Štefan Znám

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VECTOR-COVERING SYSTEMS WITH A SINGLE TRIPLE OF EQUAL MODULI

ŠTEFAN ZNÁM, Bratislava

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This is a continuation of the paper [7], where the notion of the vector-covering system of congruences was introduced and studied. We shall prove here a theorem on such vector-covering systems in which there exists exactly one triple or exactly two couples of equal moduli (the remaining being distinct), presenting a generalization of results concerning disjoint covering systems in [4], [5] and [6].

I. PRELIMINARY FACTS

In our considerations we shall need some properties of the roots of unity. The n-th roots of unity are the numbers

$$\exp \frac{2\pi i}{n} s , \quad s = 1, 2, \ldots, n .$$

Consider the equality

$$(1) \qquad \sum_{t=1}^{r} a_t \zeta_t = 0 ,$$

where a_t are rational integers and ζ_t are roots of unity. The left-hand side of (1) is said to be irreducible if none of its proper subsums is equal to 0. H. B. Mann in [2] proved the following very strong theorem:

If the left-hand side of (1) is irreducible, then there exist such primes $p_1 < p_2 < \ldots \ldots < p_s \leqq r$ and such $p_1 p_2 \ldots p_s$-th roots of unity η_t $(t = 1, 2, \ldots, r)$ that

$$\zeta_t = \alpha \eta_t ,$$

where α is a fixed number (if, moreover, all a_t's are positive and we cannot choose $p_s < r$, then we have $s = 1$ and the numbers a_t coincide).

Now we shall give the definition of the vector-covering systems (see [7]). Let Z be the set of all integers; $a, n \in Z$ with $0 \leqq a < n$. The set $a(n)$ of all numbers of the form $a + sn$, $s \in Z$ will be called a congruence (in [7], an arithmetic sequence) with modulus n. Let a vector

$$\varepsilon = (v_1, v_2, \ldots, v_m)$$

with real v_i's be given. A system of congruences

$$a_j(n_j)\,, \quad j = 1, 2, \ldots, m\,, \quad n_1 \leqq n_2 \leqq \ldots \leqq n_m \tag{2}$$

is called an ε-covering (or vector-covering) if for all $r \in Z$ we have

$$\sum_{j=1}^{m} v_j f_j(r) = 1\,,$$

where f_j is the characteristic function of $a_j(n_j)$.

(If $v_1 = v_2 = \ldots = v_m = 1$, then we get the well-known notion of a disjoint covering system introduced in [1] and studied for example in [3–6].)

We have to recall some properties of vector-covering systems from [7]:

1 ([7], Corollary 1 of Theorem 2). For an ε-covering system (2) we have

$$\sum_{j=1}^{m} \frac{v_j}{n_j} = 1\,.$$

2 ([7], (7)). If (2) is an ε-covering system, then for all $j = 1, 2, \ldots, m$ we have

$$\sum_{\substack{t=1 \\ n_j | sn_t}}^{m} \frac{v_t}{n_t} \exp\{2\pi i s a_t/n_j\} = \begin{cases} 0 & \text{if} \quad s = 1, \ldots, n_j - 1\,, \\ 1 & \text{of} \quad s = n_j \end{cases}. \tag{3}$$

(this is proved in [3] for disjoint covering systems.)

3 ([7], Corollary of Theorem 4). In every vector-covering system with $v_m \neq 0$ we have $n_{m-1} = n_m$.

The ε-covering system (2) is said to be of type 1 if

$$n_1 < n_2 < \ldots < n_{m-2} < n_{m-1} = n_m\,.$$

4 ([7], Theorem 6). Every $(v_1, \ldots, v_m)$-covering system of type 1 with all $v_j \neq 0$ is a disjoint covering system with the moduli

$$n_j = 2^j \quad \text{for} \quad j = 1, 2, \ldots, m - 2\,, \quad n_{m-1} = n_m = 2^{m-1}$$

(which is a generalization of Stein's theorem concerning the disjoint covering systems in [5]).

In [6] the author solved the case when in a disjoint covering system there exists exactly one triple of equal moduli (the remaining being distinct) as follows: the moduli are uniquely determined and

$$n_j = 2^j \quad \text{for} \quad j = 1, 2, \ldots, m - 3\,, \quad n_{m-2} = n_{m-1} = n_m = 3.2^{m-3}\,.$$

Our article aims at generalizing the last result for vector-covering systems of congruences.

Remark. We have to mention here that for disjoint covering systems Š. Porubský in [4] solved the cases when there exist exactly 4, 5, 6 or 7 equal moduli (the remaining being distinct).

Definition 1. A non-empty system (2) is called *ε-vanishing* if for all $r \in Z$ we have

$$\sum_{j=1}^{m} v_j f_j(r) = 0 .$$

By a very similar argument as in the proof of (7) in [7] we get:

Lemma 1. *If (2) is a $(v_1, \ldots, v_m)$-vanishing system, then for all $j = 1, 2, \ldots, m$ and $s = 1, 2, \ldots, n_j$ we have*

$$\sum_{\substack{t=1 \\ n_j | s n_t}}^{m} \frac{v_t}{n_t} \exp\{2\pi i s a_t / n_j\} = 0 . \tag{4}$$

Lemma 2. *If (2) is $(v_1, \ldots, v_m)$-vanishing and $v_j \neq 0$, then we have $n_{m-1} = n_m$.*

Proof. Put $j = m$, $s = 1$ in (4).

Lemma 3. *Let (2) be a $(v_1, \ldots, v_m)$-vanishing system with all $v_j \neq 0$ and $n_1 < < n_2 < \ldots < n_{m-2} < n_{m-1} = n_m$, then for all $j = 1, 2, \ldots, m$ we have $n_j \mid n_m$.*

Proof. First of all we prove that to every n_j there exists such n_w that $n_j \mid n_w$. Indeed, if this were not the case, then putting $s = 1$ in (4) we should have

$$\frac{v_j}{n_j} \exp\{2\pi i a_j / n_j\} = 0 ,$$

which is impossible. Now, if $w = m$, the proof is finished. If this is not the case, then repeat the same argument for w instead of j, and so on.

II. THE MAIN RESULTS

In what follows we shall always suppose that the v_j's are rational and all different from 0.

Definition 2. A $(v_1, \ldots, v_m)$-covering system (2) with distinct congruence is called *reduced* if it does not contain any non-empty vanishing subsystem.

Definition 3. A reduced vector-covering system (2) is said to be *of type* 2 if there exists in it one triple of equal moduli and the others are distinct, hence (see I.3) we have

$$n_1 < n_2 < \ldots < n_{m-3} < n_{m-2} = n_{m-1} = n_m . \tag{5}$$

Definition 4. A reduced vector-covering system (2) is *of type* 3 if there exist in it two couples of equal moduli, the others being distinct; hence (see I.3) we have

$$n_1 < n_2 < \ldots < n_{s-1} = n_s < \ldots < n_{m-2} < n_{m-1} = n_m . \tag{6}$$

Now we shall generalize the results of Theorem 3 of [6]:

Theorem. *Let* (2) *be a vector-covering system of type* 2 *or* 3. *Then all its moduli are of the form*

$$n_j = 3^a 2^b \,, \tag{7}$$

where $a = 0$ *or* 1 *and* b *is a nonnegative integer.*

Proof. Obviously we have $m \geqq 3$ and for $m = 3$ the assertion holds.

We shall prove by induction: Suppose that $m > 3$ and that the assertion of Theorem is true for all natural numbers smaller than m. We shall distinguish two cases:

A. Let (2) be of type 2. Then putting $j = k$, $s = 1$ in (3) we get

$$v_{m-2} \exp\{2\pi i a_{m-2}/n_m\} + v_{m-1} \exp\{2\pi i a_{m-1}/n_m\} + \\ + v_m \exp\{2\pi i a_m/n_m\} = 0 \,. \tag{8}$$

Because the v_j's are rational, (8) is of the form (1). The only primes not exceeding $r = 3$ are 2 and 3, thus according to Mann's theorem, there exist such 6-th roots of unity η_t that

$$\zeta_t = \exp\{2\pi i a_t/n_m\} = \alpha\eta_t \,, \quad t = m-2,\, m-1,\, m$$

and α is fixed. Therefore the equality (8) can be rewritten as follows:

$$v_{m-2}\eta_{m-2} + v_{m-1}\eta_{m-1} + v_m\eta_m = 0 \,,$$

where the numbers η_t are from the set

$$\{\exp \pi i/3,\ \exp 2\pi i/3,\ -1,\ \exp 4\pi i/3,\ \exp 5\pi i/3,\ 1\} \,.$$

Considering these numbers η_t as vectors in the plane and using elementary geometrical considerations we can easily show that their linear combination with real coefficients may be 0 only in the following cases:

a) The vectors η_t are edges of a directed equilateral triangle.

b) They are edges of an equilateral triangle in which one edge is directed oppositely to the others.

c) All η_t's are parallel. However, then at least two of them are equal, which is impossible, because we have supposed that (2) is a reduced system.

First, consider the case a). Here also the vectors ζ_t are edges of an equilateral triangle; hence possibly after a suitable permutation of numbers a_{m-2}, a_{m-1}, a_m we get

$$\frac{2\pi i}{n_m} a_{m-2} = \frac{2\pi i}{n_m} a_m + \frac{2\pi i}{3} \,, \quad \text{thus} \quad a_{m-2} = a_m + n_m/3 \,;$$

$$\frac{2\pi i}{n_m} a_{m-1} = \frac{2\pi i}{n_m} a_m + \frac{4\pi i}{3} \,, \quad \text{thus} \quad a_{m-1} = a_m + 2n_m/3 \,.$$

Therefore, the last three congruences in (1) are:

(9) $$a_m + n_m/3(n_m),\ a_m + 2n_m/3\,(n_m),\ a_m(n_m)\,.$$

The above mentioned elementary geometrical considerations imply that $v_{m-2} = v_{m-1} = v_m$. Therefore, replacing the congruences (9) in (2) by a single congruence $a_m(n_m/3)$ we get a $(v_1, \ldots, v_{m-2})$-covering system.

This new system is, due to I.3, of type 1, hence due to I.4 it is a disjoint covering system with moduli of the form 2^c, c natural. Thus, the moduli of (2) must be of the form (7) and the assertion of Theorem follows (for this case). (Remark. In this case (2) is a disjoint covering system.)

Now take the more complicated case b). We have $|v_{m-2}| = |v_{m-1}| = |v_m|$, however, one of these numbers – say v_m – has the sign opposite to the others.

Join to (1) two equal congruences

$$a_{m+1}(n_{m+1}) = a_{m+2}(n_{m+2}) = a_m + n_m/2(n_m)$$

and put $v_{m+1} = v_m$, $v_{m+2} = -v_m$. Obviously

$$a_j(n_j) \quad j = 1, 2, \ldots, m, m+1, m+2$$

is a $(v_1, \ldots, v_m, v_{m+1}, v_{m+2})$-covering system. Due to

$$\exp\{2\pi i(a_m + n_m/2)/n_m\} = -\exp\{2\pi i a_m/n_m\}$$

(8) yields the equation

$$v_{m-2}\exp\{2\pi i a_{m-2}/n_m\} + v_{m-1}\exp\{2\pi i a_{m-1}/n_m + v_{m+2}\exp\{2\pi i a_{m+2}/n_m\} = 0\,.$$

The vectors appearing here are edges of a directed equilateral triangle (and $v_{m-2} = v_{m-1} = v_{m+2}$), hence by the same considerations as in the case a) we get that the congruences $a_{m-2}(n_{m-2})$, $a_{m-1}(n_{m-1})$ and $a_{m+2}(n_{m+2})$ can be replaced by a single congruence $a_{m-2}(n_m/3)$. By a similar argument it can be shown that the congruences $a_m(n_m)$ and $a_{m+1}(n_{m+1})$ can be replaced by a single congruence $a_m(n_m/2)$. In this way we get a new $(v_1, \ldots, v_{m-3}, v_{m-2}, v_m)$-covering system

(10) $$a_1(n_1), \ldots, a_{m-3}(n_{m-3}), a_{m-2}(n_m/3), a_m(n_m/2)\,.$$

Because (2) is of type 2, (10) contains at most two couples of equal moduli and consists of $m - 1$ congruences. Now, if (10) is reduced, then owing to the induction hypothesis its moduli as well as the moduli of (2) are of the form (7).

Suppose (10) is not reduced. According to Lemma 2 and I.3 a vanishing subsystem of (10) contains exactly one of the last two congruences, hence (due to Lemma 3) all its moduli are divisors of $n_m/2$ or $n_m/3$. The remaining congruences form a system of type 1, thus its moduli are of the form 2^c; hence also one of the numbers $n_m/2$, $n_m/3$ is of the form 2^c and therefore all the moduli of (10) are of the form (7).

If (10) contains no vanishing subsystem but contains two equal congruences – say $a_h(n_h) = a_k(n_k)$, $h < k$ – then by deleting the latter we get a $(v_1, \ldots, v_{h-1}, v_h + v_k,$

$v_{h+1}, \ldots, v_{k-1}, v_{k+1}, \ldots, v_m$)-covering system, which due to I.3 is of type 1 ($v_h + v_k \neq 0$ since we have supposed that (10) does not contain a vanishing subsystem), hence its moduli are (see I.4) of the form 2^c and the assertion of Theorem follows for the case A.

B. If (2) is of type 3, then using similar considerations as in part A (see also the proof of Theorem 6 in [7]) we can prove that

$$v_{m-1} = v_m, \; a_{m-1} = a_m + n_m/2 \quad (\text{or } a_{m-1} = a_m - n_m/2)\,.$$

Thus, replacing $a_{m-1}(n_{m-1})$, $a_m(n_m)$ in (2) by a single congruence $a_{m-1}(n_m/2)$ we get a $(v_1, \ldots, v_{m-1})$-covering system. If this new system is reduced, then it is of type 1, 2 or 3 and hence according to the induction hypothesis and I.4 all its moduli as well as the moduli of (2) are of the form (7).

Obviously this new system cannot contain any vanishing subsystem, because then (2) would, too. If it contains equal congruences, then proceed as in part A.

The proof of Theorem is complete.

Remark. Not every vector-covering system of type 2 is a disjoint covering system. For example, the system $1(2), 1(3), 0(6), 1(6), 2(6)$ is a $(1, 1, 1, -1, 1)$-covering system of type 2. Obviously, the same can be said on systems of type 3.

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Author's address: 842 15 Bratislava, Mlynská dolina, ČSSR (Matematicko-fyzikálna fakulta Univerzity Komenského).