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ON VANISHING AT INFINITY OF SOLUTIONS OF ORDINARY DIFFERENTIAL EQUATIONS

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INTRODUCTION

In 1952 M. Biernacki [1] proved that the binomial differential equation of the fourth order

$$u^{(4)} = p(t)\, u$$

with a continuously differentiable nonincreasing coefficient $p : [0, +\infty[\rightarrow] -\infty, 0[$ possesses a nonzero solution satisfying the condition

$$\lim_{t \to +\infty} u(t) = 0\,.$$

He also suggested that there exists a two-dimensional subspace of such solutions. Later M. Švec [11] succeeded in proving this hypotheses and it turned out that the conditions of continuous differentiability and monotonicity of the function can be omitted. M. Švec proved existence of a two-dimensional subspace of solutions vanishing at infinity under the only condition that p: $[0, +\infty[\rightarrow]-\infty, 0[$ is a continuous function bounded from above by a negative constant

$$p(t) \leqq -\delta \quad \text{for} \quad t \geqq 0\,.$$

The question of validity of the theorem of Biernacki-Švec type for the differential equation

$$u^{(n)} = p(t)\, u \tag{0.1}$$

in the case of $n \neq 4$ has remained open and the analogous question for the nonhomogeneous equations

$$u^{(n)} = p(t)\, u + q(t) \tag{0.2}$$

and

$$u^{(n)} = \sum_{k=0}^{n-1} p_k(t)\, u^{(k)} + q(t) \tag{0.3}$$

has not been studied even in the case of $n = 4$.

In this paper the above questions are answered which has required the study of the problems of existence of solutions of the above equations under boundary conditions of the following three forms:

(0.4_1)

$$u^{(i)}(a) = c_i \ (i = 0, \ldots, n_0 - 1), \quad \int_a^{+\infty} g_k(t)\, |u^{(k)}(t)|^2 \, dt < +\infty \ (k = 0, \ldots, n_0),$$

(0.4_2)

$$u^{(i)}(a) = c_i \ (i = 0, \ldots, n_0), \quad \int_0^{+\infty} g_k(t)\, |u^{(k)}(t)|^2 \, dt < +\infty \ (k = 0, \ldots, n_0)$$

and

(0.4_3)

$$u^{(i)}(a) = c_i \ (i = 1, \ldots, n_0 - 1), \quad \int_a^{+\infty} g_k(t)\, |u^{(k)}(t)|^2 \, dt < +\infty \ (k = 0, \ldots, n_0)$$

where n_0 is the entire part of the number $\frac{1}{2}n$.

In § 1 some auxiliary statements are given. In §§ 2–4 the theorems on existence and uniqueness of solutions of the problems (0.2), (0.4_m) and (0.3), (0.4_m) $(m = 1, 2, 3)$ are proved and the conditions under which the equations (0.2) and (0.3) have families of solutions vanishing at infinity are established. In § 5 these results are applied to the equation (0.1). In particular, the theorems proved here imply that if p: $[0, +\infty[\to \to R$ is locally integrable and

$$(0.5) \qquad (-1)^{n-n_0-1} t^n \, p(t) \to +\infty \quad \text{for} \quad t \to +\infty ,$$

then the equation (0.1) has an n_0-dimensional subspace of solutions vanishing at infinity. In the case of $n = 4$ this statement implies the above mentioned theorem of M. Švec, but in contrast to his theorem, ours is the best possible since, as one easily verifies, (0.5) cannot be replaced by the condition

$$\liminf_{t \to +\infty} (-1)^{n-n_0-1} t^n \, p(t) > 0 .$$

In this paper we use the following notation.

R is the set of real numbers; $R_+ = [0, +\infty[$.

$\dim X$ is the dimension of the linear space X.

$L(R_+)$ is the set of all Lebesgue integrable functions f: $R_+ \to R$.

$L_{\text{loc}}(R_+)$ is the set of functions f: $R_+ \to R$ which are Lebesgue integrable on $[0, b]$ for any $b \in R_+$.

$\tilde{C}^k([a, b])$ is the set of functions f: $[a, b] \to R$ which are absolutely continuous along with their derivatives up to the order k inclusively.

$\tilde{C}^k_{\text{loc}}(R_+)$ is the set of functions f: $R_+ \to R$ whose restrictions to $[0, b]$ belong to $\tilde{C}^k([0, b])$ for any $b \in R_+$; $\tilde{C}_{\text{loc}}(R_+) = \tilde{C}^0_{\text{loc}}(R_+)$.

Γ_i^k $(i = 0, 1, \dots; k = 2i, 2i + 1, \dots)$ are positive numbers defined by the recurrent relations

$$\Gamma_0^{i+1} = \tfrac{1}{2}, \quad \Gamma_i^{2i} = 1, \quad \Gamma_{i+1}^k = \Gamma_{i+1}^{k-1} + \Gamma_i^{k-2} \tag{0.6}$$
$$(i = 0, 1, \dots; k = 2i + 3, \dots).$$

The notation $\int^{+\infty} \alpha(t)\, dt < +\infty$ means that $\int_a^{+\infty} \alpha(t)\, dt < +\infty$ for any sufficiently large a.

In what follows, unless otherwise stated, we assume that $n \geqq 2$, $a \in R_+$, $c_i \in R$, $g_k \in L_{loc}(R_+)$, $g_k(t) \geqq 0$ for $t \in R_+$ $(k = 0, \dots, n_0)$ and

$$p \in L_{loc}(R_+), \quad q \in L_{loc}(R_+), \quad p_0 \in L_{loc}(R_+), \quad p_k \in \tilde{C}_{loc}^{k-1}(R_+) \tag{0.7}$$
$$(k = 1, \dots, n - 1).$$

By a solution of the problem (0.3), (0.4_m) we mean a function $u \in \tilde{C}_{loc}^{n-1}(R_+)$ which satisfies the differential equation (0.3) almost everywhere in R_+, as well as the boudary conditions (0.4_m).

1. AUXILIARY STATEMENTS

1.1. Certain integral identities and inequalities. Let Γ_i^k $(i = 0, 1, \dots; k = 2i, 2i + 1, \dots)$ be the numbers defined in Introduction and let Γ_{ij}^k $(i = 0, 1, \dots; j = i, i + 1, \dots; k = i + j + 1, \dots)$ be numbers satisfying the recurrent relations

$$\Gamma_{00}^k = \tfrac{1}{2}, \quad \Gamma_{0j}^k = 1 \quad (k = 1, 2, \dots; j = 1, \dots, k - 1), \tag{1.1}$$
$$\Gamma_{ii}^{2i+1} = \tfrac{1}{2}, \quad \Gamma_{ik-i-1}^k = 1 \quad (i = 1, 2, \dots; k = 2i + 2, 2i + 3, \dots)$$
$$\Gamma_{ij}^k = \Gamma_{ij}^{k-1} + \Gamma_{i-1j-1}^{k-2} \quad (i = 1, 2, \dots; j = i, i + 1, \dots;$$
$$k = i + j + 2, i + j + 3, \dots).$$

Lemma 1.1. *Let k be a positive integer, $-\infty < a < b < +\infty$, $u \in \tilde{C}^{k-1}([a, b])$ and $v \in \tilde{C}^{k-1}([a, b])$. Then*

$$\int_a^b v(t)\, u(t)\, u^{(k)}(t)\, dt = \sum_{i=0}^{[(k-1)/2]} \sum_{j=i}^{k-1-i} (-1)^{k-1-j}\, \Gamma_{ij}^k (v^{(k-1-i-j)}(b)\, u^{(i)}(b)\, u^{(j)}(b) - \tag{1.2}$$
$$- v^{(k-1-i-j)}(a)\, u^{(i)}(a)\, u^{(j)}(a)) + \sum_{i=0}^{[k/2]} (-1)^{k-i}\, \Gamma_i^k \int_a^b v^{(k-2i)}(t)\, |u^{(i)}(t)|^2\, dt,$$

where $[\frac{1}{2}(k - 1)]$ and $[\frac{1}{2}k]$ are entire parts of $\frac{1}{2}(k - 1)$ and $\frac{1}{2}k$.

Proof. For $k = 1$ the identity (1.2) is obviously true. Assume that it is true for all

$k \in \{1, \dots, m\}$ where m is an arbitrary positive integer. Then for any u and $v \in \in \tilde{C}^m([a, b])$ we have

$$(1.3) \qquad \int_a^b v(t)\, u(t)\, u^{(m+1)}(t)\, dt = v(b)\, u(b)\, u^{(m)}(b) - v(a)\, u(a)\, u^{(m)}(a) - \\ - \int_a^b v'(t)\, u(t)\, u^{(m)}(t)\, dt - \int_a^b v(t)\, u'(t)\, u^{(m)}(t)\, dt\,.$$

According to our assumption

$$(1.4)$$
$$\int_a^b v'(t)\, u(t)\, u^{(m)}(t)\, dt = \sum_{i=0}^{[(m-1)/2]} \sum_{j=i}^{m-1-i} (-1)^{m-1-j}\, \Gamma_{ij}^m (v^{(m-i-j)}(b)\, u^{(i)}(b)\, u^{(j)}(b) - \\ - v^{(m-i-j)}(a)\, u^{(i)}(a)\, u^{(j)}(a)) + \sum_{i=0}^{[m/2]} (-1)^{m-i}\, \Gamma_i^m \int_a^b v^{(m+1-2i)}(t)\, |u^{(i)}(t)|^2\, dt$$

and

$$(1.5)$$
$$\int_a^b v(t)\, u'(t)\, u^{(m)}(t)\, dt = \sum_{i=0}^{[(m-2)/2]} \sum_{j=1}^{m-2-i} (-1)^{m-2-j}\, \Gamma_{ij}^{m-1} (v^{(m-2-i-j)}(b)\, u^{(1+i)}(b)\,. \\ .\, u^{(1+j)}(b) - v^{(m-2-i-j)}(a)\, u^{(1+i)}(a)\, u^{(1+j)}(a)) + \sum_{i=0}^{[(m-1)/2]} (-1)^{m-1-i}\, \Gamma_i^{m-1}\,. \\ .\int_a^b v^{(m-1-2i)}(t)\, [u^{(i+1)}(t)]^2\, dt = \sum_{i=1}^{[m/2]} \sum_{j=i}^{m-i} (-1)^{m-1-j}\, \Gamma_{i-1\, j-1}^{m-1} (v^{(m-i-j)}(b)\, u^{(i)}(b)\,. \\ .\, u^{(j)}(b) - v^{(m-i-j)}(a)\, u^{(i)}(a)\, u^{(j)}(a)) + \\ + \sum_{i=1}^{[(m+1)/2]} (-1)^{m-i}\, \Gamma_{i-1}^{m-1} \int_a^b v^{(m+1-2i)}(t)\, |u^{(i)}(t)|^2\, dt\,.$$

From (1.3)–(1.5) by (0.6) and (1.1) we conclude that (1.2) is valid for $k = m + 1$. This completes the proof.

Lemma 1.2. *Let $k \geqq 2$ be a positive integer, $0 < a < b < +\infty$, $\sigma \geqq 0$, $c_0 \geqq 0$ and let $u : [a, b] \to R$ be a k times continuously differentiable function such that*

$$(1.6) \qquad b^{2i-\sigma}\, u^{(i)}(b)\, u^{(i-1)}(b) - \left(i - \frac{\sigma}{2}\right) b^{2i-\sigma-1} |u^{(i-1)}(b)|^2 - \\ - a^{2i-\sigma}\, u^{(i)}(a)\, u^{(i-1)}(a) + \left(i - \frac{\sigma}{2}\right) a^{2i-\sigma-1} |u^{(i-1)}(a)|^2 \leqq c_0 \quad (i = 1, \dots, k-1)\,.$$

Then for any $\lambda > \frac{1}{3}(k - 2)(4k^2 - k + 3 + 3\sigma^2)$ the following estimates hold

$$(1.7)\qquad \int_a^b t^{2i-\sigma}|u^{(i)}(t)|^2\,dt \leqq 2(\lambda+\sigma(2k^2-3k-2))^{k-2}c_0 +$$

$$+\ \alpha_i(\lambda,k)\int_a^b t^{-\sigma}|u(t)|^2\,dt + \beta_i(\lambda,k)\int_a^b t^{2k-\sigma}|u^{(k)}(t)|^2\,dt \quad (i=1,\ldots,k-1),$$

where

$$(1.8)\qquad \alpha_1(\lambda,k) = (k-1)\left(\left(1-\frac{\sigma}{2}\right)(1-\sigma)+\frac{\lambda}{4}\right),$$

$$\alpha_i(\lambda,k) = \frac{k-i}{k-1}\,\alpha_1(\lambda,k)\prod_{j=1}^{i-1}\left(\lambda-\frac{(j-1)(4j^2+7j+6)}{3}+\sigma(j-1)(2j+3-\sigma)\right)$$

$$(i=2,\ldots,k-1),$$

$$\beta_i(\lambda,k) = \prod_{j=i}^{k-1}\left(\lambda-\frac{(j-1)(4j^2+7j+6)}{3}+\sigma(j-1)(2j+3-\sigma)\right)^{-1}$$

$$(i=1,\ldots,k-1).$$

Proof. Set

$$\gamma_0 = \left(1-\frac{\sigma}{2}\right)(1-\gamma)+\frac{\lambda}{4},$$

$$\gamma_i = \lambda-\frac{(i-1)(4i^2+7i+6)}{3}+\sigma(i-1)(2i+3-\sigma) \quad (i=1,\ldots,k-1)$$

and

$$\varrho_i = \int_a^b t^{2i-\sigma}|u^{(i)}(t)|^2\,dt \quad (i=0,\ldots,k-1).$$

Then according to (1.6), for $i\in\{1,\ldots,k-1\}$ we have

$$\varrho_i = b^{2i-\sigma}u^{(i)}(b)\,u^{(i-1)}(b) - \left(i-\frac{\sigma}{2}\right)b^{2i-\sigma-1}|u^{(i-1)}(b)|^2 - a^{2i-\sigma}u^{(i)}(a)\,u^{(i-1)}(a) +$$

$$+\left(i-\frac{\sigma}{2}\right)a^{2i-\sigma-1}|u^{(i-1)}(a)|^2 + \left(i-\frac{\sigma}{2}\right)(2i-\sigma-1)\,\varrho_{i-1} -$$

$$-\int_a^b t^{2i-\sigma}u^{(i+1)}(t)\,u^{(i-1)}(t)\,dt \leqq$$

$$\leqq c_0 + \left(i-\frac{\sigma}{2}\right)(2i-\sigma-1)\,\varrho_{i-1} + \int_a^b t^{2i-\sigma}|u^{(i+1)}(t)\,u^{(i-1)}(t)|\,dt\,.$$

Furthermore, taking into consideration that

$$t^{2-\sigma}|u(t)\,u''(t)| \leqq \frac{\lambda}{4}t^{-\sigma}|u(t)|^2 + \frac{1}{\lambda}t^{4-\sigma}|u''(t)|^2,$$

$$t^{2i-\sigma}\left|u^{(i+1)}(t)\,u^{(i-1)}(t)\right| \leqq \frac{\gamma_i}{2}\,t^{2i-2-\sigma}\left|u^{(i-1)}(t)\right|^2 + \frac{1}{2\gamma_i}\,t^{2i+2-\sigma}\left|u^{(i+1)}(t)\right|^2$$

and

$$\left(i-\frac{\sigma}{2}\right)(2i-\sigma-1)+\frac{\gamma_i}{2}=\frac{\gamma_{i-1}}{2} \quad (i=2,\ldots,k-1),$$

we obtain

(1.9) $$\varrho_1 \leqq c_0 + \gamma_0\varrho_0 + \frac{1}{\gamma_1}\varrho_2,$$

(1.10) $$\varrho_i \leqq c_0 + \frac{\gamma_{i-1}}{2}\varrho_{i-1} + \frac{1}{2\gamma_i}\varrho_{i+1} \quad (i=2,\ldots,k-1).$$

If $k = 2$, then the inequality (1.9) coincides with (1.7). So it remains to consider the case when $k > 2$. In this case

$$0 < \gamma_i < \lambda_0 - 2 \quad (i = 2, \ldots, k-1)$$

where $\lambda_0 = \lambda + \sigma(2k^2 - 3k - 2)$. Hence (1.9) and (1.10) imply

(1.11) $$\varrho_i \leqq \lambda_0^{i-1}c_0 + \frac{\alpha_i(\lambda,k)}{k-i}\varrho_0 + \frac{1}{\gamma_i}\varrho_{i+1} \quad (i=1,\ldots,k-1).$$

Now we show that for any $i \in \{1, \ldots, k-1\}$

(1.12) $$\varrho_i \leqq 2(1-2^{i-k})\,\lambda_0^{k-2}c_0 + \alpha_i(\lambda,k)\,\varrho_0 + \beta_i(\lambda,k)\,\varrho_k.$$

For $i = k - 1$ this estimate coincides with (1.11). Assume that (1.12) is valid for a certain $i \in \{2, \ldots, k-1\}$. Then according to (1.11)

(1.13) $$\varrho_{i-1} \leqq \lambda_0^{i-2}c_0 + \frac{\alpha_{i-1}(\lambda,k)}{k-i+1}\varrho_0 + \frac{2}{\gamma_{i-1}}(1-2^{i-k})\,\lambda_0^{k-2}c_0 + \\ + \frac{\alpha_i(\lambda,k)}{\gamma_{i-1}}\varrho_0 + \frac{\beta_i(\lambda,k)}{\gamma_{i-1}}\varrho_k.$$

But, as follows from (1.8),

$$\frac{\alpha_{i-1}(\lambda,k)}{k-i+1}+\frac{\alpha_i(\lambda,k)}{\gamma_{i-1}}=\alpha_{i-1}(\lambda,k), \quad \frac{\beta_i(\lambda,k)}{\gamma_{i-1}}=\beta_{i-1}(\lambda,k).$$

On the other hand,

$$\lambda_0^{i-2} \leqq \lambda_0^{k-2} \quad \text{and} \quad 1 + \frac{2}{\gamma_{i-1}}(1-2^{i-k}) \leqq 2(1-2^{i-1-k}).$$

Therefore (1.13) implies

$$\varrho_{i-1} \leqq 2(1-2^{i-1-k})\,\lambda_0^{k-2}c_0 + \alpha_{i-1}(\lambda)\,\varrho_0 + \beta_{i-1}(\lambda)\,\varrho_k.$$

Hence the validity of the estimate (1.12) for any $i \in \{1, \ldots, k-1\}$ is proved by induction. So the estimates (1.7) are also valid. This completes the proof.

Lemma 1.3. *Let $k \geqq 2$ be a positive integer, $a > 0$, $\sigma \geqq 0$, $c_0 \geqq 0$, and let $u : [a, +\infty[\to R$ be a k times continuously differentiable function such that*

$$(1.14) \qquad \int_a^{+\infty} t^{2i-\sigma} |u^{(i)}(t)|^2 \, dt < +\infty \quad (i = 0, 1, \ldots, k)$$

and

$$(1.15) \qquad \left(i - \frac{\sigma}{2}\right) a^{2i-\sigma-1} |u^{(i-1)}(a)|^2 - a^{2i-\sigma} u^{(i)}(a)\, u^{(i-1)}(a) \leqq c_0$$
$$(i = 1, \ldots, k-1)\,.$$

Then for any $\lambda > \frac{1}{3}(k-2)(4k^2 - k + 3 + 3\sigma^2)$ the following estimates hold

$$(1.16) \qquad \int_a^{+\infty} t^{2i-\sigma} |u^{(i)}(t)|^2 \, dt \leqq 2(\lambda + \sigma(2k^2 - 3k - 2))^{k-2} c_0 +$$
$$+ \alpha_i(\lambda, k) \int_a^{+\infty} t^{-\sigma} |u(t)|^2 \, dt + \beta_i(\lambda, k) \int_a^{+\infty} t^{2k-\sigma} |u^{(k)}(t)|^2 \, dt \quad (i = 1, \ldots, k-1)$$

where $\alpha_i(\lambda, k)$ and $\beta_i(\lambda, k)$ are the numbers defined by the identities (1.8).

Proof. By (1.14)

$$\int_a^{+\infty} t^{2i-\sigma-1} |u^{(i)}(t)\, u^{(i-1)}(t)| \, dt \leqq$$
$$\leqq \left(\int_a^{+\infty} t^{2i-\sigma} |u^{(i)}(t)|^2 \, dt\right)^{1/2} \left(\int_a^{+\infty} t^{2i-2-\sigma} |u^{(i-1)}(t)|^2 \, dt\right)^{1/2} < +\infty$$
$$(i = 1, \ldots, k)\,.$$

Thus there exist sequences $(b_{im})_{m=1}^{+\infty}$ $(i = 1, \ldots, k-1)$ of points from $[a, +\infty[$ which tend to $+\infty$ and

$$(1.17) \qquad b_{im}^{2i-\sigma} |u^{(i)}(b_{im})\, u^{(i-1)}(b_{im})| \leqq \frac{1}{m} \quad (i = 1, \ldots, k-1;\ m = 1, 2, \ldots)\,.$$

For any positive integer m, according to Lemma 1.2 and to the inequalities (1.15) and (1.17), we have

$$\int_a^{b_{im}} t^{2i-\sigma} |u^{(i)}(t)|^2 \, dt \leqq 2(\lambda + \sigma(2k^2 - 3k - 2))^{k-2} \left(c_0 + \frac{1}{m}\right) +$$
$$+ \alpha_i(\lambda, k) \int_a^{b_{im}} t^{-\sigma} |u(t)|^2 \, dt + \beta_i(\lambda, k) \int_a^{b_{im}} t^{2k-\sigma} |u^{(k)}(t)|^2 \, dt \quad (i = 1, \ldots, k-1)\,.$$

From these inequalities, by letting m pass to $+\infty$, we obtain (1.16). This completes the proof.

1.2. On some properties of functions satisfying the conditions (1.14).

Lemma 1.4. *Let m be a positive integer, and let k be the entire part of $m/2$. Suppose that $a > 0$, $\sigma \geqq 0$ and that $u : [a, +\infty[\to R$ is a $m-1$ times differentiable function satisfying the conditions (1.14). Then for any constants c_{ij} ($i = 0, \ldots, k$; $j = i, \ldots, m_i$) the function*

$$w(t) = \sum_{i=0}^{k} \sum_{j=i}^{m_i} c_{ij} t^{i+j+1-\sigma} u^{(i)}(t)\, u^{(j)}(t)\,,$$

where $m_0 = m - 1$ and $m_i = m - i$ for $i \neq 0$ satisfies the condition

$$\liminf_{t \to +\infty} |w(t)| = 0\,. \tag{1.18}$$

Proof. Admit on the contrary that for certain c_{ij} ($i = 0, \ldots, k$; $j = i, \ldots, m_i$) (1.18) is violated. Then, without loss of generality, we may assume that

$$\sum_{i=0}^{k} \sum_{j=i}^{m_i} c_{ij} t^{i+j-\sigma} u^{(i)}(t)\, u^{(j)}(t) \geqq \frac{\delta}{t} \quad \text{for} \quad t \geqq t_1$$

where $\delta > 0$ is sufficiently small and $t_1 > a$ is sufficiently large. According to Lemma 1.1, by integrating both sides of this inequalities from t_1 to t ,we obtain

$$\sum_{i=0}^{[(m-1)/2]} \sum_{j=i}^{m-1-i} c_{ij}^{(1)} t^{i+j+1-\sigma} u^{(i)}(t)\, u^{(j)}(t) + \sum_{i=0}^{k} c_i \int_{t_1}^{t} \tau^{2i-\sigma} |u^{(i)}(\tau)|^2 \, d\tau \geqq c + \delta \ln \frac{t}{t_1}$$
$$\text{for} \quad t \geqq t_1$$

where $c_{ij}^{(1)}$, c_i and c are constants. Thus (1.14) implies

$$\sum_{i=0}^{[(m-1)/2]} \sum_{j=i}^{m-1-i} c_{ij}^{(1)} t^{i+j-\sigma} u^{(i)}(t)\, u^{(j)}(t) \geqq \frac{1}{t} \quad \text{for} \quad t \geqq t_2$$

with a certain $t_2 > t_1$. Now applying Lemma 1.1 it is easy to verify that for any $\nu \in \{1, \ldots, m\}$

$$\sum_{i=0}^{[(m-\nu)/2]} \sum_{j=i}^{m-\nu-i} c_{ij}^{(\nu)} t^{i+j-\sigma} u^{(i)}(t)\, u^{(j)}(t) \geqq \frac{1}{t} \quad \text{for} \quad t \geqq t_\nu$$

where $c_{ij}^{(\nu)} = \text{const}$ and $t_\nu > t_{\nu-1} > \ldots > t_1$. Hence

$$c_{00}^{(m)} t^{-\sigma} |u(t)|^2 \geqq \frac{1}{t} \quad \text{for} \quad t \geqq t_m\,.$$

But in view of (1.14) this is impossible. This completes the proof.

Lemma 1.5. *If $u \in \tilde{C}^{k-1}_{\mathrm{loc}}(R_+)$ satisfies the condition* (1.14) *with $\sigma \in R$, then*

$$\lim_{t\to+\infty} t^{i+(1-\sigma)/2}\, u^{(i)}(t) = 0 \quad (i = 0, \dots, k-1)\,.$$

Proof. Let $i \in \{0, \dots, k-1\}$. Set

$$v(t) = t^{2i-\sigma+1}\left|u^{(i)}(t)\right|^2 .$$

According to (1.14) there exists a sequence $t_m \in [a, +\infty[$ $(m = 1, 2, \dots)$ such that

$$\lim_{t\to+\infty} t_m = +\infty\,, \quad \lim_{t\to+\infty} v(t_m) = 0\,.$$

On the other hand,

$$v(t) \leqq v(t_m) + \int_t^{t_m} \left|v'(\tau)\right| \mathrm{d}\tau \leqq v(t_m) + \left|2i - \sigma\right| \int_t^{+\infty} \tau^{2i-\sigma}\left|u^{(i)}(\tau)\right|^2 \mathrm{d}\tau +$$

$$+ 2\left(\int_t^{+\infty} \tau^{2i-\sigma}\left|u^{(i)}(\tau)\right|^2 \mathrm{d}\tau\right)^{1/2} \left(\int_t^{+\infty} \tau^{2i+2-\sigma}\left|u^{(i+1)}(\tau)\right|^2 \mathrm{d}\tau\right)^{1/2} \quad \text{for} \quad 0 < t \leqq t_m\,.$$

Thus, by passing to the limit first for $m \to +\infty$ and then for $t \to +\infty$ we obtain

$$\lim_{t\to+\infty} v(t) = 0\,.$$

This completes the proof.

1.3. A lemma on solvability of problems of (0.3), (0.4$_j$) **type.** In this section we establish conditions of existence of a solution of the equation (0.3) satisfying the boundary conditions

$$l_i(u(a), \dots, u^{(n-1)}(a)) = c_i \quad (i = 0, \dots, m-1)\,, \tag{1.19}$$

$$\int_a^{+\infty} g_k(t)\left|u^{(k)}(t)\right|^2 \mathrm{d}t < +\infty \quad (k = 0, \dots, m_0)\,,$$

where m and $m_0 \in \{1, \dots, n-1\}$, $c_i \in R$ $(i = 0, \dots, m-1)$, $a \in R_+$, $g_k \in L_{\mathrm{loc}}(R_+)$ $(k = 0, \dots, m_0)$ are nonnegative and $l_i : R^n \to R$ $(i = 0, \dots, m-1)$ are linear homogeneous functions.

In addition to (0.3) we consider the corresponding homogeneous equations

$$v^{(n)} = \sum_{k=0}^{n-1} p_k(t)\, v^{(k)}\,. \tag{1.20}$$

Lemma 1.6. *Let g_0 be distinct from zero on a subset of positive measure of the interval $[a, +\infty[$, and let there exist a continuous function $r : R_+ \to R_+$ and linear homogeneous functions $l_i : R^n \to R$ $(i = m, \dots, n-1)$ such that for any sufficiently large b all solutions u and v of the equations* (0.3) *and* (1.20), *respectively, satisfy the inequalities*

$$(1.21)\quad \int_a^b g_k(t)\,|u^{(k)}(t)|^2\,dt \leqq r(a)\,(1+\sum_{i=0}^{n-1}|u^{(i)}(a)|\sum_{i=0}^{m-1}|l_i(u(a),\ldots,u^{(n-1)}(a))|)+$$
$$+\,r(b)\sum_{i=0}^{n-1}|u^{(i)}(b)|\sum_{i=m}^{n-1}|l_i(u(b),\ldots,u^{(n-1)}(b))|\quad (k=0,\ldots,m_0)$$

and

$$(1.22)\qquad \int_a^b g_0(t)\,|v(t)|^2\,dt \leqq r(a)\sum_{i=0}^{n-1}|v^{(i)}(a)|\sum_{i=0}^{m-1}|l_i(v(a),\ldots,v^{(n-1)}(a))|+$$
$$+\,r(b)\sum_{i=0}^{n-1}|v^{(i)}(b)|\sum_{i=m}^{n-1}|l_i(v(b),\ldots,v^{(n-1)}(b))|\,.$$

Then the problem (0.3), (1.19) *has at least one solution.*

Proof. Choose $b_0 > a$ such that

$$(1.23)\qquad \operatorname{mes}\{t\in[a,b_0]:\ g_0(t)>0\}>0$$

and that for any $b \geqq b_0$ the inequalities (1.21) and (1.22) hold.

Set $b_j = b_0 + j$ and for any positive integer j consider the equation (0.3) under the boundary conditions

$$(1.24)\qquad \begin{aligned} &l_i(u(a),\ldots,u^{(n-1)}(a)) = c_i\quad (i=0,\ldots,m-1)\,,\\ &l_k(u(b_j),\ldots,u^{(n-1)}(b_j)) = 0\quad (k=m,\ldots,n-1)\,. \end{aligned}$$

It is well-known (see e.g. [2]) that the problem (0.3), (1.24) is uniquely solvable if and only if the equation (1.20) under the homogeneous boundary conditions

$$(1.24_0)\qquad \begin{aligned} &l_i(v(a),\ldots,v^{(n-1)}(a)) = 0\quad (i=0,\ldots,m-1)\,,\\ &l_k(v(b_j),\ldots,v^{(n-1)}(b_j)) = 0\quad (k=m,\ldots,n-1) \end{aligned}$$

has only the zero solution. Let v be an arbitrary solution of the problem (0.3), (1.24_0). Then according to (1.22)

$$\int_a^{b_j} g_0(t)\,|v(t)|^2\,dt = 0\,.$$

This by (1.23) implies that $v(t)\equiv 0$. Hence the problem (0.3), (1.24) has a unique solution. Denote it by u_j. As follows from (1.21), for any positive integer j the inequalities

$$(1.25)\qquad \int_a^{b_j} g_k(t)\,|u_j^{(k)}(t)|^2\,dt \leqq r_0(1+\varrho_j)\quad (k=0,\ldots,m_0)$$

hold, where

$$r_0 = r(a)\,(1+\sum_{i=0}^{m-1}|c_i|)\,,\quad \varrho_j = \sum_{i=0}^{n-1}|u_j^{(i)}(a)|\,.$$

Now we show that the sequence $(\varrho_j)_{j=1}^{+\infty}$ is bounded. Admit on the contrary that there exists a subsequence $(\varrho_{j_\nu})_{\nu=1}^{+\infty}$ satisfying the condition

$$\varrho_{j_\nu} > \nu \quad (\nu = 1, 2, \ldots) . \tag{1.26}$$

Put

$$v_\nu(t) = \frac{u_{j_\nu}(t)}{\varrho_{j_\nu}} .$$

Obviously

$$\sum_{i=0}^{n-1} \left| v_\nu^{(i)}(a) \right| = 1 \quad (\nu = 1, 2, \ldots) \tag{1.27}$$

and v_ν is a solution of the equation

$$v^{(n)} = \sum_{k=1}^{n-1} p_k(t)\, v^{(k)} + \frac{1}{\varrho_{j_\nu}}\, b(t) .$$

On the other hand, from (1.25) and (1.26) we obtain

$$\int_a^{b_0} g_0(t) \left| v_\nu(t) \right|^2 \mathrm{d}t \leqq \frac{2r_0}{\nu} \quad (\nu = 1, 2, \ldots) . \tag{1.28}$$

By (1.27) without loss of generality we may assume that the sequences $(v_\nu^{(i)}(a))_{\nu=1}^{+\infty}$ $(i = 0, \ldots, n-1)$ converge. Then according to the theorem on continuous dependence of the solution of the Cauchy problem on the parameter (see [6], Theorem 1) and to the conditions (1.26) and (1.27) we have

$$\lim_{\nu \to +\infty} v_\nu(t) = v(t)$$

uniformly on $[a, b_0]$, where v is a solution of the equation (1.20) satisfying the condition

$$\sum_{i=0}^{n-1} \left| v^{(i)}(a) \right| = 1 .$$

On the other hand, by (1.23) and (1.28), $v(t) \equiv 0$. This contradiction proves the boundedness of the sequence $(\varrho_j)_{j=1}^{+\infty}$.

Since $(\varrho_j)_{j=1}^{+\infty}$ is bounded, without loss of generality we may assume that the sequences $(u_j^{(i)}(a))_{j=1}^{+\infty}$ $(i = 0, \ldots, n-1)$ converge. Then we have

$$\lim_{j \to +\infty} u_j^{(i)}(t) = u^{(i)}(t) \quad (i = 0, \ldots, n-1)$$

uniformly on each finite segment of R_+ where u is a solution of the equation (0.3). It obviously follows from (1.24) and (1.25), that u satisfies the boundary conditions (1.19). Therefore, u is a solution of the problem (0.3), (1.19). This completes the proof.

Recall that n_0 is the entire part of $\frac{1}{2}n$.

Theorem 2.1. *Let g_0 differ from zero on a subset of positive measure of the interval $[a, +\infty[$, and let there exist nonnegative functions $h \in \tilde{C}^{n-1}_{\mathrm{loc}}(R_+)$ and $\alpha \in L(R_+)$ such that the inequalities*

$$|h(t)\, q(t)|^2 \leqq \alpha(t)\, g_0(t)\,, \tag{2.1}$$

$$\sum_{k=2i}^{n-1} (-1)^{n-n_0-1-i-k}\, \Gamma_i^k (h(t)\, p_k(t))^{(k-2i)} + (-1)^{n_0-i}\, \Gamma_i^n h^{(n-2i)}(t) \geqq g_i(t) \tag{2.2}$$
$$(i = 0, 1, \ldots, n - n_0 - 1)$$

hold on $[a, +\infty[$ and if $n = 2n_0$, then

$$h(t) \geqq g_{n_0}(t)\,. \tag{2.3}$$

Then the problem (0.3), (0.4_1) has at least one solution.

Proof. According to Lemma 1.6, it will suffice to show that for any $b \in]a, +\infty[$ all solutions u and v of the equations (0.3) and (1.20), respectively, satisfy the inequalities (1.21) and (1.22) where

$$m = m_0 = n_0\,, \quad l_i(x_1, \ldots, x_n) = x_{i+1} \quad (i = 0, \ldots, n_0 - 1)\,, \tag{2.4}$$
$$l_i(x_1, \ldots, x_n) = x_{i-n_0+1} \quad (i = n_0, \ldots, n - 1)\,,$$

and a continuous function $r\colon R_+ \to R_+$ does not depend on u and v.

Due to Lemma 1.1, by multiplying (0.3) by $(-1)^{n-n_0}\, h(t)\, u(t)$ and integrating from a to b we obtain

$$\sum_{i=0}^{n_0} \int_a^b h_i(t)\, |u^{(i)}(t)|^2\, dt = l(u)(b) - l(u)(a) + (-1)^{n-n_0} \int_a^b q(t)\, h(t)\, u(t)\, dt \tag{2.5}$$

where

$$l(u)(t) = \sum_{i=0}^{n-n_0-1} \sum_{j=i}^{n-1-i} (-1)^{n_0-j}\, \Gamma_{ij}^n\, h^{(n-1-i-j)}(t)\, u^{(i)}(t)\, u^{(j)}(t) + \tag{2.6}$$
$$+ \sum_{k=1}^{n-1} \sum_{i=0}^{[(k-1)/2]} \sum_{j=i}^{k-1-i} (-1)^{n-n_0+k-1-j}\, \Gamma_{ij}^k (h(t)\, p_k(t))^{(k-1-i-j)}\, u^{(i)}(t)\, u^{(j)}(t)\,,$$

$$h_i(t) = \sum_{k=2i}^{n-1} (-1)^{n-n_0-1-i-k}\, \Gamma_i^k (h(t)\, p_k(t))^{(k-2i)} + (-1)^{n_0-i}\, \Gamma_i^n\, h^{(n-2i)}(t) \tag{2.7}$$
$$(i = 0, \ldots, n - n_0 - 1)$$

and if $n = 2n_0$, then $h_{n_0}(t) = h(t)$. Put

$$(2.8) \qquad r_0(t) = \sum_{i=0}^{n-n_0-1} \sum_{j=i}^{n-1-i} \Gamma^n_{ij} \left| h^{(n-1-i-j)}(t) \right| + $$
$$+ \sum_{k=1}^{n-1} \sum_{i=0}^{[(k-1)/2]} \sum_{j=i}^{k-1-i} \Gamma^k_{ij} \left| (h(t)\, p_k(t))^{(k-1-i-j)} \right| .$$

It is clear that

$$(2.9) \qquad -r_0(t) \sum_{i=0}^{n-1} \left| u^{(i)}(t) \right| \sum_{i=0}^{n_0-1} \left| u^{(i)}(t) \right| \leqq l(u)(t) \leqq$$
$$\leqq r_0(t) \sum_{i=0}^{n-1} \left| u^{(i)}(t) \right| \sum_{i=0}^{n-n_0-1} \left| u^{(i)}(t) \right| .$$

By (2.1)

$$(2.10) \qquad \int_a^b \left| q(t)\, h(t)\, u(t) \right| \mathrm{d}t \leqq \frac{1}{2} \int_a^b g_0(t) \left| u(t) \right|^2 \mathrm{d}t + \frac{1}{2} \int_a^{+\infty} \alpha(t)\, \mathrm{d}t .$$

From (2.5), in virtue of (2.2), (2.3) and (2.10), we obtain

$$\frac{1}{2} \int_a^b g_0(t) \left| u(t) \right|^2 \mathrm{d}t + \sum_{i=1}^{n_0} \int_a^b g_i(t) \left| u^{(i)}(t) \right|^2 \mathrm{d}t \leqq l(u)(b) - l(u)(a) + \frac{1}{2} \int_a^{+\infty} \alpha(t)\, \mathrm{d}t .$$

By (2.4) and (2.9) this implies the estimates (1.21), where

$$r(t) = 2\, r_0(t) + \int_a^{+\infty} \alpha(\tau)\, \mathrm{d}\tau .$$

Similarly we can show that an arbitrary solution v of the equation (1.20) satisfies the estimate (1.22). This completes the proof.

Now we consider the case when the boundary conditions (0.4_1) are of the form

$$(2.11) \qquad u^{(i)}(a) = c_i \quad (i = 0, \ldots, n_0 - 1), \quad \int_a^{+\infty} t^{2k} \left| u^{(k)}(t) \right|^2 \mathrm{d}t < +\infty$$
$$(k = 0, \ldots, n_0) .$$

Theorem 2.2. *Let*

$$(2.12) \qquad h_0(t) = \sum_{k=0}^{n-1} (-1)^{n-n_0-1-k}\, \Gamma^k_0 (t^n\, p_k(t))^{(k)} \to +\infty \quad \textit{for} \quad t \to +\infty ,$$

$$(2.13) \qquad \limsup_{t \to +\infty} t^{-2i} \sum_{k=2i}^{n-1} (-1)^{n-n_0-i-k}\, \Gamma^k_i (t^n\, p_k(t))^{(k-2i)} < +\infty$$
$$(i = 1, \ldots, n_0 - 1) ,$$

$$(2.14) \qquad \int^{+\infty} \frac{[t^n\, q(t)]^2}{h_0(t)}\, \mathrm{d}t < +\infty$$

and, moreover, let in the case of odd n

$$\liminf_{t\to+\infty} t p_{n-1}(t) > -\frac{n^2}{2}. \tag{2.15}$$

Then for sufficiently large a the problem (0.3), (2.11) *has at least one solution.*

Proof. Let $h(t) = t^n$, and let the operator l and the functions r_0 and h_i ($i = 0, \ldots, n - n_0 - 1$) be defined by the equalities (2.6)–(2.8). In the case of $n = 2n_0 + 1$ we obtain from (2.7)

$$h_{n_0}(t) = \Gamma_{n_0}^{2n_0} t^n\, p_{n-1}(t) + \Gamma_{n_0}^{2n_0+1} n t^{n-1} = t^{n-1}\left(t\, p_{n-1}(t) + \frac{n^2}{2}\right).$$

In the case of $n = 2n_0$ set

$$h_{n_0}(t) = t^n .$$

According to (2.13) and (2.15) there exist constants $a_0 > 0$, $\gamma_0 > 0$ and $\gamma > 0$ such that

$$h_{n_0}(t) > \gamma t^{2n_0}, \quad h_i(t) > -\gamma_0 t^{2i} \quad \text{for} \quad t \geqq a_0 \quad (i = 1, \ldots, n_0 - 1). \tag{2.16}$$

Let $\alpha_i(\lambda, n_0)$ and $\beta_i(\lambda, n_0)$ ($i = 1, \ldots, n_0 - 1$) be constants defined by the identities (1.8) where $\sigma = 0$, and let $\lambda > n^3$ be so large that

$$2\gamma_0 \sum_{i=1}^{n_0-1} \beta_i(\lambda, n_0) < \gamma . \tag{2.17}$$ [1])

Due to (2.12) and (2.14) without loss of generality we may assume that

$$h_0(t) > 2 + 2\gamma_0 \sum_{i=1}^{n_0-1} \alpha_i(\lambda, n_0) \quad \text{for} \quad t \geqq a_0 \tag{2.18}$$

and

$$\delta = \int_{a_0}^{+\infty} \frac{|t^n q(t)|^2}{h_0(t)}\, dt < +\infty . \tag{2.19}$$

Our aim is to show that if $a \geqq a_0$, then the problem (0.3), (2.11) is solvable.

Let $a_0 \leqq a < b < +\infty$, and let u be an arbitrary solution of the equation (0.3). Then, as we have already noticed, the identity (2.5) and the inequality (2.9) hold.

From (2.5) by virtue of (2.16) and (2.19) we get

$$\int_a^b h_0(t)\, |u(t)|^2\, dt + \gamma \int_a^b t^{2n_0} |u^{(n_0)}(t)|^2\, dt \leqq l(u)(b) - l(u)(a) + \tag{2.20}$$

$$+ \frac{\delta}{2} + \frac{1}{2}\int_a^b h_0(t)\, |u(t)|^2\, dt + \gamma_0 \sum_{i=1}^{n_0-1} \int_a^b t^{2i} |u^{(i)}(t)|^2\, dt .$$

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[1]) In the case of $n_0 = 1$ here and in what follows, by sums of the type $\sum_{i=1}^{n_0-1}$ we mean zero.

According to Lemma 1.2 and inequality (2.17),

$$(2.21)\quad \gamma_0 \sum_{i=1}^{n_0-1} \int_a^b t^{2i}|u^{(i)}(t)|^2 \, dt \leqq r_1(a) \Big(\sum_{i=0}^{n_0-1} |u^{(i)}(a)| \Big)^2 + r_1(b) \Big(\sum_{i=0}^{n_0-1} |u^{(i)}(b)| \Big)^2 + $$
$$+ \gamma_1 \int_a^b |u(t)|^2 \, dt + \frac{\gamma}{2} \int_a^b t^{2n_0}|u^{(n_0)}(t)|^2 \, dt ,$$

where $r_1(t) = n\gamma_0\lambda^n(1 + t)^n$, $\gamma_1 = \gamma_0 \sum_{i=1}^{n_0-1} \alpha_i(\lambda, n_0)$.

By (2.9) and (2.18) the inequalities (2.20) and (2.21) imply (1.21) where

$$g_k(t) = t^{2k} \quad (k = 0, \ldots, n_0), \quad r(t) = \left(\frac{\gamma_1 + 1}{\gamma_0} + \frac{2}{\gamma} + 1\right)(r_0(t) + r_1(t) + \delta)$$

and m, m_0 and l_i $(i = 0, \ldots, n - 1)$ are the constants defined in (2.4).

It follows from these arguments that an arbitrary solution v of the equation (1.20) satisfies the inequality (1.22). Thus all the conditions of Lemma 1.6 are fulfilled, which guarantees the solvability of the problem (0.3), (2.11). This completes the proof.

Remark. As the proof shows, in the case of $n \in \{2, 3\}$ the condition (2.12) may be somewhat relaxed, namely we may assume that

$$\liminf_{t \to +\infty} h_0(t) > 0 .$$

If a solution u satisfies the conditions (2.11), then according to Lemma 1.5,

$$(2.22)\qquad \lim_{t \to +\infty} t^{i+1/2}|u^{(i)}(t)| = 0 \quad (i = 0, \ldots, n_0 - 1) .$$

Hence, Theorem 2.2 implies the following statement.

Corollary. *Under the conditions of Theorem 2.2 the equation (0.3) has an n_0-parameter family of solutions vanishing at infinity together with their derivatives up to the order $n_0 - 1$ inclusively (to be precise, satisfying the conditions (2.22)).*

In the conclusion consider the boundary value problem

$$(2.23)\qquad u^{(i)}(a) = c_i \quad (i = 0, \ldots, n_0 - 1), \quad \int_a^{+\infty} |u^{(n_0)}(t)|^2 \, dt < +\infty$$

for the equation (0.2).

Theorem 2.3. *Let the inequalities*

$$(2.24)\qquad (-1)^{n-n_0-1} p(t) \geqq 0 , \quad t^{n-2n_0}|q(t)|^2 \leqq \alpha(t) \, |p(t)| ,$$

where $\alpha \in L(R_+)$ hold in $[a, +\infty[$. Then the problem (0.2), (2.23) has one and only one solution.

Proof. If $p(t) \equiv 0$, then $q(t) \equiv 0$ and the problem (0.2), (2.23) has the unique solution

$$u(t) = \sum_{i=0}^{n_0 - 1} \frac{c_i}{i!} (t - a)^{i-1} .$$

Consider the case when p differs from zero on a subset of positive measure of the interval $[a, +\infty[$. Put

$$h(t) = t^{n-2n_0}, \quad g_0(t) = t^{n-2n_0}|p(t)|, \quad g_i(t) = 0 \quad (i = 0, \ldots, n_0 - 1), \quad g_{n_0}(t) = 1,$$
$$p_0(t) = p(t), \quad p_k(t) = 0 \quad (k = 1, \ldots, n - 1).$$

Then in virtue of (2.24) the conditions of Theorem 2.1 are fulfilled. Therefore the problem (0.3), (0.4_1) has a solution u. It is obvious that u is a solution of the problem (0.3), (2.23).

Now it remains to show that the problem we are considering has at most one solution, i.e. that the differential equation (0.1) under the homogeneous boundary conditions

$$(2.23_0) \qquad u^{(i)}(a) = 0 \quad (i = 0, \ldots, n_0 - 1), \quad \int_a^{+\infty} |u^{(n_0)}(t)|^2 \, dt < +\infty$$

has only the zero solution. Admit on the contrary that the problem (0.1), (2.23_0) has a nonzero solution u. Clearly,

$$(2.24) \qquad \lim_{b \to +\infty} b^{2i-n} \int_a^b |u^{(i)}(t)|^2 \, dt = 0 \quad (i = 0, \ldots, n - n_0 - 1).$$

Let

$$w(t) = \sum_{i=0}^{n_0 - 1} (-1)^{n-n_0-1-i} u^{(n-1-i)}(t) \, u^{(i)}(t)$$

in the case of even n and

$$w(t) = \sum_{i=0}^{n_0 - 1} (-1)^{n-n_0-1-i} u^{(n-1-i)}(t) \, u^{(i)}(t) + \tfrac{1}{2}|u^{(n_0)}(t)|^2$$

in the case of odd n. Then $w(a) \geqq 0$ and

$$w'(t) \geqq (-1)^{n-n_0-1} p(t) \, |u(t)|^2 \geqq 0 \quad \text{for} \quad t \geqq a ,$$

w' being distinct from zero on a set of positive measure. So there exist constants $t_0 > a$ and $\gamma > 0$ such that

$$(2.25) \qquad w(t) > \gamma \quad \text{for} \quad t \geqq t_0 .$$

According to Lemma 1.1 and the conditions (2.23_0) we have

$$\int_a^b (b - t)^{n-1} w(t) \, dt = \sum_{i=0}^{n-n_0-1} l_i \int_a^b (b - t)^{2i} |u^{(i)}(t)|^2 \, dt$$

for any $b > a$ where the constants l_i $(i = 0, \ldots, n - n_0 - 1)$ do not depend on b. If we divide both sides of this identity by b^n and let b pass to $+\infty$, then by (2.24) and (2.25) we obtain the contradiction $0 < \gamma \leqq 0$. Therefore, the problem (0.1), (2.23_0) has no nonzero solution. This completes the proof.

Theorems 2.2 and 2.3 imply the following statement.

Theorem 2.4. *Let*

$$(-1)^{n-n_0-1}\, t^n\, p(t) \to +\infty \quad \text{for} \quad t \to +\infty$$

and

$$\int^{+\infty} t^n \frac{|q(t)|^2}{|p(t)|}\, \mathrm{d}t < +\infty\,.$$

Then for sufficiently large a the problem (0.2), (2.23) *has the unique solution u. This solution satisfies the conditions*

$$\int_a^{+\infty} t^{2i} |u^{(i)}(t)|^2\, \mathrm{d}t < +\infty \quad (i = 0, \ldots, n_0)$$

and, consequently, the condition (2.22).

3. THE PROBLEM (0.3), (0.4_2)

This section is concerned with the problem (0.3), (0.4_2) in the general case as well as in the case when the boundary conditions (0.4_2) are of the form

$$(3.1) \qquad u^{(i)}(a) = c_i \quad (i = 0, \ldots, n_0)\,, \quad \int_a^{+\infty} t^{2k-n-\varepsilon} |u^{(k)}(t)|^2\, \mathrm{d}t < +\infty \quad (k = 0, \ldots, n_0)$$

with $\varepsilon > 0$.

Theorem 3.1. *Let $n = 2n_0 + 1$ and let g_0 differ from zero on a subset of positive measure of the interval $[a, +\infty[$. Suppose that there exist nonnegative functions $h \in \tilde{C}^{n-1}_{\mathrm{loc}}(R_+)$ and $\alpha \in L(R_+)$ such that the inequalities*

$$(3.2) \qquad \sum_{k=2i}^{n-1} (-1)^{n_0+1-i-k}\, \Gamma_i^k (h(t)\, p_k(t))^{(k-2i)} + (-1)^{n_0+1-i}\, \Gamma_i^n\, h^{(n-2i)}(t) \geqq g_i(t) \quad (i = 0, \ldots, n_0)$$

hold in $[a, +\infty[$. Then the problem (0.3), (0.4_2) *has at least one solution.*

Proof. Let the operator l and the functions h_i $(i = 0, \ldots, n_0)$ and r_0 be defined by the identities (2.6)–(2.8). Then, as we have verified above, for any $b \in]a, +\infty[$ and any solution u of the equation (0.3) the identity (2.5) and the inequalities (2.9) and (2.10) are valid.

According to (2.7) and (3.2),

$$-h_i(t) \geqq g_i(t) \quad \text{for} \quad t \geqq a \quad (i = 0, \ldots, n_0)\,.$$

Hence, taking into consideration the inequality (2.10), we obtain

$$\frac{1}{2}\int_a^b g_0(t)\,|u(t)|^2\,\mathrm{d}t + \sum_{i=1}^{n_0}\int_a^b g_i(t)\,|u^{(i)}(t)|^2\,\mathrm{d}t \leqq l(u)(a) - l(u)(b) + \frac{1}{2}\int_a^{+\infty}\alpha(t)\,\mathrm{d}t\,.$$

By (2.9) this implies the estimates (1.21) where

$$(3.3) \qquad m = n_0 + 1\,, \quad m_0 = n_0\,, \quad l_i(x_1, \ldots, x_n) = x_{i+1} \quad (i = 0, \ldots, n_0)\,,$$
$$l_i(x_1, \ldots, x_n) = x_{i-n_0} \quad (i = n_0 + 1, \ldots, n - 1)$$

and $r(t) = 2\,r_0(t) + \int_a^{+\infty}\alpha(t)\,\mathrm{d}t$. Similarly, we show that for any solution v of the equation (1.20) the estimate (1.22) holds. But according to Lemma 1.6 the estimates (1.21) and (1.22) gurantee the solvability of the problem (0.3), (0.4_2). This completes the proof.

Theorem 3.2. *Let* $n = 2n_0 + 1$,

$$(3.4) \qquad g_0(t) = \sum_{k=0}^{n-1}(-1)^{n_0+1-k}\,\Gamma_0^k(t^{-\varepsilon}\,p_k(t))^{(k)}\,. \quad \lim_{t\to+\infty} t^{n+\varepsilon}\,g_0(t) = +\infty\,,$$

$$(3.5) \qquad \limsup_{t\to+\infty} t^{n-2i+\varepsilon}\sum_{k=2i}^{n-1}(-1)^{n_0-k-i}\,\Gamma_i^k(t^{-\varepsilon}\,p_k(t))^{(k-2i)} < +\infty$$
$$(i = 1, \ldots, n_0 - 1)\,,$$

$$\limsup_{t\to+\infty} t\,p_{n-1}(t) < \frac{n\varepsilon}{2}$$

and

$$(3.6) \qquad \int^{+\infty}\frac{|t^{-\varepsilon}\,q(t)|^2}{g_0(t)}\,\mathrm{d}t < +\infty\,.$$

Then for sufficiently large a the problem (0.3), (3.1) *has a solution u such that*

$$(3.7) \qquad \int_a^{+\infty} g_0(t)\,|u(t)|^2\,\mathrm{d}t < +\infty\,.$$

Proof. Let $h(t) = t^{-\varepsilon}$, $\sigma = n + \varepsilon$, and let the operator l and the functions h_i $(i = 0, \ldots, n_0)$ and r_0 be defined by the identities (2.6)–(2.8). By (3.5) there exist constants $a_0 > 1$, $\gamma_0 > 1$ and $\gamma > 0$ such that

$$(3.8) \qquad h_i(t) \leqq \gamma_0 t^{2i-\sigma} \quad (i = 1, \ldots, n_0 - 1)\,,$$
$$h_{n_0}(t) = -t^{-1-\varepsilon}\left(\frac{n\varepsilon}{2} - t\,p_n(t)\right) \leqq -\gamma t^{2n_0-\sigma} \quad \text{for} \quad t \geqq a_0\,.$$

Choose $\lambda > (n + \sigma)^3$ satisfying the inequality (2.17) where the constants $\beta_i(\lambda, n_0)$ $(i = 1, \ldots, n_0 - 1)$ are defined by the identities (1.8). By (3.4) and (3.5) without loss of generality we may assume that

$$h_0(t) = -g_0(t) < -4\gamma_0 \sum_{i=1}^{n_0-1} \alpha_i(\lambda, n_0)\, t^{-\sigma} \quad \text{for} \quad t \geqq a_0 \tag{3.9}$$

and

$$\delta = \int_{a_0}^{+\infty} \frac{|t^{-\varepsilon} q(t)|^2}{g_0(t)}\, dt < +\infty\,. \tag{3.10}$$

Let $a_0 \leqq a < b < +\infty$, and let u be a solution of the equation (0.3). Then the identity (2.5) and the inequality (2.9) hold. According to (3.8) and (3.10) the identity (2.5) yields

$$\begin{aligned} &\frac{1}{2}\int_a^b g_0(t)\,|u(t)|^2\, dt + \gamma \int_a^b t^{2n_0-\sigma}|u^{(n_0)}(t)|^2\, dt \leqq \\ &\leqq l(u)(a) - l(u)(b) + \delta + \gamma_0 \sum_{i=1}^{n_0-1} \int_a^b t^{2i-\sigma}|u^{(i)}(t)|^2\, dt\,. \end{aligned} \tag{3.11}$$

By Lemma 1.2 and the inequalities (2.17) and (3.9) we have

$$\begin{aligned} \gamma_0 \sum_{i=1}^{n_0-1} \int_a^b t^{2i-\sigma}|u^{(i)}(t)|^2\, dt \leqq \gamma_1 \Big(\sum_{i=0}^{n_0-1} |u^{(i)}(a)|\Big)^2 + \gamma_1\Big(\sum_{i=0}^{n_0-1} |u^{(i)}(b)|\Big)^2 + \\ + \frac{1}{4}\int_a^b g_0(t)\,|u(t)|^2\, dt + \frac{\gamma}{2}\int_a^b t^{2n_0-\sigma}|u^{(n_0)}(t)|^2\, dt \end{aligned} \tag{3.12}$$

with $\gamma_1 = 2\sigma(\lambda + n^2\sigma)^n\, \gamma_0$.

According to (2.9) the inequalities (3.11) and (3.12) imply the estimates (1.21) where

$$r(t) = \left(4 + \frac{2}{\gamma}\right)\left(r_0(t) + \gamma_1 + \delta\right), \quad g_k(t) = t^{2k-\sigma} \quad (k = 1, \ldots, n_0)$$

and the constants m and m_0 and the functions l_i $(i = 0, \ldots, n-1)$ are defined by the identities (3.3). Similarly we show that for an arbitrary solution v of the equation (1.20) the estimate (1.22) holds. Therefore, by Lemma 1.6 the problem (0.3), (3.1) has a solution u satisfying the condition (3.7). This completes the proof.

Corollary. *Let $n = 2n_0 + 1$, and let the conditions (3.5) and (3.6) hold. Suppose that there exists a positive constant δ such that*

$$\sum_{k=0}^{n-1} (-1)^{n_0+1-k}\, \Gamma_0^k (t^{-\varepsilon}\, p_k(t))^{(k)} \geqq \delta t^{n+\varepsilon-2}$$

for sufficiently large t. Then the equation (0.3) *has a* $(n_0 + 1)$-*parametric family of solutions vanishing at infinity.*

Proof. Since all the conditions of Theorem 3.2 are fulfilled, there exists $a > 0$ such that for any $c_i \in R$ $(i = 0, \ldots, n_0)$ the problem (0.3), (3.1) has at least one solution satisfying the condition (3.7). Let u be such a solution. Then

$$\int_a^{+\infty} t^{n+\varepsilon-2}|u(t)|^2 \, dt < +\infty, \quad \int_a^{+\infty} t^{2-n-\varepsilon}|u'(t)|^2 \, dt < +\infty.$$

Thus

$$|u(t)|^2 \leqq 2\left(\int_t^{+\infty} \tau^{n+\varepsilon-2}|u(\tau)|^2 \, d\tau\right)^{1/2} \left(\int_t^{+\infty} \tau^{2-n-\varepsilon}|u'(\tau)|^2 \, d\tau\right)^{1/2} \to 0$$

$$\text{for} \quad t \to +\infty.$$

This completes the proof.

Theorem 3.3. *Let* $n = 2n_0 + 1$, *and let the inequalities*

$$(3.13) \qquad (-1)^{n_0+1} p(t) \geqq 0, \quad |q(t)|^2 \leqq \alpha(t) |p(t)|$$

where $\alpha \in L(R_+)$ *hold in* $[a, +\infty[$. *Then the equation* (0.2) *has at least one solution satisfying the boundary conditions*

$$(3.14) \qquad u^{(i)}(a) = c_i \quad (i = 0, \ldots, n_0), \quad \int_0^{+\infty} |p(t)| \, |u(t)|^2 \, dt < +\infty.$$

Moreover, if the conditions

$$(3.15) \qquad \liminf_{t \to +\infty} |p(t)| > 0, \quad \limsup_{t \to +\infty} |p(t)| < +\infty$$

are fulfilled together with (3.13), *then such a solution u is unique and*

$$(3.16) \qquad \lim_{t \to +\infty} u^{(i)}(t) = 0 \quad (i = 0, \ldots, n-1).$$

Proof. If ϱ differs from zero on a subset of positive measure of the interval $[a, +\infty[$, then by (3.13) all the conditions of Theorem 3.1 are fulfilled where $h(t) = 1$, $p_0(t) = p(t)$, $g_0(t) = |p(t)|$, $p_k(t) = 0$ $(k = 1, \ldots, n-1)$, $g_i(t) = 0$ $(i = 1, \ldots, n_0)$. Thus the problem (0.2), (3.14) is solvable. If $p(t) \equiv 0$, then the solvability of this problem is obvious.

Now assume that the conditions (3.15) hold and that u is an arbitrary solution of the problem (0.2), (3.14). Then

$$\int_0^{+\infty} |u(t)|^2 \, dt < +\infty, \quad \int_0^{+\infty} |u^{(n)}(t)|^2 \, dt < +\infty.$$

Hence, according to the Kolmogorov-Horny inequalities ([3], p. 393), u satisfies the conditions (3.16). It remains to show that the problem (0.2), (3.14) has at most one solution. Let u_1 and u_2 be arbitrary solutions of the problem. Then, as we have already proved, $v = u_1 - u_2$ is a solution of the equation (0.1) satisfying the conditions

$$v^{(i)}(a) = 0 \ (i = 0, \ldots, n_0), \quad \lim_{t \to +\infty} v^{(k)}(t) = 0 \ (k = 0, \ldots, n - 1).$$

On the other hand,

$$\sum_{i=0}^{n_0 - 1} (-1)^i \, v^{(n-1-i)}(t) \, v^{(i)}(t) + \tfrac{1}{2}(-1)^{n_0} \left|v^{(n_0)}(t)\right|^2 = \int_a^t p(\tau) \left|v(\tau)\right|^2 \mathrm{d}\tau .$$

When t tends to $+\infty$ in this identity, we get

$$\int_a^{+\infty} p(\tau) \left|v(\tau)\right|^2 \mathrm{d}\tau = 0 .$$

Thus, $v(t) \equiv 0$. This completes the proof.

For the problem

$$u''' = p(t) \, u + q(t) , \tag{3.17}$$

$$u^{(i)}(a) = c_i \quad (i = 0, 1) , \quad \int_0^{+\infty} |p(t)| \, |u(t)|^2 \, \mathrm{d}t < +\infty \tag{3.18}$$

we get a result on uniqueness which is the best possible in a certain sense.

Theorem 3.3′. *Let the inequalities*

$$p(t) \geqq 0 , \quad |q(t)|^2 \leqq \alpha(t) \, p(t) \tag{3.19}$$

where $\alpha \in L(R_+)$ hold in $[a, +\infty[$. Then the condition

$$\int_a^{+\infty} t^4 \, p(t) \, \mathrm{d}t = +\infty \tag{3.20}$$

is necessary and sufficient for the unique solvability of the problem (3.17), (3.18). *Moreover, if*

$$\liminf_{t \to +\infty} p(t) > 0 , \tag{3.21}$$

then the solution u of this problem satisfies the conditions

$$\lim_{t \to +\infty} u(t) = 0 , \quad \limsup_{t \to +\infty} |u'(t)| < +\infty . \tag{3.22}$$

Proof. Under the conditions (3.19) the solvability of the problem (3.17), (3.18) follows from Theorem 3.3. We show that if (3.20) is fulfilled together with (3.19), then the solution is unique. Admit on the contrary that the problem has two distinct solutions u_1 and u_2. Set $v(t) = u_1(t) - u_2(t)$. Then

$$v(a) = v'(a) = 0\,, \quad v''(a) = \delta \neq 0\,, \quad v'''(t)\,v(t) = p(t)\,|v(t)|^2 \geqq 0 \quad \text{for} \quad t \geqq a\,.$$

Thus

$$|v(t)| \geqq \tfrac{1}{2}|\delta|\,(t-a)^2 \quad \text{for} \quad t \geqq a\,.$$

According to (3.20) this implies

$$\int_a^{+\infty} p(t)\,|v(t)|^2\,\mathrm{d}t = +\infty\,,$$

which contradicts the definition of v. Hence (3.20) is sufficient for the uniqueness of the solution of the problem (3.17), (3.18). Now we show the necessity. Indeed, if

$$\int_a^{+\infty} t^4\,p(t)\,\mathrm{d}t < +\infty\,,$$

then according to the theorem of Sobol [10] any solution of the equation $v''' = p(t)\,v$ satisfies the condition

$$\int_a^{+\infty} p(t)\,|v(t)|^2\,\mathrm{d}t < +\infty\,.$$

Therefore, in this case the problem (3.17), (3.18) has an infinite set of solutions.

Finally, consider the case when (3.21) holds. Then the solution u of the problem (3.17), (3.18) satisfies the condition

$$\int_a^{+\infty} |u(t))|^2\,\mathrm{d}t < +\infty\,. \tag{3.23}$$

Thus (3.22) will be proved if we show that u' is a bounded function. When u' is monotone in a certain neighborhood of $+\infty$, the boundedness immediately follows from (3.23). Let u' be non-monotone in any neighborhood of $+\infty$. Then there exists a sequence $t_k \in [a, +\infty[$ $(k = 1, 2, \ldots)$ tending to $+\infty$ such that

$$u''(t_k) = 0\,, \quad |u'(t_k)| = \max\{|u'(t)|: t_1 \leqq t \leqq t_k\} \quad (k = 1, 2, \ldots)\,.$$

Multiplying both sides of (3.17) by $-2\,u(t)$ and integrating from t_1 to t_k, we obtain

$$|u'(t_k)|^2 = |u'(t_1)|^2 - 2\int_{t_1}^{t_k} p(\tau)\,|u(\tau)|^2\,\mathrm{d}\tau - 2\int_{t_1}^{t_k} q(\tau)\,u(\tau)\,\mathrm{d}\tau \leqq c_0 \quad (k = 1, 2, \ldots)$$

where $c_0 = |u'(t_1)|^2 + \int_a^{+\infty} \alpha(t)\,\mathrm{d}t$. Therefore, u' is bounded. This completes the proof.

Theorem 3.4. *Let* $n = 2n_0 + 1$

$$(-1)^{n_0+1}\,t^n\,p(t) \to +\infty \quad \textit{for} \quad t \to +\infty$$

and

$$(3.24) \qquad \int^{+\infty} t^{-\varepsilon} \frac{|q(t)|^2}{|p(t)|} \, dt < +\infty .$$

Then for sufficiently large a the problem (0.2), (3.1) *has one and only one solution u and*

$$(3.25) \qquad \int_a^{+\infty} t^{-\varepsilon} |p(t)| \, |u(t)|^2 \, dt < +\infty .$$

Proof. For sufficiently large a the existence of a solution of the problem (0.2), (3.1) satisfying the condition (3.25) follows from Theorem 3.2. It remains to show that the solution is unique.

Let

$$\sigma = n + \varepsilon, \quad v_i = \Gamma_i^n \prod_{j=i}^{n-2i} (\sigma - n + j - 1) \quad (i = 0, \ldots, n_0).$$

Choose $\lambda > (n + \sigma)^3$ and $a_0 > 0$ such that

$$(3.26) \qquad \sum_{i=0}^{n_0 - 1} v_i \beta_i(\lambda, n_0) < v_{n_0},$$

$$(-1)^{n_0 + 1} t^n p(t) > \sum_{i=0}^{n_0 - 1} v_i \alpha_i(\lambda, n_0) + 1 \quad \text{for} \quad t \geqq a_0,$$

where $\alpha_0(\lambda, n_0) = 1$, $\beta_0(\lambda, n_0) = 0$ and the constants $\alpha_i(\lambda, n)$ and $\beta_i(\lambda, n)$ $(i = 1, \ldots, n_0 - 1)$ are defined by the identities (1.8).

Let $a \geqq a_0$, and let u_1 and u_2 be arbitrary solutions of the problem (0.2), (3.1). Then $u = u_1 - u_2$ is a solution of the equation (0.1) under the boundary conditions

$$u^{(i)}(a) = 0 \ (i = 0, \ldots, n_0) \quad \int_a^{+\infty} t^{2k-\sigma} |u^{(k)}(t)|^2 \, dt < +\infty \quad (k = 0, \ldots, n_0).$$

Multiplication of both sides of (0.1) by $(-1)^{n_0+1} t^{n-\sigma} u(t)$ and integration from a to t yields

$$w(t) + \sum_{i=0}^{n_0-1} (-1)^{n_0 - i - 1} v_i \int_a^t \tau^{2i-\sigma} |u^{(i)}(\tau)|^2 \, d\tau =$$

$$= (-1)^{n_0+1} \int_a^t \tau^{n-\sigma} p(\tau) \, |u(\tau)|^2 \, dt + v_{n_0} \int_a^t \tau^{2n_0 - \sigma} |u^{(n_0)}(\tau)|^2 \, d\tau$$

where $w(t) = \sum_{i=0}^{n_0} \sum_{j=i}^{n-1-i} c_{ij} t^{i+j+1-\sigma} u^{(i)}(t) \, u^{(j)}(t)$, $c_{ij} = \text{const}$. According to Lemma 1.4

$$\liminf_{t \to +\infty} w(t) = 0 .$$

Therefore, the last identitity implies

(3.27)
$$\sum_{i=0}^{n_0-1}(-1)^{n_0-i-1}\,\nu_i\int_a^{+\infty}t^{2i-\sigma}\left|u^{(i)}(t)\right|^2\,\mathrm{d}t=(-1)^{n_0+1}\int_a^{+\infty}t^{n-\sigma}\,p(t)\,\left|u(t)\right|^2\,\mathrm{d}t+$$
$$+\,\nu_{n_0}\int_a^{+\infty}t^{2n_0-\sigma}\left|u^{(n_0)}(t)\right|^2\,\mathrm{d}t\,.$$

By Lemma 1.3 and the inequalities (3.26)

$$\sum_{i=0}^{n_0-1}\nu_i\int_a^{+\infty}t^{2i-\sigma}\left|u^{(i)}(t)\right|^2\,\mathrm{d}t\leqq\int_a^{+\infty}\left((-1)^{n_0+1}\,t^{n-\sigma}\,p(t)-1\right)\left|u(t)\right|^2\,\mathrm{d}t+$$
$$+\,\nu_{n_0}\int_a^{+\infty}t^{2n_0-\sigma}\left|u^{(n_0)}(t)\right|^2\,\mathrm{d}t\,.$$

Hence from (3.27) we obtain

$$\int_a^{+\infty}\left|u(t)\right|^2\,\mathrm{d}t=0\,,$$

i.e. $u(t)\equiv 0$. So if $a\geqq a_0$, the problem (0.2), (3.1) is uniquely solvable. This completes the proof.

Corollary. *Let $n=2n_0+1$, and let the conditions (3.24) hold. Suppose that there exists a positive constant δ such that for large t*

$$(-1)^{n_0+1}\,p(t)\geqq\delta t^{n+2\varepsilon-2}\,.$$

Then the equaiion (0.2) has a (n_0+1)-parametric family of solutions vanishing at infinity.

4. THE PROBLEM (0.3), (0.4_3)

In contrast to §§ 2 and 3 throughout this section we assume that the conditions

$$q\in L_{\mathrm{loc}}(R_+)\,,\quad p\in\tilde{C}_{\mathrm{loc}}(R_+)\,,\quad p_0\in\tilde{C}_{\mathrm{loc}}(R_+)\,,\quad p_1\in L_{\mathrm{loc}}(R_+)\,,\quad p_k\in\tilde{C}^{k-2}_{\mathrm{loc}}(R_+)$$
$$(k=2,\ldots,n-1)$$

hold instead of (0.7).

Theorem 4.1. *Let $n=2n_0\geqq 4$, and let g_0 differ from zero on a subset of positive measure of the interval $[a,+\infty[$. Suppose that there exist nonnegative functions $h\in\tilde{C}^{n-2}_{\mathrm{loc}}(R_+)$ and $\alpha\in L(R_+)$ such that the inequalities*

(4.1)
$$\sum_{k=2i-1}^{n-1}(-1)^{n_0-i-k+1}\,\Gamma^{k-1}_{i-1}\left(h(t)\,p_k(t)\right)^{(k+1-2i)}+$$

$$+ (-1)^{n_0 - i} \Gamma_{i-1}^{n-1} h^{(n+1-2i)}(t) \geqq g_i(t) \quad (i = 1, \dots, n_0),$$
$$(-1)^{n_0} p_0(t) \geqq 0, \quad (-1)^{n_0} (p_0(t) h(t))' \geqq g_0(t), \quad |h(t) q(t)|^2 \leqq \alpha(t) g_1(t)$$

are fulfilled on $[a, +\infty[$. *Then the problem* (0.3), (0.4_3) *has at least one solution.*

Proof. Let $b \in]a, +\infty[$, and let u be an arbitrary solution of the equation (0.3). Due to Lemma 1.1, when multiplying both sides of (0.3) by $(-1)^{n_0} h(t) u'(t)$ and integrating from a to b, we obtain

$$\sum_{i=0}^{n_0} \int_a^b h_i(t) |u^{(i)}(t)|^2 \, dt = l(u)(b) - l(u)(a) + (-1)^{n_0} \int_a^b q(t) h(t) u'(t) \, dt \tag{4.2}$$

where

(4.3)
$$h_0(t) = \frac{(-1)^{n_0}}{2} (p_0(t) h(t))', \quad h_i(t) = \sum_{k=2i-1}^{n-1} (-1)^{n_0 - i - k + 1} \Gamma_{i-1}^{k-1} (h(t) p_k(t))^{(k+1-2i)} +$$
$$+ (-1)^{n_0 - i} \Gamma_{i-1}^{n-1} h^{(n+1-2i)}(t) \quad (i = 1, \dots, n_0),$$

(4.4)
$$l(u)(t) = \frac{(-1)^{n_0}}{2} p_0(t) h(t) |u(t)|^2 + \sum_{i=1}^{n_0} \sum_{j=i}^{n-1} (-1)^{n_0 - j} \Gamma_{i-1\, j-1}^{n-1} h^{(n-i-j)}(t) u^{(i)}(t) u^{(j)}(t) +$$
$$+ \sum_{k=2}^{n-1} \sum_{i=1}^{[k/2]} \sum_{j=i}^{k-i} (-1)^{n_0 - k - j + 1} \Gamma_{i-1\, j-1}^{k-1} (h(t) p_k(t))^{(k-i-j)} u^{(i)}(t) u^{(j)}(t).$$

According to (4.1) and (4.3), from (4.2) and (4.4) we get

$$\sum_{i=0}^{n_0} \int_a^b g_i(t) |u^{(i)}(t)|^2 \, dt \leqq 2\, l(u)(b) - 2\, l(u)(a) + \delta, \tag{4.5}$$

$$-r_0(t) \sum_{i=0}^{n-1} |u^{(i)}(t)| \sum_{i=1}^{n_0 - 1} |u^{(i)}(t)| \leqq l(u)(t) \leqq r_0(t) \sum_{i=0}^{n-1} |u^{(i)}(t)| \sum_{i=0}^{n_0} |u^{(i)}(t)|, \tag{4.6}$$

where

$$\delta = \int_a^{+\infty} \alpha(t) \, dt, \quad r_0(t) = \tfrac{1}{2} |p_0(t)| h(t) + \sum_{i=1}^{n_0} \sum_{j=i}^{n-1} \Gamma_{i-1\, j-1}^{n-1} |h^{(n-i-j)}(t)| +$$
$$+ \sum_{k=2}^{n-1} \sum_{i=1}^{[k/2]} \sum_{j=i}^{k-i} \Gamma_{i-1\, j-1}^{k-1} |(h(t) p_k(t))^{(k-i-j)}|.$$

Setting $m = n_0 - 1$, $m_0 = n_0$,

$$r(t) = 2 r_0(t) + \delta, \quad l_i(x_1, \dots, x_n) = x_{i+2} \quad (i = 0, \dots, n_0 - 2),$$
$$l_k(x_1, \dots, x_n) = x_{k+2-n_0} \quad (k = n_0 - 1, \dots, n - 1),$$

from (4.5) and (4.6) we obtain the estimates (1.21). Similarly we show that an arbitrary solution v of the equation (1.20) satisfies the estimate (1.22). Therefore, by Lemma 1.6 the problem (0.3), (0.4_3) is solvable. This completes the proof.

Now consider the special case of the problem (0.3), (0.4_3) when the boundary conditions (0.4_3) are of the form

(4.7)
$$u^{(i)}(a) = c_i \quad (i = 1, \ldots, n_0 - 1), \quad \int_a^{+\infty} t^{2k} |u^{(k)}(t)|^2 \, dt < +\infty \quad (k = 0, \ldots, n_0).$$

Theorem 4.2. *Let* $n = 2n_0 \geqq 4$,

$$(-1)^{n_0} (t^{n+1} p_0(t))' \to +\infty \quad \textit{for} \quad t \to +\infty,$$

$$\limsup_{t \to +\infty} t^{-2i} \sum_{k=2i-1}^{n-1} (-1)^{k+n_0-i} \, \Gamma_{i-1}^{k-1} (t^{n+1} p_k(t))^{(k+1-2i)} < +\infty \quad (i = 1, \ldots, n_0)$$

and

(4.8)
$$\int^{+\infty} t^{2n} |q(t)|^2 \, dt < +\infty .$$

Then for sufficiently large a the problem (0.3), (4.7) *has at least one solution.*

We omit the proof of this theorem since it is quite similar to that of Theorem 2.2. The only difference is that instead of (2.5)–(2.7) one has to apply the identities (4.2)–(4.4) with $h(t) = t^{n+1}$.

According to Lemma 1.5, Theorem 4.2 implies the following statement.

Corollary. *Under the conditions of Theorem* 4.2 *the equation* (0.3) *has a* $(n_0 - 1)$-*parametric family of solutions vanishing at infinity together with its derivatives up to the order* $n_0 - 1$ *inclusively* (*to be precise, satisfying the conditions* (2.22)).

Theorem 4.3. *Let* $n = 2n_0 \geqq 4$,

$$(-1)^{n_0} (t^{n+1} p(t))' \to +\infty \quad \textit{for} \quad t \to +\infty,$$

and let the condition (4.8) *hold. Then for sufficiently large a the problem* (0.2), (4.7) *has one and only one solution.*

Proof. For sufficiently large a the solvability of the problem (0.2), (4.7) follows from Theorem 4.2. It remains to show that the solution is unique, i.e. we have to prove that if a is sufficiently large, then the differential equation (0.1) with the boundary conditions

(4.9)
$$u^{(i)}(a) = 0 \quad (i = 1, \ldots, n_0 - 1), \quad \int_a^{+\infty} t^{2k} |u^{(k)}(t)|^2 \, dt < +\infty \quad (k = 0, \ldots, n_0)$$

has only the zero solution.

Choose $a_0 > 0$ and $\lambda > n^3$ such that

$$(4.10) \qquad (-1)^{n_0} t^n p(t) > v_1 \quad \text{for} \quad t \geqq a_0 \,,$$

$$(4.11) \qquad \sum_{i=1}^{n_0-1} v_i \beta_i(\lambda, n_0) < 1 \,,$$

$$(-1)^{n_0} (t^{n+1} p(t))' > 2 + \sum_{i=1}^{n_0-1} v_i \alpha_i(\lambda, n_0) \quad \text{for} \quad t \geqq a_0$$

where

$$v_i = 2 \frac{(n+1)!}{(2i)!} \Gamma^{n-1}_{i-1}$$

and the constants $\alpha_i(\lambda, n_0)$ and $\beta_i(\lambda, n_0)$ are defined by the identities (1.8).

Admit that for a certain $a > a_0$ the problem (0.1), (4.9) has a nonzero solution u. According to Lemmas 1.1 and 1.4 and to the condition (4.10),

$$w(t) - w(a) + \sum_{i=0}^{n_0} (-1)^{n_0 - i} \frac{n!}{(2i)!} \Gamma^n_i \int_a^t \tau^{2i} |u^{(i)}(\tau)|^2 \, d\tau = \int_a^t \tau^n |p(\tau)| \, |u(\tau)|^2 \, d\tau$$

where

$$w(t) = \sum_{i=0}^{n_0-1} \sum_{j=i}^{n-1-i} (-1)^{n_0-1-j} \frac{n!}{(i+j+1)!} \Gamma^n_{ij} t^{1+i+j} u^{(i)}(t) \, u^{(j)}(t)$$

and

$$\liminf_{t \to +\infty} w(t) = 0 \,.$$

Therefore

$$(4.12) \qquad \int_a^{+\infty} t^n |p(t)| \, |u(t)|^2 \, dt < +\infty \,.$$

Due to Lemma 1.1, multiplication of both sides of (0.1) by $(-1)^{n_0-1} 2t^{n+1} u'(t)$ and integration from a to t yield

$$(4.13) \qquad \sum_{i=1}^{n_0} \sum_{j=i}^{n-i} v_{ij} t^{i+j+1} u^{(i)}(t) \, u^{(j)}(t) = a^{n+1} |p(a)| \, |u(a)|^2 + a^{n+1} |u^{(n_0)}(a)|^2 -$$

$$- t^{n+1} |p(t)| \, |u(t)|^2 + (-1)^{n_0} \int_a^t (\tau^{n+1} p(\tau))' \, |u(\tau)|^2 \, d\tau + v_{n_0} \int_a^t \tau^{2n_0} |u^{(n_0)}(\tau)|^2 \, d\tau +$$

$$+ \sum_{i=1}^{n_0-1} (-1)^{n_0-i} v_i \int_a^t \tau^{2i} |u^{(i)}(\tau)|^2 \, d\tau$$

where

$$v_{ij} = (-1)^{n_0-j} 2 \frac{(n+1)!}{(i+j+1)!} \Gamma^{n-1}_{i-1\, j-1} \,.$$

By Lemma 1.3 and the inequalities (4.11) there exists $a_1 > a_0$ such that

(4.14)
$$(-1)^{n_0} \int_a^t (\tau^{n+1} p(\tau))' |u(\tau)|^2 \, d\tau + \nu_{n_0} \int_a^t \tau^{2n_0} |u^{(n_0)}(\tau)|^2 \, d\tau > \delta - \nu_1 a |u(a)|^2 + $$
$$+ \sum_{i=1}^{n_0-1} \nu_i \int_a^t \tau^{2i} |u^{(i)}(\tau)|^2 \, d\tau \quad \text{for} \quad t \geqq a_1$$

where
$$\delta = \int_a^{+\infty} |u(\tau)|^2 \, d\tau > 0 \, .$$

According to (4.10) and (4.14) the relation (4.13) implies
$$\sum_{i=1}^{n_0} \sum_{j=i}^{n-i} \nu_{ij} t^{i+j} u^{(i)}(t) \, u^{(j)}(t) \geqq \frac{\delta}{t} - t^n |p(t)| \, |u(t)|^2 \quad \text{for} \quad t \geqq a_1 \, .$$

Due to Lemma 1.1 by integrating this inequality from a_1 to t we obtain

(4.15)
$$w_1(t) \geqq w_1(a) + \delta \ln \frac{t}{a_1} - \int_{a_1}^t \tau^n |p(\tau)| \, d\tau - \sum_{i=1}^{n_0} c_i \int_{a_1}^t \tau^{2i} |u^{(i)}(\tau)|^2 \, d\tau \quad \text{for} \quad t \geqq a_1$$

where
$$w_1(t) = \sum_{i=1}^{n_0-1} \sum_{j=i}^{n-1-i} c_{ij} t^{i+j+1} u^{(i)}(t) \, u^{(j)}(t)$$

and c_{ij} and c_i are constants.

By (4.9) and (4.12) the inequality (4.15) implies
$$\lim_{t \to +\infty} w_1(t) = +\infty \, .$$

On the other hand, by Lemma 1.4
$$\liminf_{t \to +\infty} w_1(t) = 0 \, .$$

This contradiction shows that for any $a \geqq a_0$ the problem (0.1), (4.9) has only the zero solution. This completes the proof.

5. ON SOLUTIONS OF THE EQUATION (0.1)

Let n_0 be an entire part of $n/2$. We denote by $U_p^{(n)}$ the set of all solutions of the equation (0.1) satisfying the condition
$$\int^{+\infty} |u^{(n_0)}(t)|^2 \, dt < +\infty$$

and by $V_p^{(n,\sigma)}$ the set of all solutions of the same equation satisfying the condition

$$\int^{+\infty} t^{2i-\sigma}|u^{(i)}(t)|^2\, \mathrm{d}t < +\infty \quad (i = 0, 1, \ldots, n_0)\,.$$

It is obvious that $U_p^{(n)}$ and $V_p^{(n,\sigma)}$ are linear spaces. According to Lemma 1.5 any solution $u \in V_p^{(n,0)}$ vanishes at infinity together with its derivatives up to the order $n_0 - 1$ inclusively and, moreover,

$$\lim_{t\to+\infty} t^{i+1/2}\, u(t) = 0 \quad (i = 0, \ldots, n_0 - 1)\,. \tag{5.1}$$

Theorems 2.3, 2.4, 3.4 and 4.3 imply the following statements.

Theorem 5.1. *If the inequality*

$$(-1)^{n-n_0-1}\, p(t) \geqq 0$$

holds for sufficiently large t, then

$$\dim U_p^{(n)} = n_0\,. \tag{5.2}$$

Furthermore, if

$$(-1)^{n-n_0-1}\, t^n\, p(t) \to +\infty \quad \textit{for} \quad t \to +\infty\,, \tag{5.3}$$

then we have

$$U_p^{(n)} = V_p^{(n,0)} \tag{5.4}$$

together with (5.2).

Theorem 5.2. *If* $n = 2n_0 + 1$ *and*

$$(-1)^{n_0+1}\, t^n\, p(t) \to +\infty \quad \textit{for} \quad t \to +\infty\,, \tag{5.5}$$

then for any $\sigma > n$

$$\dim V_p^{(n,\sigma)} = n_0 + 1\,.$$

Furthermore, if

$$\liminf_{t\to+\infty} (-1)^{n_0+1}\, t^{2-2\sigma+n}\, p(t) > 0 \tag{5.6}$$

for a certain $\sigma > n$, *then every solution* $u \in V_p^{(n,\sigma)}$ *vanishes at infinity.*

Theorem 5.3. *If* $n = 2n_0 \geqq 4$, $p \in \tilde{C}_{\mathrm{loc}}(R_+)$ *and*

$$(-1)^{n_0}\, (t^{n+1}\, p(t))' \to +\infty \quad \textit{for} \quad t \to +\infty\,,$$

then

$$\dim V_p^{(n,0)} = n_0 - 1\,.$$

We shall separately consider the case when

$$\limsup_{t\to+\infty} |p(t)| < +\infty . \tag{5.7}$$

It turns out that, if together with (5.7) either the condition (5.3) holds or $n = 2n_0 + 1$ and

$$\liminf_{t\to+\infty} (-1)^{n_0+1}\, p(t) > 0 , \tag{5.8}$$

then each solution of the equation (0.1) is either unbounded or vanishes at infinity as well as its derivatives up to the order n inclusively.

Theorem 5.4. *If the conditions (5.3) and (5.7) are fulfilled, then the set of all bounded solutions of the equation (0.1) forms a n_0-dimensional linear space which coincides with $V_p^{(n,0)}$. Moreover, each bounded solution of the equation (0.1) satisfies the conditions*

$$\lim_{t\to+\infty} t^{i+1/2}\, u^{(i)}(t) = 0 \quad (i = 0, \dots, n_0 - 1), \tag{5.9}$$
$$\lim t^{[(n-i)(n_0+1)]/(n-n_0+1)+1/2}\, u^{(i)}(t) = 0 \quad (i = n_0, \dots, n) .$$

Proof. According to Theorem 5.1, the conditions (5.2) and (5.4) hold. Thus any solution $u \in U_p^{(n)}$ satisfies the conditions (5.1). So by (5.7) we have

$$\lim_{t\to+\infty} t^{1/2}\, u^{(n)}(t) = 0 . \tag{5.10}$$

However, according to the Kolmogorov-Horny inequalities ([3], p. 393), (5.1) and (5.10) imply the conditions (5.9).

Hence it remains to show that the set of all bounded solutions of the equation (0.1) coincides with $U_p^{(n)}$. Due to Theorem 2.3, it will suffice to prove that for large a the equation (0.1) with the boundary conditions

$$u^{(i)}(a) = 0 \quad (i = 0, \dots, n_0 - 1), \qquad \sup\{|u(t)| : t \in R_+\} < +\infty \tag{5.11}$$

has only the zero solution.

Choose $a_0 > 0$ such that

$$(-1)^{n-n_0-1}\, p(t) > 0 \quad \text{for} \quad t \geqq a_0 \tag{5.12}$$

and

$$\sup\{|p(t)| : t \geqq a_0\} < +\infty . \tag{5.13}$$

Let $a \geqq a_0$, and let u be a solution of the problem (0.1), (5.11). Then by (5.13)

$$\sup\{|u^{(n)}(t)| : t \geqq a_0\} < +\infty .$$

Thus according to the Kolmogorov-Horny inequalities, there exists a positive constant c_0 such that

$$(5.14) \qquad |u^{(i)}(t)| \leqq c_0 \quad \text{for} \quad t \geqq a_0 \quad (i = 0, \ldots, n).$$

If $n = 2n_0$, then according to (5.11), (5.12) and (5.14) we have

$$\int_a^t |u^{(n_0)}(\tau)|^2 \, d\tau + \int_a^t |p(\tau)| \, |u(\tau)|^2 \, d\tau = \sum_{i=0}^{n_0-1} (-1)^{n_0-1-i} u^{(n-1-i)}(t) \, u^{(i)}(t) \leqq n_0 c_0^2$$
$$\text{for} \quad t \geqq a .$$

So u is a solution of the problem (0.1), (2.23_0). Therefore, according to Theorem 2.3, $u(t) \equiv 0$.

Now we show that $u(t) \equiv 0$ also in the case of $n = 2n_0 + 1$. Admit on the contrary that $u(t) \not\equiv 0$. Then the identity

$$\sum_{i=0}^{n_0-1} (-1)^{n_0+i} u^{(n-1-i)}(t) \, u^{(i)}(t) + \tfrac{1}{2}|u^{(n_0)}(t)|^2 = \tfrac{1}{2}|u^{(n_0)}(a)|^2 + \int_a^t |p(\tau)| \, |u(\tau)|^2 \, d\tau$$

implies

$$(5.15) \qquad \int_a^{+\infty} |p(\tau)| \, |u(\tau)|^2 \, d\tau < +\infty$$

and

$$(5.16) \qquad \sum_{i=0}^{n_0-1} (-1)^{n_0+i} u^{(n-i)}(t) \, u^{(i)}(t) + \tfrac{1}{2}|u^{(n_0)}(t)|^2 \geqq \delta \quad \text{for} \quad t \geqq a_1$$

where $\delta = \frac{1}{2}\int_a^{+\infty} |p(\tau)| \, |u(\tau)|^2 \, d\tau > 0$ and the constant $a_1 > a$ is sufficiently large.

By (5.13) and (5.15)

$$\eta^2 = \int_a^{+\infty} |u^{(n)}(\tau)|^2 \, d\tau < +\infty .$$

Thus, taking into consideration (5.14), from the identities

$$\int_a^t |u^{(n_0+1)}(\tau)|^2 \, d\tau = \sum_{i=1}^{n_0} (-1)^{n_0-i} u^{(n-i)}(t) \, u^{(i)}(t) - u^{(n_0+1)}(a) \, u^{(n_0)}(a) +$$
$$+ (-1)^{n_0} \int_a^t u^{(n)}(\tau) \, u'(\tau) \, d\tau$$

and

$$\int_a^t |u^{(n_0)}(\tau)|^2 \, d\tau = u^{(n_0)}(t) \, u^{(n_0-1)}(t) - \int_a^t u^{(n_0+1)}(\tau) \, u^{(n_0-1)}(\tau) \, d\tau$$

we derive

$$\int_a^t |u^{(n_0+1)}(\tau)|^2 \, d\tau \leqq nc_0^2 + \left(\int_a^t |u^{(n)}(\tau)|^2 \, d\tau\right)^{1/2} \left(\int_a^t |u'(\tau)|^2 \, d\tau\right)^{1/2} \leqq (nc_0^2 + \eta c_0) \, t^{1/2}$$
$$\text{for} \quad t \geqq a$$

and

$$(5.17)\int_a^t |u^{(n_0)}(\tau)|^2\, d\tau \leqq c_0^2 + c_0 t^{1/2}\left(\int_a^t |u^{(n_0+1)}(\tau)|^2\, d\tau\right)^{1/2} \leqq \eta_0 t^{3/4} \quad \text{for} \quad t \geqq a$$

where $\eta_0 = c_0^2 + c_0(nc_0^2 + \eta c_0)^{1/2}$.

On the other hand, by integrating the inequality (5.16) from a_1 to t and applying (5.14), we obtain

$$(n_0 + \tfrac{1}{2})\int_{a_1}^t |u^{(n_0)}(\tau)|^2\, d\tau \geqq \delta(t - t_1) - n_0(n_0 + 1)c_0^2 \quad \text{for} \quad t \geqq a_1\,,$$

which contradicts the inequality (5.17). This contradiction shows that $u(t) \equiv 0$. Therefore, for any $a \geqq a_0$ the problem (0.1), (5.11) has only zero solution. This completes the proof.

In the case of $n = 4$ Theorems 5.1 and 5.4 imply Theorems 1–6 from the paper by M. Švec [11]. To verify this fact it suffices to note that under the conditions imposed on the function p in [11] the set S introduced there coincides with $U_p^{(n)} \setminus \{0\}$.

Theorem 5.5. *Let $n = 2n_0 + 1$, and let the conditions (5.7) and (5.8) hold. Then the set of all bounded solutions of the equation (0.1) forms a $(n_0 + 1)$-dimensional linear space which coincides with $V_p^{(n,\sigma)}$ for any $\sigma > n$. Moreover, each bounded solution of the equation (0.1) satisfies the conditions*

$$(5.18) \qquad \lim_{t \to +\infty} u^{(i)}(t) = 0 \quad (i = 0, 1, \ldots, n)\,.$$

Proof. Let $a > 0$ be so large that

$$(-1)^{n_0+1}\, p(t) \geqq 0 \quad \text{for} \quad t \geqq a\,.$$

Consider the equation (0.1) under the boundary conditions

$$(5.19) \qquad u^{(i)}(a) = c_i \;(i = 0, \ldots, n_0)\,, \quad \sup\{|u(t)| : t \in R_+\} < +\infty\,.$$

According to Theorem 3.3 for any $c_i \in R$ $(i = 0, \ldots, n_0)$ the problem (0.1), (3.14) has a unique solution which satisfies the conditions (5.18) and, obviously, is also a solution of both the problems (0.1), (5.19) and (0.1), (3.1) where $\varepsilon = \sigma - n$. By Theorem 3.4 the problem (0.1), (3.1) has a unique solution. Thus

$$V_p^{(n,\sigma)} = \{u(\cdot\,; c_0, \ldots, c_{n_0}) : (c_0, \ldots, c_{n_0}) \in R^{n_0+1}\}$$

where $u(t) = u(t; c_0, \ldots, c_{n_0})$ is a solution of the problem (0.1), (3.14). So it remains to show that for any $c_i \in R$ $(i = 0, \ldots, n_0)$ each solution u of the problem (0.1), (5.19) is also a solution of the problem (0.1), (3.14), i.e. satisfies the condition (5.15).

According to the Kolmogorov-Horny inequalities, (5.16) and (5.19) imply the boundedness of the functions $u^{(i)}$ $(i = 0, \ldots, n - 1)$. Therefore, (5.15) follows from the identity

$$\int_a^t |p(\tau)|\,|u(\tau)|^2\,d\tau = \sum_{i=0}^{n_0-1} (-1)^{n_0+1-i}\, u^{(n-1-i)}(t)\, u^{(i)}(t) - \tfrac{1}{2}|u^{(n_0)}(t)|^2 + \tfrac{1}{2}|u^{(n_0)}(a)|^2\,.$$

This completes the proof.

Let $Z_p^{(n)}$ be the set of all solutions of the equation (0.1) satisfying the condition

$$\lim_{t\to+\infty} u(t) = 0\,.$$

Theorems 5.1–5.5 imply the following statements.

If the condition (5.3) holds, then $\dim Z_p^{(n)} \geqq n_0$, and if the condition (5.7) is fulfilled together with (5.3), then $\dim Z_p^{(n)} = n_0$.

If $n = 2n_0 + 1$ and the condition (5.6) holds, then $\dim Z_p^{(n)} \geqq n_0 + 1$, and if the conditions (5.7) and (5.8) are fulfilled instead of (5.6), then $\dim Z_p^{(n)} = n_0 + 1$.

Under the conditions of Theorem 5.3 $\dim Z_p^{(n)} \geqq n_0 - 1$.

These statements together with the theorems of M. Matell on asymptotic representation of solutions of high order linear differential equations [7] (see also [4]) and with the theorems of H. Milloux [8], Armellini-Tonelli-Sansone ([9], p. 56–63) and J. Kurzweil [5] give ground to formulate the following conjectures.

1. If the condition (5.3) holds, then $\dim Z_p^{(n)} = n_0$.

2. If $n = 2n_0 + 1$ and the condition (5.5) holds, then $\dim Z_p^{(n)} = n_0 + 1$.

3. If $n = 2n_0$, $p \in \tilde{C}_{\mathrm{loc}}(R_+)$, $(-1)^{n_0} p'(t) \geqq 0$, $(-1)^{n_0} p(t) \to +\infty$ for $t \to +\infty$ then $\dim Z_p^{(n)} \geqq n_0$.

4. Let $n = 2n_0$, $p \in \tilde{C}_{\mathrm{loc}}(R_+)$, $(-1)^{n_0} p'(t) \geqq 0$, $(-1)^{n_0} p(t) \to +\infty$ for $t \to +\infty$ and

$$\int_I d \ln a(t) = +\infty$$

for any open set $I \subset R_+$ satisfying the condition $\operatorname{mes}(I \cap]t, t+1[) \to 1$ for $t \to +\infty$. Then $\dim Z_p^{(n)} = n_0 + 1$.

Note that in the case of $n = 2$ the statements 3 and 4 are not mere conjectures. They have been proved by H. Milloux [8] and J. Kurzweil [5] respectively.

References

[1] *Biernacki M.:* Sur l'équation differentialle $Y^{(4)} + A(x)\,Y = 0$. Annales Univ. M. Curie-Sklodowska *6* (1952), 65–78.

[2] *Conti R.:* Recent trends in the theory of boundary value problems for ordinary differential equations. Boll. Unione Mat. Ital., *22* (1967), 135–178.

[3] *Харди Г. Г., Литтльвуд Дж. Е., Полиа Г.:* Неравенства. Издат. иностр. литературы, Москва, 1948.

[4] *Кигурадзе И. Т.:* Об асимптотическом представлении решений линейных дифференциальных уравнений. Труды Тбилисского университета *102* (1964), 149–167.

[5] *Kurzweil J.:* Sur l'équation $x'' + f(t)\,x = 0$. Čas. Pěst. Mat. *82* (1957), 218–226.

[6] *Курцвейль Я., Ворел З.:* О непрерывной зависимости решений дифференциальных уравнений от параметра. Чехослов. матем. ж. *82* (1957), 568—583.
[7] *Matell M.:* Asymptotische Eigenschaften gewisser linearer Differentialgleichungen. Appelbergs Boktryckeri Aktiebolag, Uppsala, 1924.
[8] *Milloux H.:* Sur l'équation differentielle $x'' + A(t)\,x = 0$. Prace Mat. Fiz. *41* (1934), 39—54.
[9] *Сансоне Дж.:* Обыкновенные дифференциальные уравнения, том 2, Издат. иностран. литературы, Москва, 1954.
[10] *Соболь И. М.:* Об асимптотическом поведении решений линейных дифференциальных уравнений. Докл. АН СССР *61* (1948), 219—222.
[11] *Švec M.:* Sur le comportement asymptotique des intégrales de l'équation differentielle $Y^{(4)} + Q(x)\,Y = 0$. Чехослов. матем. журнал *8* (*83*), 1958, 230—245.

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