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A NOTE ON THE STRONGLY COMPATIBLE TOLERANCES ON AN ARBITRARY SEMIGROUP

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In an earlier paper [3] the author proved that in a regular semigroup a strongly compatible tolerance is an idempotent separating congruence if it is contained in Green's relation $\mathscr{H}$. However, in an aribitrary semigroup it is not so. In this note we obtain sufficient conditions for a strongly compatible tolerance to be an idempotent separating congruence on a semigroup in general.

A reflexive and symmetric relation on a semigroup S is called a *tolerance on* S [4]. If ξ is a tolerance on S and if $(a, b), (c, d) \in \xi$ implies that $(ac, bd) \in \xi$ then ξ is called a *strongly compatible tolerance on* S. A non empty subset B of S is called a *block* [1] *of* ξ provided (i) $B \times B \subseteq \xi$ (ii) B is a maximal subset of S with respect to (i), that is, if $B \subseteq C$ and $C \times C \subseteq \xi$ then $B = C$.

For undefined terms and notions the reader is referred to [2].

Let ξ be a strongly compatible tolerance on a semigroup S. We observe that the condition "$\xi \subseteq \mathscr{H}$" is neither necessary nor sufficient for ξ to be an idempotent separating congruence on S.

This is illustrated by the following examples.

Example 1 [2]. Let $S = \{x, o\}$ be a null semigroup. Here $\mathscr{H} = 1_S$ and $S \times S$ is an idempotent separating congruence. Thus it is not necessary for any congruence, in particular a strongly compatible tolerance, to be contained in $\mathscr{H}$ in order that it should be an idempotent separating congruence.

Example 2. $S = \{a, b, c, d, e, f, g\}$ is a semigroup with the multiplication table given below:

	a	b	c	d	e	f	g
a	a	a	a	d	d	d	d
b	a	b	c	d	d	d	d
c	a	c	b	d	d	d	d
d	d	d	d	a	a	a	a
e	d	e	e	a	a	a	a
f	d	d	d	a	a	a	a
g	d	d	d	a	a	a	a

$\mathscr{H}$-classes of S are $\{a, d, f, g\}$, $\{b, c\}$ and $\{e\}$. Let ξ be a relation defined on S such that $\xi = \{(a, a), (a, d), (d, a)\ (d, d), (d, f), (d, g), (f, d), (f, f), (f, g), (g, d), (g, f), (g, g), (b, b), (c, c), (e, e)\}$. It is easy to see that ξ is contained in $\mathscr{H}$ and that it is a strongly compatible tolerance on S. However, ξ is not a congruence since (a, d), $(d, f) \in \xi$ but $(a, f) \notin \xi$. This shows that "$\xi \subseteq \mathscr{H}$" is not a sufficient condition for a strongly compatible tolerance to be an idempotent separating congruence on an arbitrary semigroup.

In the following theorem we obtain sufficient conditions for a strongly compatible tolerance to be an idempotent separating congruence on an arbitrary semigroup.

Theorem. *Let S be an arbitrary semigroup. A strongly compatible tolerance ξ on S is an idempotent separating congruence on S if* (i) $\xi \subseteq \mathscr{H}$ *and* (ii) *for the blocks $\{B_i\}_{i \in I}$ of ξ either $B_i \cap B_j = \emptyset$ $(i \neq j)$, or $B_i \cap B_j$ $(i \neq j)$ contains an idempotent.*

Proof. The conditions $\xi \subseteq \mathscr{H}$ and $B_i \cap B_j = \emptyset$ $(i \neq j)$ imply that the blocks form a partition of S and hence ξ is an idempotent separating congruence. Alternatively if $\xi \subseteq \mathscr{H}$ and $B_i \cap B_j$ contains an idempotent then H, the $\mathscr{H}$-class containing B_i and B_j, contains an idempotent and hence a subgroup of S ([2] Theorem 2.5).

If $a, b \in B_i$ and $b, c \in B_j$ we have $(a, b) \in \xi$ and $(b, c) \in \xi$. Now $(a, b) \in \xi$, $(\bar{b}, \bar{b}) \in \xi$ (where $\bar{b}$ is the group inverse of b in H) imply $(a\bar{b}, \bar{b}\bar{b}) \in \xi$; $(a\bar{b}, b\bar{b}) \in \xi$ and $(b, c) \in \xi$ together imply $(a\bar{b}b, b\bar{b}c) \in \xi$, that is $(a, c) \in \xi$, $\bar{b}b$ and $b\bar{b}$ being equal to the group identity in H. ξ is therefore a congruence on S. "$\xi \subseteq \mathscr{H}$" implies that ξ is an idempotential separating congruence on S.

Example 2 shows that condition (i) of the above theorem alone is not sufficient for a strongly compatible tolerance to be an idempotent separating congruence on an arbitrary semigroup.

The example which follows illustrates that condition (i) is an essential condition for a strongly compatible tolerance to be an idempotent separating congruence on an arbitrary semigroup. It further shows that condition (ii) of the theorem by itself is not sufficient for a strongly compatible tolerance on an arbitrary semigroup to be an idempotent separating congruence.

Example 3. Let $S = \{x, y, o\}$ be a null semigroup; clearly $\mathscr{H} = 1_S$ on S. Let a relation ξ on S be defined as $\xi = \{(x, x), (x, o), (o, x), (o, o), (y, y), (y, o), (o, y)\}$. Obviously ξ is a strongly compatible tolerance on S which is not contained in $\mathscr{H}$. Clearly (x, o), $(o, y) \in \xi$ but $(x, y) \notin \xi$. Thus ξ is not a congruence. However, the two blocks of ξ, i.e. $\{x, o\}$, $\{o, y\}$ have an idempotent in their intersection.

We note that in the case of regular semigroups condition (ii) is implied by condition (1) of the above theorem.

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