\theta^n_\Lambda(X, A; Y, B; G) = \theta^n_{\Lambda,CW}(X, A; Y, B; G)$ if the pair (X, A) has the CW homotopy type.

We shall now prove that the homomorphisms (2.1) just defined have all the properties claimed in Theorem 2.1.

Ad (a). This property is an immediate consequence of the definition of the homomorphisms θ^n_Λ and of the naturality of isomorphisms $\theta^n_{\Lambda,CW}$.

Ad (b). Using the property (a) one can easily see from our definition of θ^n_Λ, $\theta^n_{\Lambda,CW}$ and $\theta^n_{\Lambda,ss}$ that it suffices to prove the homotopy commutativity of the diagram

$$\begin{array}{ccc} S(Y', B'; |K(G', n)|, *) & \xrightarrow{\vartheta^n_{\Lambda'}(Y', B'; G')} & \prod\limits_{i=0}^{n} K(\hat{h}^{n-i}(Y', B'; G'), i) \\ \big\downarrow & & \big\downarrow \\ S(Y, B; |K(G, n)|, *) & \xrightarrow{\vartheta^n_\Lambda(Y, B; G)} & \prod\limits_{i=0}^{n} K(\hat{h}^{n-i}(Y, B; G), i) \end{array}$$

where both vertical arrows denote the ss. Γ-homomorphisms induced by g and γ. But this follows immediately from Proposition 3.14, Corollary 4.11 and the way in which the weak ss. Λ-homomorphisms ϑ^n_Λ and the weak ss. Λ'-homomorphisms $\vartheta^n_{\Lambda'}$ have been chosen.

Ad (c). It is clear that it again suffices to consider the case of (X, A) being the geometric realization of a ss. pair. Using Theorem 1.6 we further find that it is sufficient to prove commutativity of the diagram

(4.16)
$$\begin{array}{ccc} [X, A; S_L, S_K] & \longrightarrow & [X, A; S(Y, B; |\bar{K}|, *), *] \\ \big\uparrow (\varrho_1)_* & & \big\downarrow (\vartheta^{n+1}_\Lambda)_* \\ [X, A; S_{L'}, S_{K'}] & & [X, A; \prod\limits_{p+q=n} K(G_q, p+1), *] \\ \approx \big\downarrow \beta_1 & & \big\uparrow \\ [A; S_{K'}] & & [X, A; \prod\limits_{p+q=n} L(G_q, p+1), \prod\limits_{p+q=n} K(G_q, p)] \\ \big\downarrow (\vartheta^n_{\Lambda'})_* & & \approx \big\downarrow \beta_2 \\ [A; \prod\limits_{p+q=n} K(G'_q, p)] & \xrightarrow{\varrho_*} & [A; \prod\limits_{p+q=n} K(G_q, p)] \end{array}$$

where (X, A) is an arbitrary ss. pair, $K = K(G, n)$, $K' = K(G', n)$, $\bar{K} = K(G, n+1)$, $L = L(G, n+1)$, $L' = L(G', n+1)$, $S_K = S(Y; B; |K|, *)$, $S_{K'} = S(Y', B'; |K'|, *)$, $S_L = S(Y, B; |L|, *)$, $S_{L'} = S(Y', B'; |L'|, *)$, $G_q = \hat{h}^q(Y, B; G)$, $G'_q = \hat{h}^q(Y', B'; G')$, the maps ϱ and ϱ_1 are induced by g and γ, and all the other maps are defined in an obvious way. The maps β_1 and β_2 are bijective by contractibility of the Kan ss. sets $S_{L'}$ and $L(G_q, p+1)$.

By Proposition 3.17 there is a commutative diagram of ss. Γ-homomorphisms

$$
\text{(4.17)}\qquad
\begin{array}{ccccc}
S_{K'} & \xleftarrow{\alpha} & D_\Gamma P & \xrightarrow{\alpha'} & \prod\limits_{p+q=n} K(G_q, p) \\
\downarrow \subset & & \downarrow \subset & & \downarrow \subset \\
S_{L'} & \xleftarrow{\beta} & D_\Gamma Q & \xrightarrow{\beta'} & \prod\limits_{p+q=n} L(G_q, p+1) \\
\downarrow & & \downarrow & & \downarrow \\
S_{L'}/S_{K'} & \leftarrow & D_\Gamma Q/D_\Gamma P & \xrightarrow{\gamma'} & \prod\limits_{p+q=n} K(G_q, p+1) \\
 & & \downarrow & & \\
 & & D_\Gamma(Q/P) & &
\end{array}
$$

where α and β are homotopy equivalences of ss. sets, P, Q and Q/P are free Γ-complexes and α' is homotopic to the composition

$$
D_\Gamma P \xrightarrow{\alpha} S_{K'} \xrightarrow{\vartheta^n_{A'}} \prod_{p+q=n} K(G'_q, p) \xrightarrow{\varrho} \prod_{p+q=n} K(G_q, p) .
$$

Consequently, commutativity of the diagram (4.16) follows from that of the diagram

$$
\text{(4.18)}\qquad
\begin{array}{ccc}
[X, A; S_L, S_K] & \longrightarrow & [X, A; S(Y, B; |K|, *), *] \\
\uparrow (\varrho_1)_* & & \downarrow (\vartheta^{n+1}_A)_* \\
[X, A; S_{L'}, S_{K'}] & & [X, A; \prod\limits_{p+q=n} K(G_q, p+1). *] \\
\approx \uparrow \beta_* & & \uparrow \\
[X, A; D_\Gamma Q, D_\Gamma P] & \xrightarrow{\beta'_*} & [X, A; \prod\limits_{p+q=n} L(G_q, p+1), \prod\limits_{p+q=n} K(G_q, p)] .
\end{array}
$$

Since β_* is clearly a bijection we finally see that it suffices to show that the diagram

$$
\begin{array}{ccc}
S_L/S_K & \longrightarrow & S(Y, B; |\bar{K}|, *) \\
\uparrow \beta_3 & & \downarrow \vartheta^{n+1}_A \\
D_\Gamma Q/D_\Gamma P & \xrightarrow{\gamma'} & \prod\limits_{p+q=n} K(G_q, p+1)
\end{array}
$$

of pointed ss. sets, where β_3 is induced by β and ϱ_1, is homotopy commutative.

Using Theorem 3.9, Proposition 3.10 and Corollary 4.11, we derive from the diagram (4.17) Γ-isomorphisms

$$
\pi_p(D_\Gamma Q/D_\Gamma P, *) \approx \hat{h}^{n+1-p}(Y', B'; G'), \quad p = 0, 1, \ldots, n+1 .
$$

Applying now Proposition 3.14 we reduce our problem to verifying commutativity of the diagram

$$\begin{array}{ccc}
\pi_p(S_L/S_K, *) & \to & \pi_p(S(Y, B; |\bar{K}|, *) \\
\uparrow (\beta_3)_* & & \downarrow (\vartheta_A^{n+1})_* \\
\pi_p(D_\Gamma Q/D_\Gamma P, *) & \to & \pi_p(\prod_{i+j=n} K(G_j, i+1) *)
\end{array}$$

for $p = 0, 1, \ldots, n+1$.

Our further step is based on the observation that

$$[\Delta(p), \dot{\Delta}(p), *;\ D_\Gamma Q, D_\Gamma P, *] \to \pi_p(D_\Gamma Q/D_\Gamma P, *)$$

is an epimorphism. This and the diagram (4.18) imply that it suffices to prove commutativity of the diagram

$$(4.19)\qquad \begin{array}{ccc}
[\Delta(p), \dot{\Delta}(p); S_L, S_K] & \to & [\Delta(p), \dot{\Delta}(p); S(Y, B; |\bar{K}|, *), *] \\
& & \downarrow (\vartheta_A^{n+1})_* \\
\uparrow (\varrho_1)_* & & [\Delta(p), \dot{\Delta}(p); \prod_{i+j=n} K(G_j, i+1), *] \\
& & \uparrow \\
[\Delta(p), \dot{\Delta}(p), *; S_{L'}, S_{K'}, *] & \xrightarrow{\beta_4} & [\Delta(p), \dot{\Delta}(p), *; \prod_{i+j=n} L(G_j, i+1), \prod_{i+j=n} K(G_j, i), *]
\end{array}$$

where β_4 is induced by any ss. map $(S_{L'}, *) \to (\prod_{i+j=n} L(G_j, i+1), *)$ extending the composition $S_{K'} \to \prod_{i+j=n} K(G'_j, i) \to \prod_{i+j=n} K(G_j, i)$, for all $p = 0, 1, \ldots, n+1$. To this end let us consider the diagram (4.20), where $\Delta = \Delta(p)$, Δ^0 is the ss. subset of Δ generated by the face $d_0[p]$ of the fundamental simplex $[p] \in \Delta$, Λ^0 is the ss. subset of Δ generated by the faces $d_i[p]$, $i \neq 0$, $*$ is the last vertex of $[p]$, ∇ and ∇^0 correspond to Δ and Δ^0, respectively, β_5's are isomorphisms from Proposition 4.10, and β_6's are canonical isomorphisms from 3.11. Using the homotopy description of the connecting homomorphism δ^* given in Theorem 1.6, the way in which ϑ_A^{n+1}, $\vartheta_{A'}^n$ have been chosen and the relation $[\nabla] = [\nabla^0 : \nabla] \circ [\nabla^0]$, one can prove that with the exception of the subdiagram (4.19) all parts of the diagram (4.20) commute. This clearly implies commutativity of the diagram (4.19) and completes our proof of the assertion (c).

Ad (d). In a similar way as in the proof of the assertion (c) we reduce the problem to verifying homotopy commutativity of the diagram

(4.21)

$$\begin{array}{ccccccc}
S(B'; |K'|) & \xleftarrow{\varrho} & S(Y', B'; |L'|, |K'|) & \to & S(Y', B'; |\bar{K}'|, *) & \xrightarrow{\beta_2} & S(Y, B; |\bar{K}|, *) \\
\downarrow \vartheta_{A'}^n & & & & & & \downarrow \vartheta_A^{n+1} \\
\prod_{p+q=n} K(G'_q, p) & \xrightarrow{\beta_1} & \prod_{p+q=n} K(\hat{h}^{q+1}(Y', B'; G'), p) & & \xrightarrow{\beta_3} & & \prod_{p+q=n} K(G_{q+1}, p)
\end{array}$$

$$
\begin{array}{ccccccc}
\hat{h}^{n+1}((\nabla,\dot{\nabla})\times(Y,B);G) & \xrightarrow[\approx]{\beta_5} & [\Delta,\dot{\Delta};S(Y,B;|\overline{K}|,*),*] & \xrightarrow{(\vartheta_{\Lambda}^{n+1})_*} & [\Delta,\dot{\Delta};\prod\limits_{i+j=n}K(G_j,i+1),*] & \xrightarrow[\approx]{\beta_6} & G_{-p+n+1} \\
\uparrow & & \uparrow & & \uparrow & & \| \\
 & & [\Delta,\dot{\Delta};S_L,S_K] & & & & \| \\
 & & \uparrow & & & & \| \\
 & \begin{array}{c}[\nabla^0:\nabla]\circ(\mathrm{id}\times g)^*\circ\gamma_* = \\ =(\mathrm{id}\times g)^*\circ\gamma_*\circ[\nabla^0:\nabla]\end{array} & [\Delta,\dot{\Delta},*;S_{L'},S_{K'},*] & \xrightarrow{\beta_4=\eta_*} & [\Delta,\dot{\Delta},*;\prod\limits_{i+j=n}L(G_j,i+1),\prod\limits_{i+j=n}K(G_j,i),*] & & \| \\
 & & \approx\uparrow & & \approx\uparrow & & \| \\
 & & [\Delta,\dot{\Delta},\Lambda^0;S_{L'},S_{K'},*] & \xrightarrow{\eta_*} & [\Delta,\dot{\Delta},\Lambda^0;\prod\limits_{i+j=n}L(G_j,i+1),\prod\limits_{i+j=n}K(G_j,i),*] & & \| \\
 & & \approx\downarrow & & \approx\downarrow & & \| \\
\hat{h}^{n}((\nabla^0,\dot{\nabla}^0)\times(Y',B');G') & \xrightarrow[\approx]{\beta_5} & [\Delta^0,\dot{\Delta}^0;S_{K'},*] & \xrightarrow{\eta_x} & [\Delta^0,\dot{\Delta}^0;\prod\limits_{i+j=n}K(G_j,i),*] & \xrightarrow[\approx]{\beta_6} & G_{-p+n+1} \\
 & & \downarrow & & \uparrow & & \uparrow \\
 & & \searrow & \xrightarrow{(\vartheta_{\Lambda'}^{n})_*} & [\Delta^0,\dot{\Delta}^0;\prod\limits_{i+j=n}K(G'_j,i),*] & \xrightarrow[\approx]{\beta_6} & G'_{-p+n+1}
\end{array}
\tag{4.20}
$$

$[\nabla]$: $G_{-p+n+1} \to \hat{h}^{n+1}((\nabla,\dot{\nabla})\times(Y,B);G)$

$[\nabla^0]$: $G'_{-p+n+1} \to \hat{h}^{n}((\nabla^0,\dot{\nabla}^0)\times(Y',B');G')$

where $K' = K(G', n)$, $L' = L(G', n+1)$, $\bar{K} = K(G, n+1)$, $\bar{K}' = K(G', n+1)$, $G'_q = \hat{h}^q(B'; G')$, $G_q = \hat{h}^q(Y, B; G)$, β_1 is induced by the Λ'-homomorphisms

$$(-1)^{n-q}\,\delta^* : \hat{h}^q(B'; G') \to \hat{h}^{q+1}(Y', B'; G')$$

and β_2, β_3 are induced by g and γ.

All maps in (4.21) are weak ss. Γ-homomorphisms, and contractibility of the CW complex $|L'|$ and Proposition 4.10 imply that

$$\pi_i(S(Y', B'; |L'|, |K'|), *) \approx \pi_i(S(B'; |K'|), *) \approx \hat{h}^{n-i}(B'; G').$$

Consequently, by Proposition 3.14, the diagram (4.21) is homotopy commutative if and only if the corresponding diagram of the i-th homotopy groups is commutative for $i = 0, 1, \ldots, n$ (all higher homotopy groups of $S(B'; |K'|)$ are trivial). This is, however, easily seen to be equivalent to the commutativity of diagram (2.4) for all pairs $(X, A) = (\nabla(p), \dot{\nabla}(p))$, $p = 0, 1, \ldots, n$. To complete the proof it suffices to use the property (e), which will be proved later, and some well-known properties of the cross product.

Ad (e). It is clearly sufficient to prove commutativity of the diagram

$$(4.22)\qquad \begin{array}{ccc} \hat{h}^p(|X|, |A|; \Lambda) \otimes_\Lambda h^q(Y, B; G) & \xrightarrow{\otimes} & \hat{h}^p(|X|, |A|; \hat{h}^q(Y, B; G)) \\ \downarrow \times & & \uparrow pr \\ \hat{h}^n((|X|, |A|) \times (Y, B); G) & \xrightarrow{(\theta^n_{\Lambda,ss})^{-1}} & \bigoplus_{i+j=n} \hat{h}^i(|X|, |A|; \hat{h}^j(Y, B; G)) \end{array}$$

where (X, A) is an arbitrary ss. pair and $p + q = n$.

We start with a general remark concerning cross products. Let

$$\mu_{ss} : K(\Lambda, p) \times K(G, q) \to K(G, n)$$

be the Λ-bilinear ss. map induced by the Alexander-Whitney natural transformation (see e.g. [11, p. 118]), and let

$$\mu = |\mu_{ss}| \circ i^{-1} : |K(\Lambda, p)| \times |K(G, q)| \to |K(G, n)|,$$

where i is the canonical bijection of Theorem 3.2. Then μ is Λ-bilinear and continuous on compact subspaces and the diagram

$$(4.23)\qquad \begin{array}{ccc} \hat{h}^p(|X|, |A|; \Lambda) \times \hat{h}^q(Y, B; G) & \xrightarrow{\times} & \hat{h}^n((|X|, |A|) \times (Y, B); G) \\ \uparrow \hat{\imath} \times \hat{\imath} & & \uparrow \hat{\imath} \\ [|X|, |A|; |K(\Lambda, p)|, *] \times [Y, B; |K(G, q)|, *] & & [(|X|, |A|) \times (Y, B); |K(G, n)|, *] \\ \uparrow \mathrm{id} \times \beta_1 & & \uparrow \beta_2 \\ [|X|, |A|; |K(\Lambda, p)|, *] \times [Y, B; |K(G, q)|, *]_{LCR} & \xrightarrow{\mu_*} & [(|X|, |A|) \times (Y, B); |K(G, n)|, *]_{LCR*} \end{array}$$

where $\hat\imath$ is the natural transformation of Theorem 1.6, β_1 is given by Proposition 4.4 and β_2 is given by Proposition 4.4 and Proposition 4.7, can be proved to be commutative. Similarly one can prove commutativity of the diagram

$$(4.24)\qquad \begin{array}{ccc} \hat h^p(|X|,|A|;\Lambda) & \xrightarrow{\otimes v} & \hat h^p(|X|,|A|;G_q) \\ \uparrow{\scriptstyle \hat\imath} & & \uparrow{\scriptstyle \hat\imath} \\ [|X|,|A|;|K(\Lambda,p)|,*] & \xrightarrow{|\lambda_v|_*} & [|X|,|A|;|K(G_q,p)|,*] \end{array}$$

where $v \in G_q = \hat h^q(Y,B;G)$ and

$$\lambda_v : K(\Lambda,p) \to K(G_q,p)$$

is the ss. Λ-homomorphism which is induced by the Λ-homomorphism $\Lambda \to \to \hat h^q(Y,B;G)$ sending $1 \in \Lambda$ onto v.

Now for every LCR-map $g : (Y,B) \to (|K(G,q)|,*)$ let

$$\mu_g : K(\Lambda,p) \to S(Y,B;|K(G,n)|,*)$$

be the ss. Λ-homomorphism defined by the formula

$$\mu_g(s)(t,y) = \mu(\chi_s(t),g(y)), \quad t \in \nabla(\dim s), \quad y \in Y,$$

where $\chi_s : \nabla(\dim s) \to |K(\Lambda,p)|$ is the characteristic map corresponding to the simplex s. Supposing g has been chosen, let us put $v = \hat\imath([g])$ and consider the diagram

$$(4.25)\qquad \begin{array}{ccccc} \hat h^p(|X|,|A|;G_q) & \xleftarrow{\otimes v} & \hat h^p(|X|,|A|;\Lambda) & \xrightarrow{\times v} & \hat h^n(Z,C;G) \\ & & & & \approx\uparrow{\scriptstyle \hat\imath} \\ \approx\uparrow{\scriptstyle \hat\imath} & \text{II} & \approx\uparrow{\scriptstyle \hat\imath} & \text{I} & [Z,C;|K|,*] \\ & & & & \uparrow{\scriptstyle \beta_2} \\ [|X|,|A|;|K(G_q,p)|,*] & \leftarrow & [|X|,|A|;|K(\Lambda,p)|,*] & \rightarrow & [Z,C;|K|,*]_{LCR^*} \\ \approx\uparrow{\scriptstyle \beta_4} & \text{III} & \uparrow{\scriptstyle \beta_4} & \text{IV} & \approx\uparrow{\scriptstyle \beta_3} \\ [X,A;K(G_q,p),*] & \xleftarrow{(\lambda_v)_*} & [X,A;K(\Lambda,p),*] & \xrightarrow{(\mu_g)_*} & [X,A;S,*] \\ \uparrow{\scriptstyle pr} & & & & \approx\uparrow{\scriptstyle \vartheta^n_\Lambda(Y,B;G)_*} \\ \longleftarrow & \longleftarrow & [X,A;\prod_{i+j=n} K(G_j,i),*] & \longrightarrow & \longrightarrow \end{array}$$

where $K = K(G,n)$, $(Z,C) = (|X|,|A|) \times (Y,B)$, $S = S(Y,B;|K(G,n)|,*)$, β_3 is the isomorphism of Proposition 4.9 and β_4's are the isomorphisms of Corollary 3.4. The squares of this diagram are easily seen to be commutative; commutativity of the squares III and IV follows immediately from the definition of the maps μ, λ_v and μ_g, and commutativity of the squares I and II is a consequence of the commutativity of the diagrams (4.23) and (4.24). It follows that the diagram (4.22) is commutative if and only if the triangle of the diagram (4.25) commutes for all LCR-maps $g : (Y,B) \to$

$\to (|K(G, q)|, *)$ and $v = \hat{\imath}([g])$. Using this fact and applying Proposition 3.14 (d) we conclude that it is sufficient to prove commutativity of the diagram (4.22) for $(X, A) = (\Delta(p), \dot{\Delta}(p))$. Making this substitution, comparing the resulting diagram with the diagram (2.5) (here we use the property (f) which will be proved next) and using the obvious commutative diagram

$$\begin{array}{ccc} \hat{h}^p(\nabla(p), \dot{\nabla}(p); \Lambda) \otimes_\Lambda G_q & \xrightarrow{\ \otimes\ } & \hat{h}^p(\nabla(p), \dot{\nabla}(p); G_q) \\ {\scriptstyle [\nabla(p)] \otimes \mathrm{id}}\Big\uparrow \approx & \approx & \approx \Big\uparrow {\scriptstyle [\nabla(p)]} \\ \Lambda \otimes_\Lambda G_q & \longrightarrow & G_q \end{array}$$

we find that it suffices to prove commutativity of the diagram

$$\begin{array}{ccc} \hat{h}^p(\nabla(p), \dot{\nabla}(p); \Lambda) \otimes_\Lambda G_q & \xrightarrow{\ \times\ } & \hat{h}^n((\nabla(p), \dot{\nabla}(p)) \times (Y, B); G) \\ {\scriptstyle [\nabla(p)] \otimes \mathrm{id}}\Big\uparrow & \approx & \Big\uparrow {\scriptstyle [\nabla(p)]} \\ \Lambda \otimes_\Lambda G_q & \longrightarrow & G_q \end{array}$$

This, however, follows from the definition of the isomorphisms $[\nabla(p)]$ and from the well-known properties of the cross product.

Ad (f). Since we can identify $\hat{h}^*(- , - ; -)$ with $H^*(- , - ; -)$ on the category of ss. pairs, it easily follows from the definition of θ^n_Λ and from the commutative diagram in 4.12 that commutativity of the diagram (2.5) is equivalent to commutativity of the diagram

$$\begin{array}{ccc} G_q & \longleftarrow & \pi_p(K(G_q, p), *) \\ \approx \Big\downarrow {\scriptstyle [\Delta(p)]} & & \Big\| \\ H^p(\Delta(p), \dot{\Delta}(p); G_q) & \underset{\approx}{\xleftarrow{\ t\ }} & [\Delta(p), \dot{\Delta}(p); K(G_q, p), *] \end{array}$$

where $q = n - p$, $G_q = \hat{h}^q(Y, B; G)$, the above arrow denotes the canonical isomorphism of 3.11, the isomorphism t is defined by $t([f]) = f^*(c_p)$, $c_p \in H^p(K(G_q, p), G_q)$ being the fundamental cohomology class (see [11, p. 230, Korollar]), and $[\Delta(p)]$ is defined in the same way as $[\nabla(p)]$. Commutativity of this last diagram is, however, obvious.

Ad (g). It remains to prove that (2.1) is bijective if the space Y is compact. Let us put $K = K(G, n)$, $(Z, C) = (X, A) \times (Y, B)$, $(X_{\mathcal{U}}, A_{\mathcal{U}}) = (|N(\mathcal{U})|, |N(\mathcal{U} \cap A)|)$ and $(Z_{\mathcal{U}}, C_{\mathcal{U}}) = (X_{\mathcal{U}}, A_{\mathcal{U}}) \times (Y, B)$ for $\mathcal{U} \in \mathrm{cov}(X)$. In view of the commutative diagram (4.15) and Theorem 1.6, the homomorphism (2.1) is bijective if and only if the canonical homomorphism

$$\varinjlim_{\mathcal{U} \in \mathrm{cov}(X)} [Z_{\mathcal{U}}, C_{\mathcal{U}}; |K|, *] \to [Z, C; |K|, *]$$

is bijective. This last homomorphism is, however, equivalent to the homomorphism

$$\varinjlim_{\mathscr{U}\in \operatorname{cov}(X)} [X_{\mathscr{U}}, A_{\mathscr{U}}; (|K|, *)^{(Y,B)}, *] \to [X, A; (|K|, *)^{(Y,B)}, *]$$

because Y is compact and K is regular. To complete the proof it suffices to notice that the pair $((|K|, *)^{(Y,B)}, *)$ is homotopy equivalent to a CW pair by [14], and to apply the general bridge mapping theorem [2], [7, Appendix]. (Remark: J. Milnor assumes in [14] that Y is regular but this assumption is easily seen to be unnecessary.)

5. PROOF OF THEOREM 2.7

Let us write $A \Subset B$ for subsets A and B of a space X if B is a functional neighborhood of A.

5.1. Definition. (See [3, p. 148] and [4, p. 45].) Let $\mathscr{A}$ be a presheaf of Abelian groups on a space X. We say that $\mathscr{A}$ is *uniformly locally trivial* if for each pair of open subsets U and V of X with $U \Subset V$ and for each $s \in \mathscr{A}(V)$ there is a normal open covering $\{U_i | \; i \in I\}$ of the space U such that $s \mid U_i = 0$ for each $i \in I$.

5.2. Proposition. *If a presheaf $\mathscr{A}$ of Abelian groups over a space X is uniformly locally trivial then $\hat{H}^*(X; \mathscr{A}) = 0$.*

For the proof see [3, p. 149] and [4, p. 45].

5.3. Corollary. *Let $\alpha : \mathscr{A} \to \mathscr{B}$ be a homomorphism of presheaves of Abelian groups on a space X. If both presheaves* Ker α *and* Coker α *are uniformly locally trivial then α induces an isomorphism $\alpha* : \hat{H}^*(X; \mathscr{A}) \approx \hat{H}^*(X; \mathscr{B})$.*

5.4. Lemma. *If the topological pair (X, A) satisfies the conditions* (a)–(c) *of Theorem* 2.7, *then the following assertions hold*:

(a) *Every neighborhood (in X) of each closed subset $F \subset X$-int A is functional.*

(b) *For each closed subset F of X and for each family $\{U_i | \; i \in I\}$ of open subsets of X with the property F-int $A \subset \bigcup_{i\in I} U_i$ there is a normal open covering $\{V_j | \; j \in J\}$ of X such that for each $j \in J$ either $V_j \cap F = \emptyset$ or $V_j \Subset A$ or $V_j \subset U_i$ for some $i \in I$.*

The proof is an exercise on the properties of normal coverings and is left to the reader.

5.5. Lemma. *Let (X, A) be a topological pair satisfying the conditions* (a)–(c) *of Theorem* 2.7, *and let $\mathscr{A}$ be a locally trivial presheaf of Abelian groups on X. If $\mathscr{A}(U) = 0$ for all open subsets U of X with $U \Subset A$, then $\mathscr{A}$ is uniformly locally trivial.*

Proof. Let U and V be open subsets of X such that $U \Subset V$, and let $s \in \mathscr{A}(V)$. Since $\mathscr{A}$ is locally trivial, for each point $x \in \bar{U}$-int A there is an open neighborhood U_x of x such that $U_x \subset V$ and $s \mid U_x = 0$. Applying Lemma 5.4 to $F = \bar{U}$ and to the family $\{U_x | x \in U\}$ we obtain a normal open covering $\{V_j | j \in J\}$ of X such that for each $j \in J$ either $V_j \cap \bar{U} = \emptyset$ or $V_j \subset A$ or $V_j \subset U_x$ for some $x \in U$. To complete the proof it suffices to notice that $\{U \cap V_j | j \in J\}$ is a normal covering of U and that $s | U \cap V_j = 0$ for all $j \in J$.

5.6. Proposition. (See [3, p. 151].) *Let (X, A) be a topological pair satisfying the conditions* (a)–(c) *of Theorem* 2.7, *let $\mathscr{A}$ be a presheaf of Abelian groups on X, let $\tilde{\mathscr{A}}$ be the sheaf generated by $\mathscr{A}$, and let $v : \mathscr{A} \to \tilde{\mathscr{A}}$ be the canonical homomorphism. If $v : \mathscr{A}(U) \approx \tilde{\mathscr{A}}(U)$ for all open subsets U of X with $U \Subset A$, then v induces an isomorphism $v_* : \hat{H}^*(X; \mathscr{A}) \approx \hat{H}^*(X; \tilde{\mathscr{A}})$.*

Proof. This follows from Corollary 5.3 because the presheaves Ker v and Coker v are uniformly locally trivial by Lemma 5.5.

5.7. Proposition. *Let (X, A) be a topological pair satisfying the conditions* (a)–(c) *of Theorem* 2.7, *let $\mathscr{A}$ be a sheaf of Abelian groups on X, and let us put $M = \{x \in X | \mathscr{A}_x \neq 0\}$. If $M \cap \operatorname{int} A = \emptyset$ then $\hat{H}^p(X; \mathscr{A}) = 0$ for $p > rd_X M$.*

Proof. Under the assumption $B = \emptyset$ this was proved by E. G. Skljarenko in [18]. Using Lemma 5.4 one can easily verify that his proof goes through also in our more general case.

Starting from this point the notation of Theorem 2.7 is used.

5.8. Lemma. *If (X, A) is locally contractible at each point $x \in X$-int A in the sense that for each neighborhood U of x there is a smaller one V such that the inclusion map $(V, V \cap A) \hookrightarrow (U, U \cap A)$ is homotopic to a constant map onto the point x, then the homomorphism* (see 2.5)

$$r_x : Sh^q(p_1, p_1 | C)_x \to h^q(p_1^{-1}(x), p_1^{-1}(x) \cap C) \tag{5.1}$$

is bijective for all $x \in X$ and all integers q.

Proof. This is an immediate consequence of the homotopy axiom.

5.9. Proposition. *If (X, A), (Y, B) and N satisfy the conditions* (a)–(d) *of Theorem* 2.7 *and if (X, A) is locally contractible at each point $x \in X$-int A, then $T^n(Z, C)$ is bijective for $n < N$ and injective for $n = N$.*

Proof. Applying Theorem 1.11 in an appropriate way and using Proposition 5.6, we obtain the following commutative diagram

$$\begin{array}{ccccc}
\hat{H}^p(X; S\hat{H}^q(p_1, p_1 | C; G)) & \approx & E_2^{p,q} & \underset{p}{\Longrightarrow} & \hat{H}^n(Z, C; G) \\
\downarrow \psi_2^{p,q} & & \downarrow \varphi_2^{p,q} & & \downarrow T^n(Z, C) \\
\hat{H}^p(X; Sh^q(p_1, p_1 C)) & \approx & E_2'^{p,q} & \underset{p}{\Longrightarrow} & h^n(Z, C)
\end{array}$$

where the homomorphism $\varphi : E \to E'$ of spectral sequences and the homomorphisms $\psi_2^{p,q}$ are induced by T. Using the obvious fact that $\psi_2^{0,N}$ is injective and applying

Lemma 5.8, we obtain that $\varphi_2^{p,q}$ is an isomorphism for $p + q < N$ and a monomorphism for $p + q = N$. An easy inductive argument shows that the same holds for $\varphi_\infty^{p,q}$. This, however, implies that $T^n(Z, C)$ is bijective for $n < N$ and injective for $n = N$.

5.10. Remark. As a special case of the last proposition we obtain: If the spaces X and Y are paracompact and regular and if one of them is locally contractible, then $\hat{H}^*(X \times Y; G) \approx \check{H}^*(X \times Y; G)$.

5.11. Lemma. *For every pair p, q of integers there is a canonical homomorphism*

$$(5.2) \qquad \varkappa_{p,q} : \hat{h}^p(X, A; h^q(Y, B)) \to \hat{H}^p(X; Sh^q(p_1, p_1| C)) .$$

If (X, A) satisfies the conditions (a)–(c) *of Theorem* 2.7 *then* (5.2) *is an epimorphism for $p = rd_X M_q + 1$ and an isomorphism for $p > rd_X M_q + 1$.*

Proof. Let $G_q = h^q(Y, B)$, let $\mathscr{A}_q$ be the presheaf on X defined by putting $\mathscr{A}_q(U) = G_q$ for $U \cap A = \emptyset$ and $\mathscr{A}_q(U) = 0$ for $U \cap A \neq \emptyset$, and let $\tilde{\mathscr{A}}_q$ be the sheaf generated by $\mathscr{A}_q$. For each open subset U of X we have an obvious canonical monomorphism $\mathscr{A}_q(U) \to Ph^q(p_1, p_1| C)(U)$, and it is clear that these monomorphisms, induce a monomorphism $v : \tilde{\mathscr{A}}_q \to Sh^q(p_1, p_1| C)$. We now define the homomorphism (5.2) as the composition $\hat{h}^p(X, A; G_q) = \hat{H}^p(X; \mathscr{A}_q) \to H^p(X; \tilde{\mathscr{A}}_q) \xrightarrow{v_*} \hat{H}^p(X; Sh^q(p_1, p_1| C))$ of the homomorphisms induced by the canonical homomorphisms $\mathscr{A}_q \to \tilde{\mathscr{A}}_q \to Sh^q(p_1, p_1| C)$.

Now let us suppose that (X, A) satisfies the conditions (a)–(c) of Theorem 2.7. Since by Proposition 5.6 $\hat{H}^*(X; \mathscr{A}_q) \approx \hat{H}^*(X; \tilde{\mathscr{A}}_q)$, we must show that v_* is an epimorphism for $p = rd_X M_q + 1$ and an isomorphism for $p > rd_X M_q + 1$. Let $\mathscr{K}$ be the cokernel of v in the category of presheaves of Abelian groups and let $\tilde{\mathscr{K}}$ be the sheaf generated by $\mathscr{K}$. Clearly $\mathscr{K}_x = 0$ for $x \notin M_q$, and therefore by Proposition 5.6 and Proposition 5.7 we have $\hat{H}^p(X; \mathscr{K}) = \hat{H}^p(X; \tilde{\mathscr{K}}) = 0$ for $p > rd_X M_q$. Applying now the exact cohomology sequence corresponding to the exact sequence $0 \to \tilde{\mathscr{A}}_q \to Sh^q(p_1, p_1| C) \to \mathscr{K} \to 0$ of presheaves, we obtain immediately that (5.2) is an epimorphism for $p = rd_X M_q + 1$ and an isomorphism for $p > rd_X M_q + 1$.

5.12. Proof of Theorem 2.7. For each $\mathscr{U} \in \text{cov}(X)$ let us put $(X_\mathscr{U}, A_\mathscr{U}) = (|N(\mathscr{U})|, |N(\mathscr{U} \cap A)|)$ and $(Z_\mathscr{U}, C_\mathscr{U}) = (X_\mathscr{U}, A_\mathscr{U}) \times (Y, B)$. By Proposition 5.9, the homomorphism $T^n(Z_\mathscr{U}, C_\mathscr{U}) \circ \Phi^n(X_\mathscr{U}, A_\mathscr{U})$, $\mathscr{U} \in \text{cov}(X)$, is bijective for $n < N$ and injective for $n = N$. Consequently, the same argument as that used in the proof of Corollary 2.2 shows that it suffices to prove that the homomorphism

$$\varinjlim_{\mathscr{U} \in \text{cov}(X)} h^n(Z_\mathscr{U}, C_\mathscr{U}) \xrightarrow{\vec{\varphi}^n} h^n(Z), C)$$

induced by canonical projections $X \to X_\mathscr{U}$ is an epimorphism for $n = v(N)$, an isomorphism for $v(N) < n < N$ and a monomorphism for $n = N$.

Let $p_\mathscr{U} : Z_\mathscr{U} \to X_\mathscr{U}$ be the canonical projection. By Theorem 1.11 and by Lemma

5.11, for each $\mathscr{U} \in \operatorname{cov}(X)$ there is a diagram

$$(5.3) \qquad \hat{h}^p(X_{\mathscr{U}}, A_{\mathscr{U}}; h^q(Y, B)) \approx E_2^{p,q}(\mathscr{U}) \underset{p}{\Longrightarrow} h^n(Z_{\mathscr{U}}, C_{\mathscr{U}}),$$

where $E(\mathscr{U}) = \{E_r(\mathscr{U}), d_r, \iota_r\}_{r \geqq 2}$ is a first-quadrant cohomological spectral sequence and $h^*(Z_{\mathscr{U}}, C_{\mathscr{U}})$ is considered as a filtered graded Abelian group with the filtration $F_{\mathscr{U}} h^*(Z_{\mathscr{U}}, C_{\mathscr{U}})$ determined by $p_{\mathscr{U}}$ and satisfying $F^0_{\mathscr{U}} h^*(Z_{\mathscr{U}}, C_{\mathscr{U}}) = h^*(Z_{\mathscr{U}}, C_{\mathscr{U}})$ and $F^{n+1}_{\mathscr{U}} h^n(Z_{\mathscr{U}}, C_{\mathscr{U}}) = 0$. It follows further from Theorem 1.11 that every canonical projection $\varphi_{\mathscr{U}\mathscr{V}} : X_{\mathscr{V}} \to X_{\mathscr{U}}$ induces a homorphism $\psi_{\mathscr{U}\mathscr{V}} : E(\mathscr{U}) \to E(\mathscr{V})$ of spectral sequences, and that the diagram

$$(5.4) \qquad \begin{array}{ccccc} \hat{h}^p(X_{\mathscr{U}}, A_{\mathscr{U}}; h^q(Y, B)) & \approx & E_2^{p,q}(\mathscr{U}) & \underset{p}{\Longrightarrow} & h^n(Z_{\mathscr{U}}, C_{\mathscr{U}}) \\ \Big\downarrow (\varphi_{\mathscr{U}\mathscr{V}})^* & & \Big\downarrow (\psi_{\mathscr{U}\mathscr{V}})_2^{p,q} & & \Big\downarrow (\varphi_{\mathscr{U}\mathscr{V}} \times \mathrm{id})^* \\ \hat{h}^p(X_{\mathscr{V}}, A_{\mathscr{V}}; h^q(Y, B)) & \approx & E_2^{p,q}(\mathscr{V}) & \underset{p}{\Longrightarrow} & h^n(Z_{\mathscr{V}}, C_{\mathscr{V}}) \end{array}$$

commutes. This diagram implies in particular that $\psi_{\mathscr{U}\mathscr{V}}$ does not depend on the choice of the canonical projection $\varphi_{\mathscr{U}\mathscr{V}}$. Consequently, (5.3) and (5.4) form an inductive system over $\operatorname{cov}(X)$ with the limit

$$\hat{h}^p(X, A; h^q(Y, B)) \approx \vec{E}_2^{p,q} \underset{p}{\Longrightarrow} \varinjlim_{\mathscr{U} \in \operatorname{cov}(X)} h^n(Z_{\mathscr{U}}, C_{\mathscr{U}}),$$

where $\vec{E} = \varinjlim E(\mathscr{U})$ and $\varinjlim h^*(Z_{\mathscr{U}}, C_{\mathscr{U}})$ is taken in the category of filtered graded Abelian groups. It is clear that the filtration in $\varinjlim h^*(Z_{\mathscr{U}}, C_{\mathscr{U}})$ is regular.

In the same way one can show that canonical projections $\varphi_{\mathscr{U}} : X \to X_{\mathscr{U}}$, $\mathscr{U} \in \operatorname{cov}(X)$, induce the commutative diagram

$$(5.5) \qquad \begin{array}{ccccc} \hat{h}^p(X, A; h^q(Y, B)) & \xrightarrow{\approx} & \vec{E}_2^{p,q} & \underset{p}{\Longrightarrow} & \varinjlim h^n(Z_{\mathscr{U}}, C_{\mathscr{U}}) \\ \Big\downarrow \varkappa_{p,q} & & \Big\downarrow \vec{\psi}_2^{p,q} & & \Big\downarrow \vec{\varphi}^* \\ H^p(X; Sh^q(p_1, p_1| C)) & \xrightarrow{\approx} & E_2^{p,q} & \underset{p}{\Longrightarrow} & h^n(Z, C) \end{array}$$

whose second row corresponds to the pair (Z, C) and the canonical projection $p_1 : Z \to X$.

Using the obvious fact that $\varkappa_{0,q}$ is always injective we conclude from (5.5) and from Lemma 5.11 that $\vec{\psi}_2^{p,q}$ is surjective for $p + q = \nu(N)$, bijective for $\nu(N) < p + q < N$ and injective for $p + q = N$. An easy inductive argument shows that the same is true for $\vec{\psi}_r^{p,q}$, $2 < r \leqq \infty$. This, however, immediately implies that $\vec{\varphi}^n$ is surjective for $n = \nu(N)$, bijective for $\nu(N) < n < N$, and injective for $n = N$, and the proof is complete.

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