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SEMIGROUPS AND RINGS WHOSE PROPER ONE-SIDED IDEALS ARE POWER JOINED

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The main result of the present paper consists in a characterization of semigroups whose proper one-sided ideals are power joined (Th. 1.6). A theorem of Pondělíček on uniform semigroups, contained in [3], is found again as a corollary of this result. Moreover, with regard to the fact that every one-sided ideal is a biideal (biideal of a semigroup S is a subsemigroup B of S such that $BSB \subseteq B$), Th. 1.6 provides also an answer to the question put in the Mathematical Reviews (82g; 20097) by the reviewer of the note [3]. The second section of the work contains a characterization of rings whose proper one-sided ideals are multiplicatively power joined semigroups.

1. We start by proving some lemmas which will enable us to establish the main theorem. We remember that by a *power joined semigroup* we mean a semigroup S such that $a^h = b^k$ for every $a, b \in S$ and h, k positive integers (see [4], II.7.7).

Lemma 1.1. *A semigroup S whose proper left (right) ideals of the form Sb (bS) ($b \in S$) are power joined is a semilattice of archimedean semigroups.*

Proof. Let $a, b \in S$ with $a = xby$ ($x, y \in S^1$). Then $a^2 = (xbyx)\, by$ with $z = xbyx \in S$. If $Sb = S$, it follows that $z = wb$ for some $w \in S$, and therefore $a^2 = zby = wb^2y$. If on the contrary, $Sb \subset S$, since Sb is power joined, there are two positive integers h, k such that $(yzb)^h = b^{2k}$, and it results that $a^{2(h+1)} = (zby)^{h+1} = zb(yzb)^h\, y = zb^{2k+1}y$. Thus in any case b^2 divides a power of a, which suffices to conclude that S is a semilattice of archimedean semigroups (see [5], Th. 2.1).

Lemma 1.2. *A non archimedean semigroup S whose proper one-sided ideals are power joined is a semilattice of two semigroups M and $S \setminus M$, where M is power joined and coincides with the greatest ideal of S, and $S \setminus M$ is a group. Moreover, S has an identity, which is the identity of $S \setminus M$.*

Proof. By Lemma 1.1, S is a semilattice of archimedean semigroups, so it is Putcha's Q-semigroup (see [1], Definitions 1.1 and 1.4). Therefore it follows from Corollary 1.5 of [1] that S is a semilattice of two semigroups: M which is power joined and the greatest ideal of S, and $S \setminus M$. It remains to prove that $S \setminus M$ is a group, whose identity is the identity of S. In fact, if L is a proper left ideal of $S \setminus M$, it is immediate to verify that $L \cup M$ is a proper left ideal of S, and therefore a power joined semigroup. But this is a contradiction, since power joined semigroups cannot be disjoint unions of proper subsemigroups. Hence $S \setminus M$ is left simple. In the same way we find that $S \setminus M$ is right simple. Thus $S \setminus M$ is a group. Now, if u is the identity of $S \setminus M$, $Su \subset S$ implies that Su is power joined. Then, for every $x \in M$, there exists a positive integer m such that $(xu)^m = u$, but this implies $u \in M$, a contradiction. Thus $Su = S$, and analogously $uS = S$, which means that u is the identity of S.

Lemma 1.3. *A non simple archimedean semigroup whose proper one-sided ideals are power joined is power joined.*

Proof is immediate.

Lemma 1.4. *A simple semigroup S whose proper one-sided ideals are power-joined has at least an idempotent.*

Proof. The lemma is obvious if S is a group. Otherwise, we may suppose that S is not left simple. Therefore, there exists $a \in S$ such that $Sa \subset S$. Moreover, S being simple, we have $Sa^2S = S$, hence $a = xa^2y$ for some $x, y \in S$. Since Sa is power joined, there are two positive integers h, k such that $(xa)^h = a^{2k}$, whence $a = (xa)\,ay = (xa)^h\,ay^h = a^{2k+1}y^h$. Hence $a^2 = a^{2k-1}a^2y^ha$ with $y^ha \in Sa$. Then there exist two positive integers m, n such that $(y^ha)^m = a^{2n}$, and consequently, $a^2 = (a^{2k-1})^m\,a^2(y^ha)^m = a^{(2k-1)m+2+2n}$. So S has an idempotent.

Lemma 1.5. *A simple semigroup S whose proper one-sided ideals are power-joined is either a group or a left (right) zero-semigroup of two periodic groups.*

Proof. Since a simple semigroup with a unique idempotent is a group, we may suppose that S contains two idempotents e, f with $e \neq f$ (Lemma 1.4). First we remark that $Se \subset S$ implies $e = (fe)^h$ for some positive integer h, since Se is a power joined semigroup containing e and fe. Hence $e = fe$. Analogously $fS \subset S$ implies $f = fe$. Since $e \neq f$, we have either $Se = S$ or $fS = S$. In the same way we find that necessarily either $eS = S$ or $Sf = S$.

If $Se = eS = S$ or $Sf = fS = S$, the semigroup S has an identity. If, on the contrary, $Se = Sf = S$ ($eS = fS = S$), S has two right (left) identities. It is immediate to verify that in the first case S cannot have a third idempotent different from e and f. In the second case, every other idempotent is a right (left) identity. In both cases one of the idempotents of S is primitive (since their number is finite when S has an identity[1]); since they are not comparable in the other case), so S is completely simple, i.e. S is a rectangular band of groups. This leads to a contradiction when S

[1]) S, being simple with order greater than 1, cannot have a zero.

has an identity, so all idempotents of S are right (left) identities. Now, let G_α and G_β be two maximal subgroups of S with $G_\alpha \neq G_\beta$. From the fact that the idempotents of S are right (left) identities it follows that S is a left (right) zero-semigroup of groups, whence $(G_\alpha \cup G_\beta) S \subseteq G_\alpha \cup G_\beta$ $(S(G_\alpha \cup G_\beta) \subseteq G_\alpha \cup G_\beta)$. Then, if $G_\alpha \cup G_\beta \subset S$, $G_\alpha \cup G_\beta$ has to be power joined, a contradiction. Thus $G_\alpha \cup G_\beta = S$ and S is a left (right) zero-semigroup of two groups G_α, G_β. Finally, G_α and G_β are periodic since they are proper right (left) ideals of S and consequently, power joined groups.

Now we are able to state the following result:

Theorem 1.6. *Let S be a semigroup whose proper one-sided ideals are power joined. Then S satisfies one of the following conditions*:

i) *S is power joined,*

ii) *S is a group,*

iii) *S is a left (right) zero-semigroup of two periodic groups,*

iv) *S is a semilattice of two semigroups M and $S \setminus M$, where M is power joined and coincides with the greatest ideal of S, and $S \setminus M$ is a group. Moreover, the identity of $S \setminus M$ is the identity for S.*

Conversely, if S is a semigroup of type i), ii), iii) *or* iv), *every proper one-sided ideal of S is power joined.*

Proof. The first part of the statement follows from Lemmas 1.2, 1.3 and 1.5. The converse is immediate.

We recall that a semigroup S is said to be *uniform* if every two left ideals of S and every two right ideals of S have a non-empty intersection (see [3], p. 331). According to this definition, we may find again Th. 1 of [3] as a corollary of the above Th. 1.6. In fact, when S is uniform, the case iii) of the statement of Th. 1.6 can not occur, since the components of a left (right) zero-semigroup of groups are disjoint right (left) ideals, and the converse is obvious.

Addendum. From the above Th. 1.6 and from Th. 4 of [7] the following Theorem can be immediately deduced: "*S is a semigroup whose proper subsemigroups are power joined if and only if S is either power joined or a band of order two*". This result extends Th. 2 of [8], which has come to the authors' knowledge when the manuscript was already sent to the Editor.

2. In this section we shall prove a theorem for rings analogous to Th. 1.6. In the sequel we shall denote by $(R, \cdot)$ the multiplicative semigroup of a ring R.

Theorem 2.1. *Let R be a ring whose proper one-sided ideals are multiplicatively power joined semigroups. Then R satisfies one of the following conditions*:

i) *R is a nilring,*

ii) *R is a ring with identity and* $(R, \cdot)$ *is a semilattice of two semigroups* M *and* $R \setminus M$, *where* M *is a nilring and coincides with the greatest ideal of* R, *and* $R \setminus M$ *is a group.*

Conversely, if R *is a ring of type* i) *or* ii), *every proper one-sided ideal of* R *is a multiplicatively power joined semigroup.*

Proof. Let R be a ring whose proper one-sided ideals are multiplicatively power joined semigroups. If $(R, \cdot)$ is archimedean, it has to be a nilsemigroup, since it contains the zero, and in this case R is a nilring. Then, let us suppose that $(R, \cdot)$ is not archimedean. We note that every left ideal of $(R, \cdot)$ of the form Rb $(b \in R)$ is an additive subgroup of R, hence it is a left ideal of R. Analogously, every right ideal of $(R, \cdot)$ of the form aR $(a \in R)$ is a right ideal of R. Therefore, by Lemma 1.1, $(R, \cdot)$ is a semilattice of archimedean semigroups. Since $(R, \cdot)$ is not archimedean, it contains a proper prime ideal M [2]) (see [1], Th. 1.3). For every $a, b \in M$, we have $aR \subseteq M$ and $Rb \subseteq M$; hence aR and Rb are proper one-sided ideals of R, so they are multiplicatively power joined semigroups. Since $ab \in aR \cap Rb$, there are three positive integers h, k, l such that $a^{2h} = (ab)^k = b^{2l}$. Thus M is power joined and, containing zero, it has to be a nilsemigroup. Then, let $a \in M$, $c \in R \setminus M$. If $Rc \subset R$, since Rc and aR are power joined and $ac \in aR \cap Rc$, we may conclude as above that $a^{2h} = c^{2l}$ for some positive integers h, l. This is a contradiction, since a and c are in disjoint semigroups. Thus, for every $c \in R \setminus M$ we have $Rc = R$. Analogously we find $cR = R$. This implies that for every $c, d \in R \setminus M$ there exist $x, y \in R$ such that $d = xc = cy$. Since $x \in M$ $(y \in M)$ implies $d \in M$, a contradiction, we have $x, y \in R \setminus M$, which enables us to conclude that $R \setminus M$ is a group. If u is the identity of $R \setminus M$, the relations $Ru = uR = R$ imply that u is the identity of R. Thus it remains to prove that M is the greatest ideal of R. In fact, since $(R, \cdot)$ is a semilattice of archimedean semigroups, such that every element has a power in a subgroup, the set of all nilpotents of R is an ideal of R (see [6], Th. 8), which obviously coincides with M. The maximality of M is guaranteed by the fact that $R \setminus M$ is a group.

The converse is immediate.

Remark 2.2. The class of rings of type ii) in the statement of Th. 2.1 contains all division rings; nevertheless it may be interesting to note that this class is wider than that of division rings. This is proved by the following example. Let R be the set of square matrices of the form

$$\begin{bmatrix} a & b \\ -b & a \end{bmatrix} + h \begin{bmatrix} 1 & \mathrm{i} \\ i & -1 \end{bmatrix}$$

$(a, b$ real numbers; h complex; $\mathrm{i} = \sqrt{-1})$. It is a routine verification to prove that R is a ring with identity with respect to the usual sum and product of matrices. Moreover, it is immediate to show that the subset M of R containing the matrices

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[2]) Here and in [1] "prime ideal" means, following Clifford, an ideal I of a semigroup S such that $S \setminus I$ is a subsemigroup. Such ideals are called "completely prime" by Petrich.

$$h\begin{bmatrix} 1 & i \\ i & -1 \end{bmatrix}$$

is a nilring of order greater than 1 and it is the greatest ideal of R, while the subset $R \setminus M$ is a multiplicative group. So R is a ring of type ii) and it is not a division ring.

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