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AMALGAMATION OVER UNIFORM MATROIDS

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INTRODUCTION

In this paper we study glueing two matroids together. It will be assumed that the reader is familiar with elementary concepts of the matroid theory. An appropriate reference is the book by Welsh [14]. In drawing matroids we follow [5].

Let $M_1(X_1)$ and $M_2(X_2)$ be matroids on sets X_1 and X_2 respectively, and let the restrictions of M_1 and M_2 to $X_1 \cap X_2$ be the same. A matroid $M(X_1 \cup X_2)$ is called an *amalgam of* M_1 *and* M_2, if M restricted to X_i is the matroid M_i for $i = 1, 2$.

Constructions related to amalgamation have been studied by many authors [1], [2], [3], [4], [9]. Geometrical description of amalgamation is given in [6]. A particular type of amalgamation was used by Seymour [12] when giving a characterization of regular matroids. Amalgamation of a greater number of matroids was studied in [7], [10].

The paper is divided into five parts. Section 1 contains basic definitions and examples of amalgams. In Section 2 we give some examples of pairs of matroids for which no amalgam exists. Relations between amalgamation and other matroid operations (as restriction, contraction, Dilworth truncation) are discussed in Section 3. In Section 4 we introduce sticky and f-sticky matroids defined in [11]. The main result is the characterization of sticky uniform matroids. The last section deals with representable matroids.

The amalgamation is closely related to the construction of Ramsey matroids. The vertex partition property for matroids was proved in [7]. The results of this paper are used in [8], where the Ramsey property for the partition of two-point lines is proved. The paper contains some results first stated in [13].

SECTION 1

Let $M(X)$ be a matroid and Y a subset of X. The closure of the set Y in M will be denoted by $\overline{Y}^M$ or $\overline{Y}$. The restriction of M to Y is the matroid $M \mid Y$ on the set Y with the rank function $r(A) = r_M(A)$ for $A \subset Y$. The contraction of M to Y is the matroid

$M . Y$ on the set Y with the rank function

$$r(A) = r_M(A \cup (X - Y)) - r_M(X - Y) \quad \text{for} \quad A \subset Y.$$

Let $M(X), N(Y)$ be matroids. An injective mapping $f : X \to Y$ is called an *embedding* if $r_M(A) = r_N(f(A))$ for every $A \subset X$.

Definition. Let $M_1(X_1)$, $M_2(X_2)$ and $M(X)$ be matroids and let $f_i : X \to X_i$ be an embedding for $i = 1, 2$. A matroid $N(Z)$ is called an *amalgam of M_1 and M_2 over M with respect to f_1 and f_2* if for $i = 1, 2$ there exist embeddings $g_i : X_i \to Z$ satisfying

(i) $g_1(X_1) \cup g_2(X_2) = Z$,

(ii) $g_1 f_1 = g_2 f_2$,

(iii) $|Z| = |X_1| + |X_2| - |X|$.

In particular, when all embeddings f_1, f_2, g_1 and g_2 are identity mappings (inclusions), the above definition converts into that given in Introduction.

In general, an amalgam needs neither be unique nor exist.

Example. A prism P_6 is a matroid of rank 4 given by three nontrivial hyperplanes $\{1, 2, 3, 4\}$, $\{1, 2, 5, 6\}$ and $\{3, 4, 5, 6\}$. (See Fig. 1).

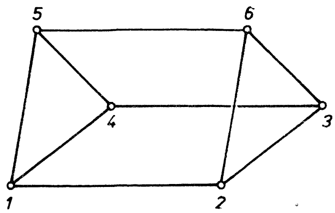

Fig. 1

Let $M_1(\{1, 2, 3, 4, 5, 6\})$ and $M_2(\{3, 4, 5, 6, 7, 8\})$ be two copies of the prism P_6 and let $M(\{3, 4, 5, 6\})$ be the common plane. There is a lot of amalgams of M_1 and M_2. Three of them are useful examples [5].

Tab. 1

	3-Flats with more than 3 points	4-Flats with more than 4 points	5-Flats with more then 5 points
V_4 Vamos rank 4	1234, 1256 3456, 5678, 3478	12345678	
V_4^+ rank 4	same as V_4 with 1278	12345678	
V_5 rank 5	same as V_4^+	123456 123478 125678 345678	12345678

In the sequel we shall use the following known fact.

Lemma. *If V is an amalgam of M_1 and M_2 and $r(V) = 5$ then $r_V(\{1, 2, 7, 8\}) = 3$.* ■

SECTION 2

In this section we give three examples in which no amalgam exists.

Example 1. Let $M_1(\{a, b, c, d, e, g, f\})$ and $M_2(\{a', b, c, d, e, f, g\})$ be matroids of rank 3 given by Fig. 2. Clearly $M_1 \mid \{b, c, d, e, f, g\} = M_2 \mid \{b, c, d, e, f, g\}$. Assume that M is an amalgam of M_1 and M_2. Denote by r the rank function of M. As $a, a' \in \overline{\{b, e\}} \cap \overline{\{c, f\}}$ we have $r(\{a, a'\}) = 1$ which contradicts $r(\{a, d, g\}) = 2$ and $r(\{a', d, g\}) = 3$.

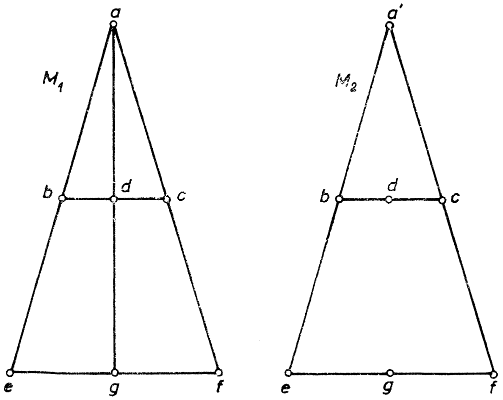

Fig. 2

Example 2. Let $M_1(\{a, b, c, x_1, x_2, x_3, x_4\})$ and $M_2(\{a', b', c', x_1, x_2, x_3, x_4\})$be matroids of rank 3 given by Fig. 3. (The matroid M_1 is the Fano matroid and M_2 is the Fano matroid without the line $\{a', b', c'\}$.) Clearly $M_1 \mid \{x_1, x_2, x_3, x_4\} = M_2 \mid \{x_1, x_2, x_3, x_4\} = U_4^3$ (uniform matroid). In any amalgam of M_1 and M_2

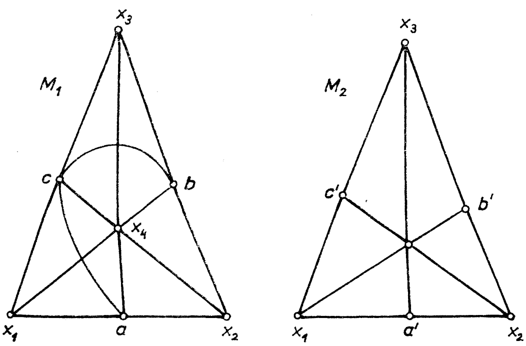

Fig. 3

with a rank function r the identities $r(a, a') = r(b, b') = r(c, c') = 1$ must hold, whereas $r(a, b, c) = 2$ and $r(a', b', c') = 3$.

Example 3. Let $M_1(\{1, 2, \ldots, 8\})$ be the matroid V_5 and let $M_2(\{3, 4, 5, 6, 9, 10, 11, 12\})$ be the Vamos matroid V_4 (see Fig. 4).

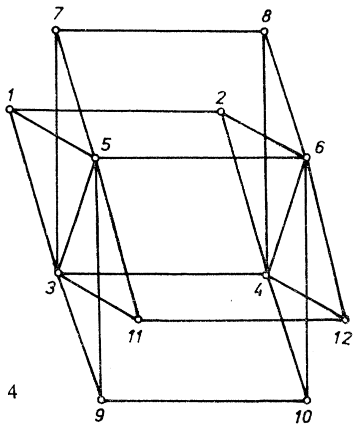

Fig. 4

Clearly $M_1 \mid \{3, 4, 5, 6\} = M_2 \mid \{3, 4, 5, 6\} = U_4^3$, and moreover, $\{3, 4, 5, 6\}$ is a flat of both M_1 and M_2. We prove that there is no amalgam of M_1 and M_2. Assume that r is the rank function of an amalgam of M_1 and M_2. Denote $a = \{1, 2\}$, $b = \{7, 8\}$, $c = \{9, 10\}$, $d = \{11, 12\}$, $e = \{3, 4, 5, 6\}$, $X = \{1, 2, \ldots, 12\}$. We distinguish three cases and show in each of them that $r(X) \leqq 4$, which contradicts $r(M_1) = 5$.

Case 1. $(r(a \cup c) = 4 \vee r(a \cup d) = 4) \,\&\, (r(b \cup c) = 4 \vee r(b \cup d) = 4)$. Suppose that e.g. $r(a \cup c) = 4$ and $r(b \cup c) = 4$. It follows from Lemma that $\overline{a \cup c} \supset \overline{a \cup c \cup e} \supset \overline{a \cup c \cup e \cup d}$ and $\overline{b \cup d} \supset \overline{b \cup d \cup e} \supset \overline{b \cup d \cup e \cup c}$. Hence $r(X) \leqq r(\overline{a \cup c}) + r(\overline{b \cup d}) - r(\overline{a \cup c} \cap \overline{b \cup d}) \leqq 4 + 4 - r(e \cup c \cup d) = 4$.

Case 2. $((r(a \cup c) = 4 \vee r(a \cup d) = 4) \,\&\, (r(b \cup c) \leqq 3 \,\&\, r(b \cup d) \leqq 3)) \vee ((r(a \cup c) \leqq 3 \,\&\, r(a \cup d) \leqq 3) \,\&\, (r(b \cup c) = 4 \vee r(b \cup d) = 4))$. Suppose that e.g. $r(a \cup c) = 4 \,\&\, r(b \cup c) \leqq 3 \,\&\, r(b \cup d) \leqq 3$. Then $r(a \cup b \cup c) \leqq r(a \cup b) + r(b \cup c) - r(b) = 4$. But it follows from Lemma that $\overline{a \cup b \cup c} \supset \overline{a \cup b \cup c \cup e} \supset X$.

Case 3. $r(a \cup c) \leqq 3 \,\&\, r(a \cup d) \leqq 3 \,\&\, r(b \cup c) \leqq 3 \,\&\, r(b \cup d) \leqq 3$. Then $r(a \cup c \cup d) \leqq r(a \cup c) + r(a \cup d) - r(a) \leqq 4$ and $r(b \cup c \cup d) \leqq r(b \cup c) + r(b \cup d) - r(b) \leqq 4$. This gives $r(a \cup b \cup c \cup d) \leqq r(a \cup d \cup c) + r(b \cup c \cup d) - r(c \cup d) \leqq 4$. But it follows from the definition of M_2 that $\overline{a \cup b \cup c \cup d} \supset X$. ■

Throughout this section let $M_1(X_1)$, $M_2(X_2)$ be matroids and $M(X_1 \cup X_2)$ their amalgam.

Proposition 1. *Let $A \subset X_1 \cup X_2$. Then $M \mid A$ is an amalgam of $M_1 \mid A$ and $M_2 \mid A$.* ∎

Proposition 2. *Let $Y \subset X_1 \cap X_2$. Then $M . ((X_1 \cup X_2) - Y)$ is an amalgam of $M_1 . (X_1 - Y)$ and $M_2 . (X_2 - Y)$.*

Proof. Set $A = (X_1 \cup X_2) - Y$. We show $(M . A) \mid (X_i - Y) = M_i . (X_i - Y)$ for $i = 1, 2$. Let $K \subset X_i - Y$. Then
K is independent in $(M . A) \mid (X_i - Y)$ iff
K is independent in $M . A$ iff
K is independent in $M_i . (X_i - Y)$. ∎

If M is a matroid with the rank function r and k is an integer, then M^k is a matroid with the rank function $\min(r, k)$.

Proposition 3. *M^k is an amalgam of M_1^k and M_2^k for any integer k.* ∎

If $M(X)$ is a matroid and $f: X \to Y$ is a mapping, then $f(M)$ is a matroid on the set Y such that $B \subset Y$ is independent in $f(M)$ iff there is an $A \subset X$ independent in M and $f(A) = B$.

Proposition 4. *Let $f: X_1 \cup X_2 \to Y$ be a mapping such that $f(X_1 \cup X_2) = Y$ and let the sets $f(X_1 \cap X_2)$, $f(X_1 - X_2)$ and $f(X_2 - X_1)$ be pairwise disjoint. Then the matroid $f(M)$ is an amalgam of matroids $f(M_1)$ and $f(M_2)$.*

Proof. We show $f(M) \mid f(X_i) = f(M_i)$ for $i = 1, 2$. Let $B \subset f(X_i)$, then B is independent in $f(M) \mid f(X_i)$ if and only if there is $A \subset X_i$, A independent in M and $B = f(A)$. This is equivalent to B being independent in $f(M_i)$. ∎

The Dilworth truncation M^D of a matroid $M(X)$ is the matroid on the set $\{F \mid F$ is a flat of M with rank $\geqq 1\}$ given by $\{F_1, \ldots, F_t\}$ is independent in M^D if for any $J \subset \{1, \ldots, t\}$

$$r(\bigcup_J F_j) \geqq |J| + \min\{r(F_j) \mid j \in J\} - 1 .$$

If $Y \subset X$ we can identify the vertices of $(M \mid Y)^D$ with the vertices of M^D, namely a flat F of $M \mid Y$ is identified with the flat $\bar{F}$ of M. From this point of view the matroid $(M \mid Y)^D$ is a restriction of M^D to the set $\{F \mid F$ is a flat of $M \mid Y$, rank $F \geqq 1\}$. These considerations immediately yield

Proposition 5. *Set $\mathscr{A} = \{F \mid F$ is a flat of M_1 or $M_2\}$. Then $M^D \mid \mathscr{A}$ is an amalgam of M_1^D and M_2^D.* ∎

A matroid $M(X)$ is called an *extension of a matroid* $N(Y)$, $Y \subset X$ if $M \mid Y = N$.

We define a matroid $M(X)$ to be sticky if there exists an amalgam of any two extensions $M_1(X_1)$ and $M_2(X_2)$ of M, with $X_1 \cap X_2 = X$.

We define a matroid $M(X)$ to be f-sticky if there exists an amalgam of any two extensions $M_1(X_1)$, $M_2(X_2)$ of M such that X is a flat of both M_1 and M_2, and $X_1 \cap \cap X_2 = X$.

In [11] we conjectured that M is sticky if and only if it is modular, and we proved (i) if M is modular then it is sticky, and (ii) if M is sticky and $r(M) \leqq 3$ then it is modular.

Clearly, (iii) if a matroid M is sticky then it is f-sticky as well, and (iv) if the conjecture is true then both properties are equivalent.

Let $1 \leqq k \leqq n$. A uniform matroid U_n^k is a matroid with vertices $\{1, \ldots, n\}$ and with the rank function

$$r(A) = \min \{|A|, k\} \quad \text{for} \quad A \subset \{1, 2, \ldots, n\} .$$

Theorem. *Let U_n^k be a uniform matroid. Then the following properties are equivalent.*

a) *U_n^k is modular,*

b) *U_n^k is sticky,*

c) *U_n^k is f-sticky,*

d) *$k \leqq 2$ or $k = n$.*

Proof. Clearly (a) is equivalent to (d). The facts mentioned above give (a) $\Rightarrow$ (b) and (b) $\Rightarrow$ (c). We show in Steps 1–5 that no uniform matroid U_n^k with $3 \leqq k < n$ is f-sticky. This gives (c) $\Rightarrow$ (d).

1. Let Y be a flat of $M(X)$ such that $M \mid Y = U_n^k$. We construct a one-point extension $M'(X \cup e)$ of M such that $M' \mid (Y \cup e) = U_{n+1}^k$ and $Y \cup e$ is a flat of M'. The rank function r of M' is given by

$$r(A) = r_M(A), \quad \text{and}$$
$$r(A \cup e) = \begin{cases} r_M(A) & \text{if} \quad Y \subset \bar{A}^M, \\ r_M(A) + 1 & \text{otherwise} \end{cases}$$

for every $A \subset X$.

2. Let $M_1(X_1)$ and $M_2(X_2)$ be matroids, $X_1 \cap X_2 = Y$, $M_1 \mid Y = M_2 \mid Y = U_n^k$, let Y be a flat of both M_1 and M_2. Denote by $M_1'(X_1 \cup e)$ and $M_2'(X_2 \cup e)$ the extensions of M_1 and M_2, respectively, constructed as in Step 1. If there is no amalgam of M_1 and M_2 then there is no amalgam of M_1' and M_2' according to Proposition 1.

3. Let Y be a flat of $M(X)$ such that M . $Y = U_n^k$. We construct a matroid $M'(X \cup e)$

such that $M' . X = M$, $M \mid (Y \cup e) = U_{n+1}^{k+1}$ and $Y \cup e$ is a flat of M'. The rank function r of M' is given by

$$r(A) = \begin{cases} r_M(A) + 1 & \text{if } A \text{ is dependent in } M\,, \\ r_M(A) & \text{if } A \text{ is independent in } M\,, \end{cases}$$

$$r(A \cup e) = r_M(A) + 1$$

for every $A \subset X$.

4. Let $M_1(X_1)$ and $M_2(X_2)$ be matroids, $X_1 \cap X_2 = Y$, let $M_1 \mid Y = M_2 \mid Y = U_n^k$ be a flat of both M_1 and M_2. Denote by $M_1'(X_1 \cup e)$ and $M_2'(X_2 \cup e)$ the extensions of M_1 and M_2, respectively, constructed as in Step 3. If there is no amalgam of M_1 and M_2, then there is also no amalgam of M_1' and M_2' according to Proposition 2.

5. Example 3 proves Theorem for $k = 3$, $n = 4$. Steps 2 and 4 extend the result for every k, n, $3 \leqq k \leqq n$. ∎

SECTION 5.

Proposition 6. *Let $M_1(X_1)$ and $M_2(X_2)$ be binary matroids and $M_1 \mid X_1 \cap X_2 = M_2 \mid X_1 \cap X_2$. Then there is a binary matroid $M(X_1 \cup X_2)$ which is an amalgam of M_1 and M_2.*

The proof follows immediately from the unique representation of binary matroids over $GF(2)$. Namely, if $v_1, \ldots, v_k$ are vectors over $GF(2)$ which form a circuit, then $v_1 + \ldots + v_k = 0$. ∎

On the other hand, it is not very difficult to find examples of matroids which are representable over a field F, $F \neq GF(2)$, but no amalgam of them is representable over F. Moreover, Example 1 shows matroids which are representable over any field with more than three elements but no amalgam of which exists.

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