Czechoslovak Mathematical Journal

Ladislav Nebeský

Upper embeddable factorizations of graphs

Czechoslovak Mathematical Journal, Vol. 34 (1984), No. 3, 432–438

Persistent URL: http://dml.cz/dmlcz/101969

Czechoslovak Mathematical Journal, 34 (109) 1984, Praha

UPPER EMBEDDABLE FACTORIZATIONS OF GRAPHS

LADISLAV NEBESKÝ, Praha

(Received July 8, 1983)

By a graph we shall mean a pseudograph in the sense of [1]. If G is a graph, then $V(G)$, $E(G)$, $C(G)$, p_G, q_G, and c_G denote its vertex set, its edge set, the set of its components, the number of its vertices, the number of its edges, and the number of its components, respectively. If G is a connected graph, then $\gamma_M(G)$ denotes the maximum genus of G, i.e. the maximum integer k with the property that there exists a 2-cell embedding of G into the closed orientable surface of genus k. If G is a connected graph, then $\gamma_M(G) \leqq [(q_G - p_G + 1)/2]$ (cf. [1] or [7], for example). A graph G is said to be upper embeddable if it is connected and $\gamma_M(G) = [(q_G - p_G + 1)/2]$.

Let G be a connected graph. We denote by $\mathscr{T}(G)$ the set of its spanning trees. If $T \in \mathscr{T}(G)$, then we denote by $x_G(T)$ the number of components F of $G - E(T)$ with the property that q_F is odd. The following theorem was proved by Homenko, Ostroverkhy, and Kusmenko [2] and independently by Xuong [8]:

Theorem A. *If G is a connected graph, then*

$$\gamma_M(G) = (q_G - p_G + 1 - \min_{T \in \mathscr{T}(G)} x_G(T))/2 .$$

The following partial case of Theorem A was also proved independently by Jungerman [3]:

Theorem B. *A connected graph G is upper embeddable if and only if there exists $T \in \mathscr{T}(G)$ such that $x_G(T) \leqq 1$.*

If H is a graph, then we denote by b_H the number of components F of H with the property that $q_F - p_F + 1$ is odd. If G is a graph and $A \subseteq E(G)$, then we denote

$$y_G(A) = c_{G-A} + b_{G-A} - 1 - |A| .$$

Theorem C ([5]). *If G is a connected graph, then*

$$\min_{T \in \mathscr{T}(G)} x_G(T) = \max_{A \subseteq E(G)} y_G(A) .$$

The following theorem is a very easy consequence of Theorems B and C:

Theorem D ([5]). *A connected graph G is upper embeddable if and only if*

$$c_{G-A} + b_{G-A} - 2 \leqq |A| \quad \text{for every} \quad A \subseteq E(G)\,.$$

In the present paper we shall generalize Theorems C and D.

Let G be a graph and let $n \geqq 1$ be an integer. By an n-factorization of G we shall mean a sequence $(G_1, \ldots, G_n)$ of edge-disjoint spanning subgraphs $G_1, \ldots, G_n$ of G with the property that $E(G) = E(G_1) \cup \ldots \cup E(G_n)$. We shall say that an n-factorization $(G_1, \ldots, G_n)$ of G is connected or upper embeddable if for each $i \in \{1, \ldots, n\}$, G_i is connected or upper embeddable, respectively.

The following theorem is due to Tutte [6]; it was also proved by Nash-Williams [4]:

Theorem E. *Let $n \geqq 1$ be an integer. A graph G has a connected n-factorization if and only if*

$$n(c_{G-A} - 1) \leqq |A| \quad \text{for every} \quad A \subseteq E(G)\,.$$

Let $n \geqq 1$ be an integer. Assume that H is a graph; then we denote by $B_{n,H}$ the set of all $F \in C(H)$ with the property that $q_F - n(p_F - 1)$ is odd; moreover, we denote $b_{n,H} = |B_{n,H}|$. Consider a graph G. We denote by $\mathscr{T}_n(G)$ the set of all sequences $(T_1, \ldots, T_n)$ of edge-disjoint spanning trees $T_1, \ldots, T_n$ of G. For every $(T_1, \ldots, T_n) \in \mathscr{T}_n(G)$ we denote

$$x_{n,G}(T_1, \ldots, T_n) = |\{F \in C(G - (E(T_1) \cup \ldots \cup E(T_n));\ q_F \text{ is odd}\}|\,.$$

For every $A \subseteq E(G)$ we denote

$$y_{n,G}(A) = n(c_{G-A} - 1) + b_{n,G-A} - |A|\,.$$

The following theorem is the main result of the present paper:

Theorem 1. *Let $n \geqq 1$ be an integer. Assume that G is a graph which has a connected n-factorization. Then*

$$\min_{(T_1,\ldots,T_n) \in \mathscr{T}_n(G)} x_{n,G}(T_1, \ldots, T_n) = \max_{A \subseteq E(G)} y_{n,G}(A)\,.$$

Combining Theorems B, E and 1, we get

Theorem 2. *Let $n \geqq 1$ be an integer and let G be a graph. Then G has an upper embeddable n-factorization if and only if*

$$(*) \qquad n(c_{G-A} - 1) + \max(0, b_{n,G-A} - n) \leqq |A| \quad \text{for every} \quad A \subseteq E(G)\,.$$

Before proving Theorems 1 and 2 we shall prove two lemmas.

Lemma 1. *Let $n \geqq 1$ be an integer and let G be a graph. Then*

$$y_{n,G}(A) \equiv q_G - n(p_G - 1) \pmod 2 \quad \text{for every} \quad A \subseteq E(G)\,.$$

Proof. For an arbitrary $A \subseteq E(G)$ we have

$$q_G - n(p_G - 1) + y_{n,G}(A) = q_G - n(p_G - 1) + n(c_{G-A} - 1) +$$
$$+ b_{n,G-A} - |A| = b_{n,G-A} + \sum_{F \in C(G-A)} (q_F - n(p_F - 1)) \equiv 0 \pmod 2 .$$

Hence, the lemma follows.

Let $n \geqq 1$ be an integer and let G be a graph. We denote

$$y_{n,G} = \max_{A \subseteq E(G)} y_{n,G}(A) .$$

Moreover, we denote by $\mathrm{MAX}_n(G)$ the set of all $A \subseteq E(G)$ with the properties that $y_{n,G}(A) = y_{n,G}$, and for each $A_0 \subseteq E(G)$, if $y_{n,G}(A_0) = y_{n,G}$, then A is not a proper subset of A_0.

Lemma 2. *Let $n \geqq 1$ be an integer. Assume that G is a graph. Let $A \in \mathrm{MAX}_n(G)$ and let $F \in C(G - A)$. Then*

(i) *if $q_F - n(p_F - 1)$ is even, then $q_F = 0$;*

(ii) *if $q_F - n(p_F - 1)$ is odd, then $q_F \geqq 1$, and for each $e \in E(F)$, $y_{n,F-e} = 0$ and $F - e$ has a connected n-factorization.*

Proof. (i) First, let $q_F - n(p_F - 1)$ be even. Clearly, $y_{n,G}(A \cup \{e\}) \geqq y_{n,G}(A)$ for each $e \in E(F)$. Since $A \in \mathrm{MAX}_n(G)$, $q_F = 0$.

(ii) Now let $q_F - n(p_F - 1)$ be odd. If $q_F = 0$, then $p_F = 1$ and $q_F - n(p_F - 1) = 0$, which is a contradiction. Thus, $q_F \geqq 1$.

Consider an arbitrary $e \in E(G)$. Let $Z \subseteq E(F - e)$. It is clear that

$$c_{G-(A \cup \{e\} \cup Z)} = c_{G-A} - 1 + c_{(F-e)-Z}$$

and

$$b_{n,G-(A \cup \{e\} \cup Z)} = b_{n,G-A} - 1 + b_{n,(F-e)-Z} .$$

We have

$$y_{n,G}(A \cup \{e\} \cup Z) = n(c_{G-(A \cup \{e\} \cup Z)} - 1) + b_{n,G-(A \cup \{e\} \cup Z)} - |A \cup Z| - 1 =$$
$$= y_{n,G}(A) + y_{n,F-e}(Z) - 2 .$$

Since $A \in \mathrm{MAX}_n(G)$, $y_{n,G}(A \cup \{e\} \cup Z) < y_{n,G}(A)$. Hence, $y_{n,F-e}(Z) \leqq 1$. Since $q_{F-e} - n(p_F - 1)$ is even, it follows from Lemma 1 that $y_{n,F-e}(Z)$ is also even, and thus $y_{n,F-e}(Z) \leqq 0$. Since $y_{n,F-e}(\emptyset) \geqq 0$, $y_{n,F-e} = 0$.

Assume that $F - e$ has no connected n-factorization. According to Theorem E, there exists $Z' \subseteq E(F - e)$ such that $|Z'| < n(c_{(F-e)-Z'} - 1)$. Since $y_{n,F-e}(Z') \leqq 0$, $n(c_{(F-e)-Z'} - 1) \leqq |Z'| - b_{n,(F-e)-Z'}$. Thus, $b_{n,(F-e)-Z'} < 0$, which is a contradiction. This means that F has a connected n-factorization, which completes the proof of the lemma.

Let $n \geqq 1$ be an integer and let G be a graph. If G has a connected n-factorization, then $\mathscr{T}_n(G) \neq \emptyset$ and we denote

$$x_{n,G} = \min_{(T_1, \ldots, T_n) \in \mathscr{T}_n(G)} x_{n,G}(T_1, \ldots, T_n) .$$

Proof of Theorem 1. We shall prove that $x_{n,G} = y_{n,G}$. If $q_G = 0$, the result is obvious. Let $q_G \geqq 1$. Assume that for every graph G' which has a connected n-factorization, it has been proved that $x_{n,G'} = y_{n,G'}$.

(I) We first prove that $x_{n,G} \leqq y_{n,G}$. Consider $A \subseteq E(G)$ such that $y_{n,G}(A) = y_{n,G}$. Let $(T_1, \ldots, T_n) \in T_n(G)$. Denote

$$B_0 = \{F \in B_{n,G-A};\ \text{for each } i \in \{1, \ldots, n\},\ \text{the subgraph of } T_i \text{ induced by } V(F) \text{ is a tree}\}$$

and

$$E_0 = E(T_1) \cup \ldots \cup E(T_n).$$

Clearly, $|E(F) - E_0|$ is odd for each $F \in B_0$. It is easy to see that for at least $|B_0| - |A - E_0|$ components H of $G - E_0$, q_H is odd. Hence,

$$x_{n,G}(T_1, \ldots, T_n) \geqq |B_0| - |A - E_0|.$$

Moreover, we have

$$c_{T_1-A} + \ldots + c_{T_n-A} \geqq nc_{G-A} + |B_{n,G-A} - B_0|.$$

Clearly, $|E(T_i) \cap A| = c_{T_i-A} - 1$ for each $i \in \{1, \ldots, n\}$. Since

$$|E_0 \cap A| = |E(T_1) \cap A| + \ldots + |E(T_n) \cap A|,$$

it is obvious that

$$0 \geqq |B_{n,G-A} - B_0| + n(c_{G-A} - 1) - |E_0 \cap A|.$$

We have

$$x_{n,G} \geqq x_{n,G}(T_1, \ldots, T_n) \geqq |B_0| - |A - E_0| \geqq$$
$$\geqq |B_0| - |A - E_0| + |B_{n,G-A} - B_0| + n(c_{G-A} - 1) - |E_0 \cap A| =$$
$$= n(c_{G-A} - 1) + b_{n,G-A} - |A| = y_{n,G}(A) = y_{n,G}.$$

(II) We now wish to prove that $x_{n,G} \leqq y_{n,G}$. We distinguish the following cases and subcases:

1. Assume that for every $A \in \mathrm{MAX}_n(G)$ and every $F \in C(G - A)$, $q_F \leqq 1$. It follows from Lemma 2 that for every $A \in \mathrm{MAX}_n(G)$ and every $F \in C(G - A)$, $p_F = 1$.

1.1. Assume that there exists no loop in G. Let $A \in \mathrm{MAX}_n(G)$. We have $A = E(G)$ and $b_{n,G-A} = 0$. Since $y_{n,G} \geqq y_{n,G}(\emptyset) \geqq 0$, $q_G \leqq n(p_G - 1)$. Since G has a connected n-factorization, there exists $(T_1, \ldots, T_n) \in \mathcal{T}_n(G)$. Since $q_G \leqq n(p_G - 1)$, $(T_1, \ldots, T_n)$ is an n-factorization of G. Hence, $x_{n,G} = 0 \leqq y_{n,G}$.

1.2. Assume that there exists a loop e in G. We denote by w the vertex incident with e in G.

1.2.1. Assume that $y_{n,G} < y_{n,G-e}$. There exists $A^* \subseteq E(G - e)$ such that $y_{n,G-e}(A^*) = y_{n,G-e}$. Obviously, $y_{n,G}(A^* \cup \{e\}) = y_{n,G-e}(A^*) - 1$. Since $y_{n,G} < y_{n,G-e} = y_{n,G-e}(A^*)$, $y_{n,G}(A^* \cup \{e\}) = y_{n,G}$. This implies that there exists $A \in \mathrm{MAX}_n(G)$ such that $e \in A$. Let F^* be the component of $G - A$ containing w.

Clearly, $q_{F^*} \leqq 1$ and $p_{F^*} = 1$. If $q_{F^*} = 0$, then $y_{n,G}(A - \{e\}) = y_{n,G}(A) + 2$, which is a contradiction. Thus $q_{F^*} = 1$. Since $p_{F^*} = 1$, the only edge of F^*, say an edge e^*, is a loop of G. Obviously, $G - e - e^*$ has a connected n-factorization. It is clear that for every $Z \subseteq E(G - e - e^*)$, $y_{n,G-e-e^*}(Z) = y_{n,G}(Z)$. Hence, $y_{n,G-e-e^*} \leqq y_{n,G}$. It follows from the induction assumption that $x_{n,G-e-e^*} = y_{n,G-e-e^*}$. Since $x_{n,G} \leqq \leqq x_{n,G-e-e^*}$, $x_{n,G} \leqq y_{n,G}$.

1.2.2. Assume that $y_{n,G-e} \leqq y_{n,G}$. It follows from Lemma 1 that $y_{n,G-e} + 1 \leqq \leqq y_{n,G}$. Since e is a loop in G, $T_n(G - e) = \mathscr{T}_n(G)$. It is easy to see that $x_{n,G} \leqq \leqq x_{n,G-e} + 1$. According to the induction assumption, $x_{n,G-e} = y_{n,G-e}$. Hence, $x_{n,G} \leqq y_{n,G}$.

2. Assume that there exists $A \in \mathrm{MAX}_n(G)$ such that for at least one $F_0 \in C(G - A)$, $q_{F_0} \geqq 2$. Denote $B = B_{n,G-A}$. As follows from Lemma 2, $B \neq \emptyset$.

Consider a graph J with the following properties:

(i) there exists a one-to-one mapping r of $C(G - A)$ onto $V(J)$;

(ii) $A \subseteq E(J)$;

(iii) if $v \in V(J)$ and $e \in A$, then v and e are adjacent in J if and only if in G the edge e is incident with a vertex of $r^{-1}(v)$;

(iv) there exists a one-to-one mapping s of B onto $E(J) - A$ such that if $F \in B$, then $s(F)$ is a loop of J and it is incident with $r(F)$.

It is easy to see that for every $Z_0 \subseteq E(J)$ and every $e_0 \in E(J) - A$, $y_{n,J}(Z_0 \cup \{e_0\}) \leqq \leqq y_{n,J}(Z_0)$. This implies that

$$y_{n,J} = \max_{Z \subseteq A} y_{n,J}(Z) .$$

Let Z' be an arbitrary subset of A. There exists a one-to-one mapping r' of $C(G - Z')$ onto $C(J - Z')$ such that for each $H \in C(G - Z')$,

$$V(r'(H)) = \{r(F);\ F \in C(H - A)\} .$$

Thus $c_{J-Z'} = c_{H-Z'}$. Consider an arbitrary $H \in C(G - Z')$; then

$$q_H - n(p_H - 1) = |E(H) \cap A| - n(c_{H-A} - 1) + \sum_{F \in C(H-A)} (q_F - n(p_F - 1)) ;$$

obviously, $|E(r'(H)) \cap A| = |E(H) \cap A|$ and $c_{r'(H)-A} = c_{H-A}$; it follows from the definition of J that

$$q_{r'(H)} - n(p_{r'(H)} - 1) \equiv q_H - n(p_H - 1) \pmod 2 .$$

This means that $b_{n,J-Z'} = b_{n,G-Z'}$, and therefore, $y_{n,J}(Z') = y_{n,G}(Z')$. Since $y_{n,G}(A) = = y_{n,G}$, we conclude that

$$y_{n,J} = y_{n,J}(A) = y_{n,G} .$$

Recall that $c_{J-Z'} = c_{G-Z'}$ for every $Z' \subseteq A$. It follows from Theorem E that J has a connected n-factorization. Since $q_J < q_G$, it follows from the induction assumption that there exists $(T_1, \ldots, T_n) \in \mathscr{T}_n(J)$ such that $x_{n,J}(T_1, \ldots, T_n) = y_{n,G}$.

Denote $E_0 = E(T_1) \cup \ldots \cup E(T_n)$. Since $n(c_{J-A}-1) = n(p_{J-A}-1) = |E_0|$, $b_{n,J-A} = |E(J) - A|$, and $y_{n,J}(A) = y_{n,G}$, it is obvious that

$$y_{n,G} = |E(J) - A| - |A - E_0| = x_{n,J}(T_1, \ldots, T_n).$$

This implies that there exists a one-to-one mapping ω of $A - E_0$ onto a subset of $E(J) - A$ such that for each $e \in A - E_0$, the edges e and $\omega(e)$ are adjacent in J. Let t be a mapping of B into $E(G - A)$ such that $t(F) \in E(F)$ for each $F \in B$, and if there exists $e \in A - E_0$ such that $\omega(e) = s(F)$, then in G the edges $t(F)$ and e are adjacent. Let $F \in B$; according to Lemma 2, $y_{n,F-t(F)} = 0$ and $F - t(F)$ has a connected n-factorization; since $q_{F-t(F)} < q_G$, it follows from the induction assumption that there exists $(T_{1,F}, \ldots, T_{n,F}) \in \mathscr{T}_n(F)$ such that $x_{n,F-t(F)}(T_{1,F}, \ldots, T_{n,F}) = 0$.

For each $i \in \{1, \ldots, n\}$, let $T_{i,G}$ denote the subgraph of G induced by

$$E(T_i) \cup \bigcup_{F \in B} E(T_{i,F}).$$

According to Lemma 2, $q_F = 0$ for each $F \in C(G - A) - B$. This implies that $(T_{1,G}, \ldots, T_{n,G}) \in \mathscr{T}_n(G)$. The fact that $x_{n,F-t(F)}(T_{1,F}, \ldots, T_{n,F}) = 0$ for each $F \in B$ implies that

$$x_{n,G} \leqq x_{n,G}(T_{1,G}, \ldots, T_{n,G}) \leqq x_{n,J}(T_1, \ldots, T_n) = y_{n,G},$$

which completes the proof of Theorem 1.

Remark 1. If we put $n = 1$ in Theorem 1, we get Theorem C. The technique used in the proof of Theorem 1 was derived from the technique used in [5] (but the structure of the proof was simplified in some points).

Proof of Theorem 2. (I) Assume that (*) holds. Then $n(c_{G-A} - 1) \leqq |A|$ for every $A \subseteq E(G)$. According to Theorem E, G has a connected n-factorization. Since $n(c_{G-A} - 1) + b_{n,G-A} - n \leqq |A|$ for every $A \subseteq E(G)$, it is obvious that $y_{n,G} \leqq n$. According to Theorem 1, there exists $(T_1, \ldots, T_n) \in \mathscr{T}_n(G)$ such that $x_{n,G}(T_1, \ldots, T_n) \leqq n$. This implies that there exists a connected n-factorization $(G_1, \ldots, G_n)$ of G with the property that $x_{G_i} \leqq 1$ for each $i \in \{1, \ldots, n\}$. Thus, according to Theorem B, G has an upper embeddable n-factorization.

(II) Assume that G has an upper embeddable n-factorization, say an n-factorization $(G_1, \ldots, G_n)$. Then $(G_1, \ldots, G_n)$ is a connected n-factorization, and according to Theorem B, there exists a spanning tree T_i of G_i such that $x_{G_i} \leqq 1$ for each $i \in \{1, \ldots, n\}$. It is obvious that $(T_1, \ldots, T_n) \in \mathscr{T}_n(G)$ and that $x_{n,G}(T_1, \ldots, T_n) \leqq n$. According to Theorem 1, $y_{n,G} \leqq n$. Combining Theorem E and the definition of $y_{n,G}$, we get (*), which completes the proof of Theorem 2.

Remark 2. We shall state one more consequence of Theorems A, E and 1 (the proof is easy): A graph G has a connected n-factorization $(G_1, \ldots, G_n)$ such that $\gamma_M(G_1) = (q_{G_1} - p_G + 1)/2, \ldots, \gamma_M(G_n) = (q_{G_n} - p_G + 1)/2$ if and only if

$$n(c_{G-A} - 1) + b_{n,G-A} \leqq |A| \quad \text{for every} \quad A \subseteq E(G).$$

References

[1] *M. Behzad, G. Chartrand,* and *L. Lesniak-Foster:* Graphs & Digraphs. Prindle, Weber & Schmidt, Boston 1979.

[2] *N. P. Homenko, N. A. Ostroverkhy,* and *V. A. Kusmenko:* The maximum genus of a graph (in Ukrainian, English summary). φ-peretvorennya grafiv (N. P. Homenko, ed.) IM AN URSR, Kiev 1973, pp. 180—210.

[3] *M. Jungerman:* A characterization of upper embeddable graphs. Trans. Amer. Math. Soc. *241* (1978), 401—406.

[4] *C. St. J. A. Nash-Williams:* Edge-disjoint spanning trees of finite graphs. J. London Math. Soc. *36* (1961), 445—450.

[5] *L. Nebeský:* A new characterization of the maximum genus of a graph. Czechoslovak Math. J. *31* (*106*) (1981), 604—613.

[6] *W. T. Tutte:* On the problem of decomposing a graph into *n* connected factors. J. London Math. Soc. *36* (1961), 221—230.

[7] *A. T. White:* Graphs, Groups, and Surfaces. North-Holland, Amsterdam 1973.

[8] *N. H. Xuong:* How to determine the maximum genus of a graph. J. Combinatorial Theory *26* B (1979), 217—225.

Author's address: 116 38 Praha 1, nám. Krasnoarmějců 2, ČSSR (Filozofická fakulta Univerzity Karlovy).