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A NOTE ON THE STABILITY OF θ-METHODS FOR VOLTERRA INTEGRAL EQUATIONS OF THE SECOND KIND

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1. INTRODUCTION

The majority of stability analyses of numerical methods for Volterra integral equations of the second kind

$$y(t) = g(t) + \int_0^t k(t, s, y(s))\, ds\,, \quad t \geqq 0\,, \tag{1}$$

were based on the simple linear test equation

$$y(t) = g(t) + \lambda \int_0^t y(s)\, ds\,, \quad t \geqq 0\,, \tag{2}$$

where λ is a complex number and $\operatorname{Re}(\lambda) < 0$ (see, for example, [1–3]). It is the purpose of this note to present a stability analysis of some methods for the numerical integration of (1) based on the more general test equation

$$y(t) = g(t) + \lambda \int_0^t k(s)\, y(s)\, ds\,, \quad t \geqq 0\,, \tag{3}$$

where $\operatorname{Re}(\lambda) < 0$ and the functions g and k satisfy certain conditions which will be given later.

Denote by $h > 0$ a fixed step size and define the grid $\{t_i\}_{i=0}^{\infty}$ by $t_i = ih$, $i = 0, 1, \ldots$. We are interested in the following class of the so called θ-methods:

$$y_n = g_n + \lambda h\big[(1 - \theta)\sum_{i=0}^{n-1} k(t_n, t_i, y_i) + \theta \sum_{i=1}^{n} k(t_n, t_i, y_i)\big]\,, \tag{4}$$

$i = 0, 1, \ldots, \theta \in [0, 1]$ $\left(\sum_{i=0}^{-1} = 0,\ \sum_{i=1}^{0} = 0\right)$. Here, $g_n = g(t_n)$ and y_n is the approximation to $Y(t_n)$, where Y is the solution of (1). For $\theta = 0$, $\theta = \frac{1}{2}$, and $\theta = 1$ these are direct quadrature methods based on the left rectangular rule, the trapezoidal rule, and the right rectangular rule, respectively. It is easy to check that the local discretization error of (4)

$$\eta_\theta(t_n, h) := Y(t_n) - g_n - \lambda h\Big[(1-\theta)\sum_{i=0}^{n-1} k(t_{n-1}, t_i, Y(t_i)) + \theta \sum_{i=1}^{n} k(t_n, t_i, Y(t_i))\Big]$$

satisfies $\eta_\theta(t_n, h) = O(h)$ for $\theta \neq \frac{1}{2}$ and $\eta_\theta(t_n, h) = O(h^2)$ for $\theta = \frac{1}{2}$ uniformly in t_n as $h \to 0$. Consequently, this method is convergent with order one for $\theta \neq \frac{1}{2}$ and with order two for $\theta = \frac{1}{2}$, i.e.

$$y_n - Y(t_n) = \begin{cases} O(h), & \theta \neq \frac{1}{2}, \\ O(h^2), & \theta = \frac{1}{2}, \end{cases}$$

as $n \to \infty$, $nh = t_n$ (see [1, 4]).

In the next section we examine the behaviour of the solution Y of (3) and the approximate solution $\{y_n\}_{n=0}^{\infty}$ when the method (4) is applied to (3), for a fixed step size $h > 0$. It turns out that both Y and $\{y_n\}_{n=0}^{\infty}$ are bounded and the last bound is uniform in h and θ.

2. STABILITY ANALYSIS

We have the following bound on the solution Y of the equation (3).

Theorem 1. *Assume that* $|g(t)| \leqq G < \infty$ *and* $k(t) \geqq 0$ *for* $t \geqq 0$. *Assume also that* $\operatorname{Re}(\lambda) < 0$. *Then the solution* Y *of* (3) *satisfies* $|Y(t)| \leqq G(1 - |\lambda|/\operatorname{Re}(\lambda))$ *for* $t \geqq 0$.

Proof. Putting $z(t) = \int_0^t k(s)\, y(s)\, ds$, the problem (3) can be written as

$$z'(t) = \lambda\, k(t)\, z(t) + k(t)\, g(t), \quad t \geqq 0,$$
$$z(0) = 0.$$

The solution Z of this equation is given by

$$Z(t) = \int_0^t k(s)\, g(s) \exp\left(\lambda \int_s^t k(\tau)\, d\tau\right) ds, \quad t \geqq 0.$$

Let $\lambda = a + bi$. It follows that

$$|Z(t)| \leqq -(G/a) \int_0^t (-a)\, k(s) \exp\left(a \int_s^t k(\tau)\, d\tau\right) ds \leqq$$

$$\leqq -(G/a)\left[1 - \exp\left(a \int_0^t k(\tau)\, d\tau\right)\right] \leqq -G/a.$$

Taking into account that the solution Y of (3) is given by $Y(t) = \lambda Z(t) + g(t)$, we obtain $|Y(t)| \leqq G(1 - |\lambda|/\operatorname{Re}(\lambda))$, $t \geqq 0$, which is our claim.

Remark. It is impossible to bound the solution Y of (3) by a constant independent of λ. To see this, let us consider the problems

$$y(t) = \lambda_n \int_0^t y(s)\,\mathrm{d}s + \sin(t)\,, \quad t \geqq 0\,,$$

$$y(t) = \mathrm{i} \int_0^t y(s)\,\mathrm{d}s + \sin(t)\,, \quad t \geqq 0\,,$$

where $\lambda_n \to \mathrm{i}$ as $n \to \infty$, $\operatorname{Re}(\lambda_n) < 0$, with solutions Y_n and Y, and observe that Y is unbounded and $Y_n \to Y$ as $n \to \infty$ uniformly on any compact interval $[0, T]$, $T > 0$.

Our next theorem establishes a bound on the solution $\{y_n\}_{n=0}^{\infty}$ of the equation (4) applied to (3).

Theorem 2. *In addition to the conditions given in Theorem 1, assume that* $0 < \omega \leqq k(t) \leqq \Omega < \infty$ *for* $t \geqq 0$. *Then there exists* $h_\theta > 0$ *and a constant* $M \geqq 0$ *independent of* h, λ *and* θ *such that* $|y_n| \leqq M(1 - |\lambda|/\operatorname{Re}(\lambda))$, $n = 0, 1, \ldots,$ *for* $h \in (0, h_\theta]$ *and* $\theta \in [0, 1]$.

Proof. The method (4) applied to (3) yields

$$y_n = g_n + \lambda h\Big[(1-\theta)\sum_{i=0}^{n-1} k_i y_i + \theta \sum_{i=1}^{n} k_i y_i\Big]\,, \tag{5}$$

$n = 0, 1, \ldots,$ where $k_i = k(t_i)$. Subtracting y_{n+1} and y_n we obtain

$$y_{n+1} = \frac{1 + (1-\theta)\,\lambda h k_n}{1 - \theta\lambda h k_{n+1}}\, y_n + \frac{g_{n+1} - g_n}{1 - \theta\lambda h k_{n+1}}\,,$$

$n = 0, 1, \ldots.$ This is a recurrent equation of the first order, its solution being given by

$$y_n = \left(\prod_{i=0}^{n-1} \frac{1 + (1-\theta)\,\lambda h k_i}{1 - \theta\lambda h k_{i+1}}\right) y_0 + \sum_{i=0}^{n-1} \frac{g_{i+1} - g_i}{1 - \theta\lambda h k_{i+1}} \prod_{j=i+1}^{n-1} \frac{1 + (1-\theta)\,\lambda h k_j}{1 - \theta\lambda h k_{j+1}}\,,$$

$n = 0, 1, \ldots$ (see [5]). Hence,

$$|y_n| \leqq \prod_{i=0}^{n-1} \frac{|1 + (1-\theta)\,\lambda h k_i|}{|1 - \theta\lambda h k_{i+1}|}\, |y_0| + 2G \sum_{i=0}^{n-1} \prod_{j=i+1}^{n-1} \frac{|1 + (1-\theta)\,\lambda h k_j|}{|1 - \theta\lambda h k_{j+1}|}\,, \tag{6}$$

$n = 0, 1, \ldots.$ On the other hand, the equation (5) can be written as $y_n = \lambda h z_n + g_n$, where

$$z_n = (1-\theta)\sum_{i=0}^{n-1} k_i y_i + \theta \sum_{i=1}^{n} k_i y_i\,,$$

$n = 0, 1, \ldots.$ Subtracting z_{n+1} and z_n and eliminating y_{n+1} and y_n from the resulting equation we obtain

$$z_{n+1} = \frac{1 + (1 - \theta)\,\lambda h k_n}{1 - \theta\lambda k_{n+1}}\, z_n + \frac{(1 - \theta)\,k_n g_n + \theta k_{n+1} g_{n+1}}{1 - \theta\lambda h k_{n+1}},$$

$n = 0, 1, \ldots$. Hence, noting that $z_0 = 0$ we get

$$z_n = \sum_{i=0}^{n-1} \frac{(1 - \theta)\,k_i g_i + \theta k_{i+1} g_{i+1}}{1 - \theta\lambda h k_{i+1}} \prod_{j=i+1}^{n-1} \frac{1 + (1 - \theta)\,\lambda h k_j}{1 - \theta\lambda h k_{j+1}},$$

$n = 0,1, \ldots$. In view of this relation and the relationship between y_n and z_n we obtain

$$|y_n| \leqq |\lambda h|\, \Omega G \sum_{i=0}^{n-1} \prod_{j=i+1}^{n-1} \frac{|1 + (1 - \theta)\,\lambda h k_j|}{|1 - \theta\lambda h k_{j+1}|} + G, \tag{7}$$

$n = 0, 1, \ldots$. For any $\xi \in [\omega, \Omega]$ (and fixed λ, $\mathrm{Re}\,(\lambda) < 0$) let us consider the functions

$$\phi_\xi(h, \theta) := |1 + (1 - \theta)\,\lambda h\xi|, \quad \psi_\xi(h, \theta) := |1 - \theta\lambda h\xi|.$$

For any fixed $\theta \in [0, 1)$ the function ϕ_ξ attains its minimum value $|\mathrm{Im}\,(\lambda)/|\lambda||$ for $h_\xi = -\mathrm{Re}\,(\lambda)/((1 - \theta)\,\xi|\lambda|^2)$. In view of the assumptions on the function k it is clear that there exists $h_\theta > 0$ such that for all $h \in (0, h_\theta]$ the following inequality holds:

$$\phi_{k_j}(h, \theta) \leqq \phi_\omega(h, \theta) < 1.$$

The largest value of h_θ for which this inequality is satisfied for any function k satisfying the assumptions of the theorem can be computed from the condition $\phi_\omega(h_\theta, \theta) = \phi_\Omega(h_\theta, \theta)$, which leads to the formula

$$h_\theta = -2\,\mathrm{Re}\,(\lambda)/((1 - \theta)\,|\lambda|^2\,(\omega + \Omega)).$$

It is also clear that $\psi_{k_{j+1}}(h, \theta) \geqq \psi_\omega(h, \theta) > 1$ for any $h > 0$ and $\theta \in (0, 1]$. Hence, putting $q_\omega(h, \theta) = \phi_\omega(h, \theta)/\psi_\omega(h, \theta)$ and $Q = \max\{2G, \Omega G\}$, in view of (6), (7), and the relation $q_\omega(h, \theta) < 1$, which holds for any $\theta \in [0, 1]$ and $h \in (0, h_\theta]$ $(h_1 = \infty)$, we obtain

$$|y_n| \leqq \begin{cases} Q\,\dfrac{|\lambda h|}{1 - q_\omega(h, \theta)} + G, & h \leqq 1/|\lambda|, \\[2ex] Q\,\dfrac{1}{1 - q_\omega(h, \theta)} + G, & h > 1/|\lambda|, \end{cases} \tag{8}$$

$n = 0, 1, \ldots$. Let us set $D := \{(h, \theta)\colon \theta \in [0, 1],\ h \in (0, h_\theta]\}$ and define the (continuous) function $\eta_\omega : D \to [0, \infty)$ by

$$\eta_\omega(h, \theta) := \begin{cases} \dfrac{|\lambda h|}{1 - q_\omega(h, \theta)}, & h \leqq 1/|\lambda|, \\[2ex] \dfrac{1}{1 - q_\omega(h, \theta)}, & h > 1/|\lambda|. \end{cases}$$

We will show that η_ω is bounded on D. We have

$$\frac{1}{1 - q_\omega(h, \theta)} \leqq \frac{2|1 - \theta\lambda h\omega|^2}{|1 - \theta\lambda h\omega|^2 - |1 + (1 - \theta)\lambda h\omega|^2},$$

hence,

$$n_\omega(h, \theta) \leqq \begin{cases} \dfrac{2|\lambda|(1 - 2\theta a h\omega + \theta^2|\lambda|^2 h^2\omega^2)}{-2a\omega + |\lambda|^2 h\omega^2(2\theta - 1)}, & h \leqq 1/|\lambda|, \\ \dfrac{2(1 - 2\theta a h\omega + \theta^2|\lambda|^2 h^2\omega^2)}{-2ah\omega + |\lambda|^2 h^2\omega^2(2\theta - 1)}, & h > 1/|\lambda|. \end{cases}$$

Here, $a = \operatorname{Re}(\lambda)$. Next we define θ^* by $|\lambda h_{\theta^*}| = 1$, i.e.

$$\theta^* = 1 + 2\operatorname{Re}(\lambda)/(|\lambda|(\omega + \Omega)).$$

We may assume without loss of generality that $\omega + \Omega > 4$, hence $\theta^* > \frac{1}{2}$. We consider the following cases:

1. $\theta \in [0, \frac{1}{2}]$. Then there exists a constant $M_1 \geqq 0$ such that

$$\eta_\omega(h, \theta) \leqq \frac{2|\lambda|(1 - 2\theta a h_\theta\omega + \theta^2|\lambda|^2 h^2\omega^2)}{-2a\omega + |\lambda|^2 h_\theta\omega^2(2\theta - 1)} \leqq -M_1(|\lambda|/a).$$

2. $\theta \in (\frac{1}{2}, 1]$ and $h \in (0, \min\{h_\theta, 1/|\lambda|\}]$. Then there exists $M_2 \geqq 0$ such that

$$\eta_\omega(h, \theta) \leqq \frac{2|\lambda|(1 - 2a\omega/|\lambda| + \omega^2)}{-2a\omega + |\lambda|^2 h\omega^2(2\theta - 1)} \leqq -M_2(|\lambda|/a).$$

3. $\theta \in [\theta^*, 1]$ and $h \in (1/|\lambda|, h_\theta]$. Then

$$\eta_\omega(h, \theta) \leqq \frac{2}{-2ah\omega + |\lambda|^2 h^2\omega^2(2\theta - 1)} + \frac{-4a\omega}{-2a\omega + |\lambda|^2 h\omega^2(2\theta - 1)} +$$
$$+ \frac{|\lambda|^2\omega^2 h}{-2a\omega + |\lambda|^2 h\omega^2(2\theta - 1)} \leqq -(|\lambda|/a\omega) + 2 + (1/(2\theta^* - 1)).$$

However, we have $(1/(2\theta^* - 1)) \leqq (\omega + \Omega)/(\omega + \Omega - 4)$, hence

$$\eta_\omega(h, \theta) \leqq -M_3(|\lambda|/a) + M_4$$

for some nonnegative constants M_3 and M_4.

Combining all these inequalities and taking into account (8) we immediately see that there exists a constant $M \geqq 0$ independent of h, θ, and λ such that

$$|y_n| \leqq M(1 - |\lambda|/\operatorname{Re}(\lambda)),$$

$n = 0, 1, \ldots$. Thus the theorem is proved.

Remark. It follows from the proof of this theorem that the approximate solution $\{y_n\}_{n=0}^{\infty}$ given by (5) for $\theta = 1$ is bounded for any $h > 0$. This property is similar to the A-stability property of numerical methods for ordinary differential equations.

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