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EXTENDED SHANNON ENTROPIES I

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We examine functionals (called extended Shannon entropies) which are defined for all probability spaces equipped with a measurable metric, coincide with the Shannon entropy for finite probability spaces endowed with the metric $\varrho(x, y) = 1$ for $x \neq y$, and satisfy certain natural conditions. It turns out that there do exist functionals of this kind which, in addition, possess various reasonable properties. An important special case of such functionals is investigated in some detail.

There are various reasons for examining extended Shannon entropies. First of all, the following simple fact seems to be important. Suppose we consider some kind of real situations, say, with finitely many possible outcomes (alternative cases, variants, etc.). Suppose we want to assign to every situation a non-negative number expressing what can be intuitively conceived as its inherent diversity or complexity or "informational content" and so on. It is intuitively clear that the number assigned should depend not only on the probabilities of possible outcomes, but also on the degree of their diversity, expressed perhaps by some kind of mutual distances of possible outcomes.

Another reason (mentioned already in [3]) stems from the fact that the entropy of finite probability spaces, the ε-entropy of metric spaces and the differential entropy have much in common. Therefore, it seems desirable to introduce a general notion from which all these concepts could be obtained (perhaps slightly modified) in a natural way. (We note, however, that the relationship of extended Shannon entropies to the ε-entropy and differential entropy will not be examined in the present paper.)

Finally, one of the reasons for investigating extended Shannon entropies is based on the following fact. In some respects, e.g. in questions of human information processing, the applications of the (Shannon) information theory have not been as fruitful as had been expected. It is possible that a broader concept will be more efficient. However, the applications will not be touched in the present paper.

The present Part I contains only basic results concerning the properties of extended Shannon entropies. Thus, the main result consists in introducing functionals φ on the class of all metrized probability spaces (actually, on a wider class, namely that of sets equipped with a finite measure and a measurable semimetric) such that (I) if P is a finite probability space $\langle\{1, \ldots, n\}, \mu\rangle$ endowed with the metric ϱ such that

$\varrho(x, y) = 1$ if $x \neq y$, then $\varphi(P) = -\sum p_i \log p_i$ where $p_i = \mu\{i\}$; (II) ϱ is continuous, in a specified sense; (III) $\varphi(P) > 0$ unless P is trivial, and $\varphi(P) < \infty$ whenever P is bounded in a specified, not too narrow sense; (IV) φ has some other convenient properties.

It is not quite easy to construct appropriate functionals φ, and the proof that they do possess properties in question is rather complicated. Therefore, the proof (except for that of (III) which will be contained in Part II) is presented, in Sections 3–5, in the form of a rather long string of propositions, some of which are of independent interest whereas many are of auxiliary character.

Many results contained in the present Part I and the forthcoming Part II have been stated, sometimes without proof, in [3]. However, this has been done for functionals C^* and C (see [3], 3.1) only. Now, we state the pertinent results with full proofs for the general case of C_τ^* and C_τ (see 3.16).

To the author's knowledge, the problem – fairly natural – of extending the Shannon entropy has been given little attention so far. Except for two notes (and a correction) of the author, there seems to exist only one paper [1], by B. Forte. However, B. Forte's approach is quite different from that presented here and concerns real random variables with finite range only.

Relatively little attention paid to the extension problem seems to have two causes. Firstly, there has not been too much investigation of probability spaces equipped with an arbitrary measurable metric (this contrasts with the relatively extensive examination of probability on e.g. Banach spaces). Secondly, the techniques employed (at least in the present paper) in extending the Shannon entropy are often considerably different from those commonly used in information theory – though, on the other hand, dyadic expansions (see Section 4) are closely related to C. F. Picard's questionnaires (see e.g. [5]).

Due to the circumstances described above, and also to the fact that all pertinent propositions on the Shannon entropy are explicitly stated (partly also proved), the paper contains few references.

Part I is organized as follows. In Section 1 we introduce semimetrized measure spaces, their partitions, etc. The concept of an extended Shannon entropy is introduced, after a detailed discussion, in Section 2. In Section 3 the functionals C_τ^* and C_τ are introduced and examined (it will be shown later that, under certain conditions, every C_τ is an extended Shannon entropy). In Section 4, dyadic expansions (see also [3]) are considered. Section 5 contains propositions concerning continuity of C_τ^* and C_τ. In Section 6 we summarize the main results.

To keep the length of Part I within reasonable limits, most examples and conterexamples are deferred to Part II.

The present paper and [3] overlap in some parts. However, the approach is different. In [3], two functionals, C^* and C, have been defined by means of dyadic expansions and their properties have been investigated. In the present paper, we start from a fairly general definition of extended Shannon entropies. Then we introduce the functionals C_τ^* and C_τ. Dyadic expansions appear later, as means for expressing (and, whenever possible, calculating) C_τ^* and C_τ, and it is shown that, for $\tau = r$, C_τ^* and C_τ coincide with C^* and C from [3].

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1

In this section we first recall some basic concepts and introduce some notational conventions. Then we present the definition of semimetrized measure spaces and of some related notions.

1.1. A) "Mapping" will have a somewhat broader meaning than usual: the domain, and also the range, of a mapping can be proper classes. – B) The domain of a mapping f is denoted by $\operatorname{dom} f$. – C) If f is a mapping and A is a class, then $f \upharpoonright A$ denotes the restriction of f to $A \cap \operatorname{dom} f$, i.e. $\operatorname{dom}(f \upharpoonright A) = A \cap \operatorname{dom} f$, $(f \upharpoonright A)(x) = f(x)$ for every $x \in A \cap \operatorname{dom} f$. – D) If f is a mapping, $\operatorname{dom} f = T \times T$, $S \subset T$, then we often write $f \upharpoonright S$ instead of $f \upharpoonright (S \times S)$.

1.2. The letter $\boldsymbol{N}$ denotes the set $\{0, 1, 2, \ldots,\}$, $\boldsymbol{R}$ denotes the set of all reals. We put $\overline{\boldsymbol{R}} = \{-\infty\} \cup \boldsymbol{R} \cup \{\infty\}$, $\boldsymbol{R}_+ = \{x \in \boldsymbol{R} : x \geqq 0\}$, $\overline{\boldsymbol{R}}_+ = \{x \in \overline{\boldsymbol{R}} : x \geqq 0\}$.

1.3. If $a, b \in \overline{\boldsymbol{R}}_+$, then $a + b$, $a - b$ and ab are defined in the usual way; in particular, $0 \,.\, \infty = 0$, $\infty - \infty$ is not defined. We put $0/0 = 0$; $a/0 = \infty$ if $a \in \overline{\boldsymbol{R}}_+$, $a > 0$; $a/\infty = 0$ if $a \in \boldsymbol{R}_+$; ∞/∞ is not defined.

1.4. A) Finite sequences are denoted by expressions like $(a_0, \ldots, a_n)$, $(a_i : i < n)$, etc. However, if, e.g., (X, μ) or (X, ϱ, μ), etc., is conceived as a set equipped with a structure, we prefer symbols like $\langle X, \mu\rangle$, $\langle X, \varrho, \mu\rangle$, etc. – B) Indexed sets (families) are denoted by expressions like $(x_t : t \in T)$ where T is the indexing set. Instead of e.g. $(x_t\colon t \in \{t \in T\colon \psi\})$, where ψ is a formula in which t occurs as a free variable, we often write $(x_t\colon t \in T, \psi)$, etc.

1.5. A mapping f of a class into $\overline{\boldsymbol{R}}$ is called a *function* or a *functional*; if $f(x) \in \boldsymbol{R}$ for each $x \in \operatorname{dom} f$, then f is called a *real-valued function* or a *real-valued functional*. As a rule, the word "functional" is used if $\operatorname{dom} f$ is a proper class or consists of sets equipped with a structure (e.g. of semimetrized measure spaces, see below).

1.6. Convention. If f, g are functions, then $f \geqq g$ means that $\operatorname{dom} f = \operatorname{dom} g$ and $f(x) \geqq g(x)$ for every $x \in \operatorname{dom} f$. If f is a non-negative function, $a \in \boldsymbol{R}_+$, then af

has its usual meaning. If f, g are non-negative functions, then $f + g$, fg are defined, in the usual way, iff $\operatorname{dom} f = \operatorname{dom} g$, whereas, in accordance with 1.3, $f - g$ and f/g are defined, iff $\operatorname{dom} f = \operatorname{dom} g$ and $f(x) = g(x) = \infty$ for no $x \in \operatorname{dom} f$.

1.7. Notational conventions. A) In formulas, we often omit parentheses $(\ldots)$, provided there is no danger of misunderstanding. Thus, e.g., if f is a mapping, $x \in \operatorname{dom} f$, we often write fx instead of $f(x)$. However, as a rule, we do not abbreviate, for instance, $C(P)$ to CP or $d(P)$ to dP (in the last case, the reason is that the letter d also occurs in expressions like $\int f \,\mathrm{d}\mu$). — B) As usual, dots are often omitted, but we never do it in expressions defined in 1.13 and 1.24. — C) We never omit brackets of the form $[\ldots]$, $\{\ldots\}$, or $\langle\ldots\rangle$, except for one case only: instead of e.g. $\{q\}$ we sometimes write q; thus $\mu(\{q\})$, where μ is a measure, can be abbreviated to $\mu\{q\}$ or to $\mu(q)$ or even to μq.

1.8. If X is a set, then $\exp X$ denotes the power set $\{Y\colon Y \subset X\}$, and $\operatorname{card} X$ denotes the cardinality of X.

1.9. Convention. If Q is a non-void set, $\mathscr{B}$ is a σ-algebra on Q and μ is a σ-additive non-negative real-valued function on $\mathscr{B}$, then μ is called a *finite measure* or simply a *measure* on Q. If μ is a measure on Q, then every $X \in \operatorname{dom} \mu$ will be called *μ-measurable* or *measurable with respect to* μ, and $\langle Q, \mu\rangle$ will be called a *measure space*. If $\mu Q = 1$, then $\langle Q, \mu\rangle$ will also be called a *probability space*.

Remarks. (1) Measures in the usual (broader) sense, i.e. possibly assuming an infinite value, will be called $\overline{\boldsymbol{R}}$-measures. In this note, they seldom occur. — (2) Measure spaces in the sense just defined are, of course, a special case of measure spaces in the usual sense, i.e. of $\langle Q, \mu\rangle$ where μ is an $\overline{\boldsymbol{R}}$-measure on Q.

1.10. If μ, ν are measures on Q, $a \in \boldsymbol{R}_+$, then $a\mu$ (see 1.6) is a measure, $\mu + \nu$ is a measure provided it is defined (i.e. provided $\operatorname{dom} \mu = \operatorname{dom} \nu$, see 1.6), and $\mu - \nu$ is a measure provided $\mu \geqq \nu$ (which implies, see 1.6, $\operatorname{dom} \mu = \operatorname{dom} \nu$).

1.11. A) If μ is a measure on Q, then $\bar{\mu}$ or $[\mu]$ denotes the *completion* of μ, i.e. $\bar{\mu}$ is the measure defined as follows: $\operatorname{dom} \bar{\mu}$ consists of all sets X of the form $X = Y \cup Z$, where $Y \in \operatorname{dom} \mu$, $Z \subset U$ for some $U \in \operatorname{dom} \mu$ such that $\mu U = 0$; for every X of this form, $\bar{\mu}X = \mu Y$. — B) If $\bar{\mu} = \mu$, we will call μ *complete*.

1.12. Let μ be a measure on Q. Let $\mathscr{B}$ be a σ-algebra on a set T. A mapping $f\colon Q \to T$ will be called *$(\mu, \mathscr{B})$-measurable* (or simply *μ-measurable*) if $f^{-1}(B) \in \operatorname{dom} \mu$ whenever $B \in \mathscr{B}$. In particular, a function $f\colon Q \to \overline{\boldsymbol{R}}$ will be called *μ-measurable* if it is $(\mu, \mathscr{B})$-measurable where $\mathscr{B}$ is the σ-algebra of Borel subsets of $\overline{\boldsymbol{R}}$.

1.13. A) Let μ be a measure on Q. Let f be a $\bar{\mu}$-measurable function on Q. Assume that $\bar{\mu}\{q \in Q : f(q) < 0\} = 0$ and $\int_Q f \,\mathrm{d}\mu < \infty$. For $X \in \operatorname{dom} \mu$, put $\nu(X) = \int_X f \,\mathrm{d}\mu$.

Then ν is a measure on Q, dom ν = dom μ. The measure ν will be denoted by $f . \mu$. — B) If μ is a measure on Q and $B \in \text{dom } \bar{\mu}$, then $i_B . \mu$, where i_B is the indicator function of B, i.e. $i_B(x) = 1$ for $x \in B$, $i_B(x) = 0$ for $x \in Q \setminus B$, will be denoted by $B . \mu$. — C) Clearly, if $X \in \text{dom } \mu$, then $(B . \mu)(X) = \bar{\mu}(B \cap X)$.

1.14. Let μ_i, $i = 1, 2$, be a measure on Q_i. Then $\mu_1 \times \mu_2$ will denote the product of μ_1 and μ_2, i.e. the measure μ on $Q_1 \times Q_2$ defined by the following conditions: (1) dom μ is the smallest σ-algebra containing all $X_1 \times X_2$, where $X_i \in \text{dom } \mu_i$, (2) if $X_i \in \text{dom } \mu_i$, $i = 1, 2$, then $\mu(X_1 \times X_2) = \mu(X_1) . \mu(X_2)$.

1.15. Definition. If Q is a set and ϱ is a non-negative realvalued function on $Q \times Q$ such that $\varrho(x, y) = \varrho(y, x)$ for all $x, y \in Q$ and $\varrho(x, x) = 0$ for all $x \in Q$, then ϱ will be called *semimetric* on Q and $\langle Q, \varrho \rangle$ will be called a *semimetric space.*

1.16. Convention. If Q is a set and $a \in \boldsymbol{R}_+$, then a_Q (or simply a) will denote the semimetric on Q defined as follows: $a_Q(x, y) = a$ if $x \neq y$, $a_Q(x, x) = 0$ for every $x \in Q$.

1.17. Basic definition. Let Q be a non-void set. Let ϱ and μ be, respectively, a semimetric and a measure on Q. If ϱ is $[\mu \times \mu]$-measurable ,then $P = \langle Q, \varrho, \mu \rangle$ will be called a *semimetrized measure space* (a *semimetrized probability space* if $\mu Q = 1$) The set Q will be called the *underlying set* of P and will be denoted by $|P|$. — Cf. [3], 1.3.

1.18. Definition. A semimetrized measure space $P = \langle Q, \varrho, \mu \rangle$ will be called *finite* if $|P|$ is finite, *separated* if $\langle Q, \mu \rangle$ is separated, i.e. if, for any $x \in Q$, $y \in Q$, $x \neq y$, there exists a set $X \in \text{dom } \mu$ such that $x \in X$, y non $\in X$.

1.19. Conventions. Semimetrized measure spaces will also be called, for short, *W-spaces* (sometimes simply "spaces"). Finite separated W-spaces will also be called *FW-spaces*. The class of all W-spaces will be denoted by $\mathfrak{W}$, that of all FW-spaces by $\mathfrak{W}_F$. If Q is a finite non-void set, then $\mathfrak{W}_F(Q)$ will denote the set of all FW-spaces of the form $\langle Q, \varrho, \mu \rangle$.

1.20. Remarks. 1) In [3], the abbreviated name for semimetrized measure spaces was WM-spaces, the class of all WM-spaces was denoted by $\{WM\}$, etc. Here we prefer a shorter notation. — 2) Nonseparated finite W-spaces are of minor importance and will occur only rarely in this paper. Most propositions on FW-spaces can be easily transformed to the corresponding propositions concerning finite W-spaces.

1.21. Examples. A) Let $Q \subset R^n$ be non-void bounded Lebesgue measurable. Let ϱ be any of the usual metrics on Q. Let λ be the Lebesgue measure on Q. Then

$\langle Q, \varrho, \lambda \rangle$ is a W-space. — B) Let $\langle Q, \mu \rangle$ be a probability space. Let x be a random variable on $\langle Q, \mu \rangle$, i.e. a $\bar{\mu}$-measurable real-valued function on Q. Put $P = \langle \boldsymbol{R}, \varrho, v \rangle$, where ϱ is the usual metric on $\boldsymbol{R}$, $vB = \bar{\mu}(x^{-1}(B))$ for any Borel set $B \subset \boldsymbol{R}$. Then $\langle \boldsymbol{R}, \varrho, v \rangle$ is a W-space. — C) If Q is a finite non-void set, ϱ is an arbitrary semimetric on Q, and μ is a measure on Q such that $\operatorname{dom} \mu = \exp Q$, then $\langle Q, \varrho, \mu \rangle$ is an FW-space. — D) Let Q be a non-void set. Let $a \in \boldsymbol{R}$, $a > 0$. Then $\langle Q, a, \mu \rangle$ is a W-space iff $\{(x, y) \in Q \times Q : x = y\}$ is $[\mu \times \mu]$-measurable. — E) Let Q be an uncountable set. If $X \subset Q$ is countable, put $\mu(X) = 0$; if $X \subset Q$, $Q \setminus X$ is countable, put $\mu(X) = 1$. Then μ is a complete measure on Q; however, $\mu \times \mu$ is not complete. It is easy to see that $\langle Q, 1, \mu \rangle$ is not a W-space. On the other hand, for every $x \in Q$, the function $y \mapsto \varrho(x, y)$ is μ-measurable.

1.22. Definition. If $S = \langle Q, \varrho, v \rangle$, $P = \langle Q, \varrho, \mu \rangle$ are W-spaces and $v \leqq \mu$, we will say that S is a *subspace* of P and write $S \leqq P$. If, in additiion, there exists a set $B \in \operatorname{dom} \bar{\mu}$ such that $v = B \,.\, \mu$, we will say that S is a *pure subspace* of P. — Cf. [3], 1.8.

1.23. Fact. *Let $P = \langle Q, \varrho, \mu \rangle$ be a W-space. Let v be a measure on Q, $v \leqq \mu$. Then $\langle Q, \varrho, v \rangle$ is a W-space, hence a subspace of P.*

Proof. It is easy to see that $\operatorname{dom} [v \times v] \supset \operatorname{dom} [\mu \times \mu]$.

1.24. Remark and notation. Let $P = \langle Q, \varrho, \mu \rangle$ be a W-space. Let f be a $\bar{\mu}$-measurable function on Q such that $f \,.\, \mu$ is defined (see 1.13). It is easy to show that $\langle Q, \varrho, f \,.\, \mu \rangle$ is a W-space. It will be denoted by $f \,.\, P$. — If $B \in \operatorname{dom} \bar{\mu}$, then $B \,.\, P$ will denote the space $\langle Q, \varrho, B \,.\, \mu \rangle$.

1.25. Fact. *If $\langle Q, \varrho, \mu_1 \rangle$, $\langle Q, \varrho, \mu_2 \rangle$ are subspaces of a W-space P, then $\langle Q, \varrho, \mu_1 + \mu_2 \rangle$ is a W-space.*

1.26. Definition. A) If $P_i = \langle Q, \varrho, \mu_i \rangle$, $i = 1, 2$, are W-spaces and there exists a W-space P such that $P_1 \leqq P$, $P_2 \leqq P$, then $\langle Q, \varrho, \mu_1 + \mu_2 \rangle$ (which is a W-space, by 1.25) will be called the *sum* of P_1 and P_2 and will be denoted by $P_1 + P_2$. — B) If $P_i = \langle Q, \varrho, \mu_i \rangle$, $i = 1, 2$, are W-spaces, $P_2 \leqq P_1$. then $\langle Q, \varrho, \mu_1 - \mu_2 \rangle$ (which is a W-space, by 1.23) will be denoted by $P_1 - P_2$. — C) If $P = \langle Q, \varrho, \mu \rangle$ is a W-space and $(P_k : k \in K)$, where $P_k = \langle Q, \varrho, \mu_k \rangle$, is a finite indexed set of subspaces of P, then we put $\sum_P(P_k\colon k \in K) = \langle Q, \varrho, \sum(\mu_k\colon k \in K) \rangle$ if $K \neq \emptyset$, $\sum_P(P_k\colon k \in K) = \langle Q, \varrho, 0_\mu \rangle$ (where $0_\mu = \mu - \mu$) if $K = \emptyset$. — D) In the case $K \neq \emptyset$, clearly, $\sum_P(P_k\colon k \in K)$ does not depend on the choice of P, and hence we omit the subscript P (provided, of course, that a space P satisfying $P_k \leqq P$ for all k does exist). If $K = \emptyset$, then $\sum_P(P_k\colon k \in K)$ depends on P, and the subscript P is omitted only if P is obvious from the context. — E) If $a \in R_+$ and $P = \langle Q, \varrho, \mu \rangle$ is a W-space, we put $aP = \langle Q, \varrho, a\mu \rangle$. Expressions like $\sum(a_k P_k\colon k \in K)$ are then introduced in the same way as $\sum(P_k\colon k \in K)$ above.

1.27. Remark. There are simple examples of W-spaces $\langle Q, \varrho, \mu_i \rangle$, $i = 1, 2$, satisfying dom μ_1 = dom μ_2 and such that $\langle Q, \varrho, \mu_1 + \mu_2 \rangle$ is not a W-space.

1.28.1. Notation. Let $P_i = \langle Q, \varrho, \mu_i \rangle$, $i = 1, 2$, be W-spaces. If there exists a W-space $P = \langle Q, \varrho, \mu \rangle$ such that $P_1 \leqq P$, $P_2 \leqq P$, then

(1) we denote by $d(P_1, P_2)$ the least $a \in \bar{\boldsymbol{R}}_+$ such that $[\mu_1 \times \mu_2] \{(x, y) \in Q \times Q : \varrho(x, y) > a\} = 0$;

(2) we put $E(P_1, P_2) = d(P_1 + P_2, \ P_1 + P_2)$ if $\mu_1 Q > 0$, $\mu_2 Q > 0$, and $E(P_1, P_2) = 0$ if $\mu_1 Q = 0$ or $\mu_2 Q = 0$;

(3) we put $\hat{r}(P_1, P_2) = \int_{Q \times Q} \varrho \, d(\mu_1 \times \mu_2)$, $r(P_1, P_2) = \hat{r}(P_1, P_2)/\mu_1 Q \cdot \mu_2 Q$.

We note that the integral in (3) does exist, for ϱ is $[\mu \times \mu]$-measurable, hence $[\mu_1 \times \mu_2]$-measurable.

1.28.2. Notation. If $P = \langle Q, \varrho, \mu \rangle$ is a W-space, then we put $wP = \mu Q$, $d(P) = d(P, P)$, $\hat{r}(P) = \hat{r}(P, P)$, $r(P) = r(P, P)$. The letter d will denote both the functional $(P_1, P_2) \mapsto d(P_1, P_2)$ introduced in 1.28.1 and the functional $P \mapsto d(P)$ defined on $\mathfrak{W}$. Similarly, $\hat{r}$ or r will denote both $(P_1, P_2) \mapsto \hat{r}(P_1, P_2)$ and $P \mapsto \hat{r}(P)$ or, respectively, $(P_1, P_2) \mapsto r(P_1, P_2)$ and $P \mapsto r(P)$. The letter E will denote the functional $(P_1, P_2) \mapsto E(P_1, P_2)$.

1.29. Fact. *Let $\langle Q, \varrho, \mu \rangle$ be a W-space. Let A be an atom of the σ-algebra* dom μ. *Then $d(A \, . \, P) = 0$, and if $\mu A > 0$, then $\varrho(x, y) = 0$ for all $x, y \in A$.*

Proof. If $\mu A = 0$, then $d(A) = 0$. Let $\mu A > 0$. Clearly, $A \times A \in \operatorname{dom}(\mu \times \mu)$ and if $X \subset A \times A$, $\emptyset \neq X \neq A \times A$, then X non $\in \operatorname{dom}(\mu \times \mu)$. Hence, if $Y \in \operatorname{dom}[\mu \times \mu]$, then either $(A \times A) \cap Y = \emptyset$ or $A \times A \subset Y$. Since $\varrho(x, x) = 0$ for $x \in A$, we get $\{(x, y) : \varrho(x, y) = 0\} \supset A \times A$, hence $d(A \, . \, P) = 0$.

1.30. Definition. Let P be a W-space. Let $\mathscr{U} = (U_k : k \in K)$, where K is finite non-void, be an indexed set of subspaces of P. If $\sum(U_k : k \in K) = P$, we will say that $\mathscr{U}$ is a *partition* of P. We will say that $\mathscr{U}$ is *pure* if U_k is a pure subspace of P for each $k \in K$. A partition $(U_k : k \in K)$ will be called *binary* if card $K = 2$. – Cf. [3], 1.10.

1.31.1. If M is a set, then a partition of M will always mean a finite partition, i.e. an indexed set $(M_x : x \in X)$, where X is finite non-void, $\bigcup(M_x : x \in X) = M$ and $M_x \cap M_y = \emptyset$ whenever $x, y \in X$, $x \neq y$.

1.31.2. Definition. Let P be a W-space. Let $\mathscr{U} = (U_k : k \in K)$ and $\mathscr{V} = (V_m : m \in M)$ be partitions of P. We will say that $\mathscr{U}$ *refines* $\mathscr{V}$ (or that $\mathscr{U}$ is a *refinement* of $\mathscr{V}$) if there exists a partition $(K_m : m \in M)$ of the set K such that $\sum(U_k : k \in K_m) = V_m$ for each $m \in M$. If, in addition, $(K_m : m \in M)$ can be chosen so that, for each

$m \in M$, $k \in K_m$, U_k is a pure subspace of V_m, then we will say that $\mathscr{U}$ is a *relatively pure refinement* of $\mathscr{V}$. If $\mathscr{U}$ refines $\mathscr{V}$ and $\mathscr{V}$ refines $\mathscr{U}$, we will say that $\mathscr{U}$ and $\mathscr{V}$ are *equivalent*.

1.32. Convention. If $(x_a : a \in A)$ and $(y_b : b \in B)$ are indexed sets and there exists a bijective mapping $f : A \to B$ such that, for any $a \in A$, $x_a = y_{fa}$, then we will say that $(x_a : a \in A)$ is *equal to* $(y_b : b \in B)$ *re-indexed*.

1.33. Fact. *Let P be a W-space. Then the relation defined by "$\mathscr{U}$ refines $\mathscr{V}$" is a transitive reflexive relation on the class of all partitions of P, and the relations defined by "$\mathscr{U}$ is equivalent to $\mathscr{V}$" and "$\mathscr{U}$ is equal to $\mathscr{V}$ re-indexed" are equivalence relations.*

1.34. We shall need the Radon-Nikodym Theorem. We recall it in the following form:

Let μ, ν be measures on Q. Assume that $\operatorname{dom} \mu = \operatorname{dom} \nu$ *and that $\mu X = 0$ implies $\nu X = 0$. Then there exists a $\bar{\mu}$-measurable function f on Q such that $\nu = f \,.\, \mu$. If $\nu \leqq \mu$, then there exists an f which possesses the properties just described and, in addition, satisfies $0 \leqq f(q) \leqq 1$ for all $q \in Q$.*

1.35. Fact. *Let $P = \langle Q, \varrho, \mu \rangle$ be a W-space and let $S \leqq P$. Then there exists a $\bar{\mu}$-measurable function f on Q such that $0 \leqq f(q) \leqq 1$ for all $q \in Q$ and $f \,.\, P = S$.*

1.36. Proposition. *If $\mathscr{U}, \mathscr{V}$ are partitions (pure partitions) of a semimetrized measure space P, then there exists a partition (pure partition) of P which refines both $\mathscr{U}$ and $\mathscr{V}$.* – Cf. [3], 2.14.

Proof. Let $P = \langle Q, \varrho, \mu \rangle$, $\mathscr{U} = (U_k : k \in K)$, $\mathscr{V} = (V_m : m \in M)$. By 1.35, there exist $\bar{\mu}$-measurable functions f_k, $k \in K$, g_m, $m \in M$, such that $U_k = f_k \,.\, P$, $V_m = g_m \,.\, P$, $0 \leqq f_k(q) \leqq 1$, $0 \leqq g_m(q) \leqq 1$ for all $q \in Q$, $k \in K$, $m \in M$. Put $h_{km} = f_k g_m$ for $(k, m) \in K \times M$. Put $T_{km} = h_{km} \,.\, P$. It is easy to see that $\mathscr{T} = (T_{km} : (k, m) \in K \times M)$ is a partition of P and that $\mathscr{T}$ refines both $\mathscr{U}$ and $\mathscr{V}$. – If $\mathscr{U}$ and $\mathscr{V}$ are pure, then we may assume that f_k and g_m assume the values 0 and 1 only. This implies that all T_{km} are pure subspaces.

1.37. Similarly as for many other kinds of sets equipped with a structure, quotients (with respect to a partition) can be introduced for W-spaces. It turns out that there is a lot of various reasonably defined quotients associated with a given partition (cf. Section 3). At this stage, we introduce only two kinds of quotients.

1.37.1. Definition. Let P be a W-space. Let $\tau = r$ or $\tau = E$. A partition $\mathscr{U} = (U_k : k \in K)$ of P will be called *τ-admissible* if $\tau(U_i, U_j) < \infty$ for $i, j \in K$, $i \neq j$. If $\mathscr{U} = (U_k : k \in K)$ is a τ-admissible partition of P, then the FW-space $\langle K, \sigma, \nu \rangle$, where $\sigma(i, j) = \tau(U_i, U_j)$ for $i, j \in K, i \neq j$, $v\{k\} = wU_k$ for each $k \in K$, will be denoted by $[\mathscr{U}]_\tau$. The space $[\mathscr{U}]_\tau$ will be called the *τ-quotient of P according to $\mathscr{U}$*. – Cf. [3], 1.12.

Convention. If $\tau = r$, then we sometimes write admissible instead of r-admissible and $[\mathscr{U}]$ instead of $[\mathscr{U}]_r$.

1.38. As suggested by 1.36, a filter can be formed, in a natural way, from partitions (pure partitions) of a W-space. We now state the relevant definitions.

1.39. Notation. Let P be a W-space. Then $\mathrm{Pt}(P)$ or $\mathrm{Pt}^*(P)$ will denote the class of all partitions or, respectively, all pure partitions of the space P. If $\mathscr{U} \in \mathrm{Pt}(P)$, then we put $\Phi_{\mathrm{Pt}}(\mathscr{U}) = \{\mathscr{V} \in \mathrm{Pt}(P)\colon \mathscr{V}$ refines $\mathscr{U}\}$, $\Phi^*_{\mathrm{Pt}}(\mathscr{U}) = \{\mathscr{V} \in \mathrm{Pt}^*(P)\colon \mathscr{V}$ refines $\mathscr{U}\}$. – Cf. [3], 2.13 (the notation in [3] is different).

1.40. Convention. $\mathrm{Pt}(P)$, $\mathrm{Pt}^*(P)$, $\Phi_{\mathrm{Pt}}(\mathscr{U})$, $\Phi^*_{\mathrm{Pt}}(\mathscr{U})$ are proper classes (since any finite non-void set K appears as the indexing set of some partition). Hence we cannot properly speak of, e.g., the "class of all $\Phi_{\mathrm{Pt}}(\mathscr{U})$". This inconvenience can be circumvented e.g. by choosing a fixed infinite set $\Omega \supset N$ such that $K \times M \subset \Omega$ whenever $K \subset \Omega$, $M \subset \Omega$ are finite, and adopting the following convention: every partition (of a W-space or of a set) is of the form $(U_k : k \in K)$ where $K \subset \Omega$. In what follows, we tacitly assume that a convention of this kind has been made, and we speak of the set $\mathrm{Pt}(P)$, of the set of all $\Phi_{\mathrm{Pt}}(\mathscr{U})$, etc.

1.41. Fact. *Let P be a W-space. Then the set of all $\Phi_{\mathrm{Pt}}(\mathscr{U})$, where $\mathscr{U} \in \mathrm{Pt}(P)$, is a filter base on $\mathrm{Pt}(P)$, and the set of all $\Phi^*_{\mathrm{Pt}}(\mathscr{U})$, where $\mathscr{U} \in \mathrm{Pt}^*(P)$, is a filter base on $\mathrm{Pt}^*(P)$.*

This follows at once from 1.36.

1.42. Definition. Let P be a W-space. The filter on $\mathrm{Pt}(P)$ generated by the sets $\Phi_{\mathrm{Pt}}(\mathscr{U})$, where $\mathscr{U} \in \mathrm{Pt}(P)$, will be denoted by $\mathscr{F}_{\mathrm{Pt}}(P)$ and called the *projective filter of partitions* of P. The filter on $\mathrm{Pt}^*(P)$ generated by the sets $\Phi^*_{\mathrm{Pt}}(\mathscr{U})$ where $\mathscr{U} \in \mathrm{Pt}^*(P)$, will be denoted by $\mathscr{F}^*_{\mathrm{Pt}}(P)$ and will be called the *semiprojective filter of pure partitions* of P. – Cf. [3], 2.16.

1.43. We recall the concept of the lower limit of a function with respect to a filter, stating it in a form somewhat broader than usual.

If $\mathscr{F}$ is a filter on a set A and g is a function (dom g can be an arbitrary class), then the *lower limit of g with respect to $\mathscr{F}$* is, by definition, the element $\sup(\inf(g(a) : a \in F \cap \operatorname{dom} g) : F \in \mathscr{F})$, where, of course, $\inf \emptyset = \infty$. This lower limit will be denoted by $\mathscr{F}\text{-}\underline{\lim}\, g$ or $\mathscr{F}\text{-}\underline{\lim}\,(g(a) : a \in A)$ or $\mathscr{F}\text{-}\underline{\lim}\, g(a)$, etc., and will be also called, e.g., the lower limit (with respect to $\mathscr{F}$) of $g(a)$ for a running through A. – If $\mathscr{F}$ is clear from the context, we sometimes write $\underline{\lim}\, g$ instead of $\mathscr{F}\text{-}\underline{\lim}\, g$, etc.

1.44. Definition. The lower limit of a function g with respect to the projective or semiprojective filter of partitions of P will be called the *projective* or, respectively, *semiprojective lower limit* of g.

2

In this section we introduce the concept of an extended Shannon entropy on $\mathfrak{W}_F$ and on $\mathfrak{W}$. To justify the approach presented here and to prepare the ground for a detailed investigation (in Sections 3 through 5) of a certain important kind of "extended entropies", we first examine some simple properties of the (trivial) extension of the Shannon entropy to the class of all $\langle Q, a_Q, \mu\rangle \in \mathfrak{W}_F$, and also of some other functions.

2.1. Notation. We write log instead of $\log_2$. We put $0 . \log 0 = 0$. If $a \in \boldsymbol{R}_+$, we put $L(a) = -a \log a$.

2.2. Notation and definitions. A) Let K be a non-void countable set. Let $\alpha = (a_k : : k \in K)$, $a_k \in R_+$, $\sum(a_k : k \in K) < \infty$. Then we put $H(\alpha) = H(a_k : k \in K) = = \sum(La_k : k \in K) - L\sum(a_k : k \in K)$. – B) Let $\langle Q, \rangle\mu$ be a measure space. Let $\mathscr{A}$ denote the set of all atoms of the σ-algebra dom μ. If $\mathscr{A}$ is countable and $Q = \bigcup\mathscr{A}$, then $H(\mu A : A \in \mathscr{A})$ will be called the *Shannon entropy* of $\langle Q, \mu\rangle$ and will be denoted by $H\langle Q, \mu\rangle$ (or by $H(\mu)$ if there is no danger of confusion).

2.3.1. Definition. If $\langle Q, \mu\rangle$ is a countable separated measure space, $a \in \boldsymbol{R}_+$, $P = = \langle Q, a_Q, \mu\rangle$, then we put $H(P) = a\, H(\mu)$ and call $H(P)$ the *Shannon entropy* of the W-space $\langle Q, a_Q, \mu\rangle$.

2.3.2. Notation. The class of all $\langle Q, a_Q, \mu\rangle \in \mathfrak{W}_F$ will be denoted by $\mathfrak{W}_{FC}$. The functional $P \mapsto H(P)$ defined on $\mathfrak{W}_{FC}$ will be denoted by H. The same symbol H will be also used, provided there is not danger of confusion, to denote the functionals $(a_k : : k \in K) \mapsto H(a_k : k \in K)$, $\langle Q, \mu\rangle \mapsto H\langle Q, \mu\rangle$.

2.4. We now list some elementary facts concerning the functional H as defined in 2.2. – A) Let $\xi = (x_k : k \in K)$ be an indexed set of reals. If $H(\xi)$ is defined, then $0 \leqq H(\xi) \leqq \infty$, $H(a\xi) = a\, H(\xi)$ for every $a \in \boldsymbol{R}_+$, $H(\xi) = 0$ iff $x_k > 0$ for at most one index $k \in K$. – B) Let K be a finite set, card $K = n > 0$. If $x_k \in \boldsymbol{R}_+$, $\sum(x_k : : k \in K) = 1$, then $H(x_k : k \in K) \leqq \log n$, and the equality holds iff $x_k = 1/n$ for all $k \in K$. – C) Let K be countable non-void. Let $\alpha \doteq (a_k : k \in K)$, $\beta = (b_k : k \in K)$, $a_k \in \boldsymbol{R}_+$, $b_k \in \boldsymbol{R}_+$, $a = \sum(a_k : k \in K) < \infty$, $b = \sum(b_k : k \in K) < \infty$. Then $H(\alpha) \leqq \leqq H(\alpha + \beta)$, $H(\alpha) + H(\beta) - H(\alpha + \beta) = \sum(H(a_k, b_k) : k \in K) - H(a, b)$. – D) Let K, M be countable non-void sets, $K \cap M = \emptyset$. Let $a_k \in \boldsymbol{R}_+$, $b_k \in \boldsymbol{R}_+$, $a = = \sum(a_k : k \in K) < \infty$, b $= \sum(b_m : m \in M) < \infty$. Then $H(a_k : k \in K) + H(b_m : m \in \in M) + H(a, b) = H(c_j : j \in K \cup M)$, where $c_j = a_j$ if $j \in K$, $c_j = b_j$ if $j \in M$.

2.5. We now introduce some elementary properties which seem to be indispensable for any functional (on some $\mathscr{X} \subset \mathfrak{W}$) that could be considered "entropy-like". In particular, regularity (see 2.8) expresses, in a restricted from, the following elementary property of a functional φ: if a point of measure zero is omitted from a space P or if the points of P are "ne-named", then the value of φP does not change. In addition, we introduce (2.9) the concept of a strongly regular functional.

2.6. Definition (cf. [3], 2.1). A non-negative functional φ defined on a class $\mathscr{X} \subset \mathfrak{W}$ will be called a *hypoentropy* (sometimes abbreviated HE) if

(HE 1) if $\langle Q, \varrho, \mu \rangle \in \mathscr{X}$, $a, b \in \boldsymbol{R}_+$, $\langle Q, a\varrho, b\mu \rangle \in \mathscr{X}$, then $\varphi\langle Q, a\varrho, b\mu \rangle = = ab\varphi\langle Q, \varrho, \mu \rangle$;

(HE 2) if $\langle Q, \varrho, \mu \rangle \in \mathscr{X}$, $i = 1, 2$, and $\varrho_1 \geqq \varrho_2$, then $\varphi\langle Q, \varrho_1, \mu \rangle \geqq \varphi\langle Q, \varrho_2, \mu \rangle$;

(HE 3) if $P = \langle \{q_1, q_2\}, \varrho, \mu \rangle \in \mathfrak{W}_F$, then $P \in \mathscr{X}$, $\varphi P \leqq H(\mu q_1, \mu q_2)\, \varrho(q_1, q_2)$;

(HE 4) if $P \in \mathscr{X}$ is finite, then $\varphi P < \infty$.

Remarks. A) Hypoentropies were called semi-subentropies in [3], 2.1. Since the term "subentropy" will not occur in the present paper, we prefer a shorter name. — B) The role of hypoentropies is purely auxiliary. In fact, they serve mainly as means for introducing the functionals C_τ and C_τ^*, see Section 3.

2.7. Definition. Let $P_i = \langle Q_i, \varrho_i, \mu_i \rangle$, $i = 1, 2$, be FW-spaces. Let a mapping $f : Q_1 \to Q_2$ satisfy the following conditions:

(Cs 1) $\mu_2 q = \mu_1(f^{-1}q)$ for each $q \in Q_2$,

(Cs 2) if $x, y \in Q_1$, $\mu x > 0$, $\mu y > 0$, then $\varrho_1(x, y) = \varrho_2(fx, fy)$. Then we will say that $f : Q_1 \to Q_2$ is *conservative with respect to* P_1 *and* P_2 or that $f : P_1 \to P_2$ is *conservative.*

2.8. Definition. Let φ be a functional, dom $\varphi \subset \mathfrak{W}$. Let the following condition hold:

(R) if $P_1, P_2 \in \mathfrak{W}_F \cap \text{dom } \varphi$, and there exists an injective conservative mapping $f : P_1 \to P_2$, then $\varphi P_1 = \varphi P_2$.

Then we will say that the functional φ is *regular.*

2.9. Definition. Let φ be a functional, dom $\varphi \subset \mathfrak{W}$. Let the following condition hold:

(SR) if $P_1, P_2 \in \mathfrak{W}_F \cap \text{dom } \varphi$ and there exists a conservative mapping $f : P_1 \to \to P_2$, then $\varphi P_1 = \varphi P_2$.

Then we will say that φ is *strongly regular.*

2.10. It is easy to see the functional H on $\mathfrak{W}_{FC}$ is continuous in the following sense: if $P_n = \langle Q, a_Q^{(n)}, \mu_n \rangle \in \mathfrak{W}_{FC}$, $P = \langle Q, a_Q, \mu \rangle \in \mathfrak{W}_{FC}$, $a^{(n)} \to a$, and $\mu_n q \to \mu q$

for $n \to \infty$ and any $q \in Q$, then $\lim H(P_n) = H(P)$. It is natural to require that an "entropy-like" functional on $\mathfrak{W}_F$ or on $\mathfrak{W}$ should be also continuous in some (perhaps weaker) sense. Therefore we introduce various kinds of continuity of functionals φ defined on some $\mathscr{X} \subset \mathfrak{W}$. (In fact, we are now interested only in the behavior of φ on $\mathfrak{W}_F \cap \operatorname{dom} \varphi$, hence only in what will be called "finite continuity", and in its various modifications.)

2.11. Notation. A) Let Q be a non-void set. If ϱ is a semimetric on Q, then $\mathfrak{W}(Q, \varrho, \cdot)$ will denote the set of all $\langle Q, \varrho, \mu \rangle \in \mathfrak{W}$. If μ is a measure on Q, then $\mathfrak{W}(Q, \cdot, \mu)$ will denote the set of all $\langle Q, \varrho, \mu \rangle \in \mathfrak{W}$. — B) Let Q be a finite non-void set. If $P_i = \langle Q, \varrho_i, \mu_i \rangle \in \mathfrak{W}_F(Q)$, $i = 1, 2$, then $\operatorname{dist}(P_1, P_2)$ will denote the number $\sum(|\mu_1 q - \mu_2 q| : q \in Q) + \max(|\varrho_1(x, y) - \varrho_2(x, y)| : x, y \in Q)$. The function $(P_1, P_2) \mapsto \operatorname{dist}(P_1, P_2)$ defined on $\mathfrak{W}_F(Q) \times \mathfrak{W}_F(Q)$ is clearly a metric on $\mathfrak{W}_F(Q)$. It will be denoted by dist_Q or simply by dist. — C) Let Q be a finite non-void set. $\mathfrak{W}_F(Q)$ (which, by 1.19, denotes the set of all $\langle Q, \varrho, \mu \rangle \in \mathfrak{W}_F$) will also denote (1) the metric space $\langle \mathfrak{W}_F(Q), \operatorname{dist}_Q \rangle$, (2) the set $\mathfrak{W}_F(Q)$ equipped with the topology induced by the metric dist_Q.

2.12. Definition. Let Q be a finite non-void set. If f is a function, $\operatorname{dom} f \subset \mathfrak{W}_F(Q)$, then f will be called (1) *feebly continuous* if it is continuous on every subspace of $\mathfrak{W}_F(Q)$ of the form $\mathfrak{W}(Q, \cdot, \mu) \cap \operatorname{dom} f$ or $\{\langle Q, \varrho, \mu \rangle \in \mathfrak{W}(Q, \varrho, \cdot) : \mu q > 0 \text{ for all } q \in Q\} \cap \operatorname{dom} f$; (2) *separately continuous* if it is continuous on every $\mathfrak{W}(Q, \cdot, \mu) \cap$ $\cap \operatorname{dom} f$ and every $\mathfrak{W}(Q, \varrho, \cdot) \cap \operatorname{dom} f$; (3) *continuous* if it is continuous with respect to the metric dist_Q.

2.13. Definition. Let φ be a functional, $\operatorname{dom} \varphi \subset \mathfrak{W}$. If, for any finite non-void set Q, $\varphi \restriction \mathfrak{W}_F(Q)$ is feebly continuous (separately continuous, continuous), then φ will be called *finitely feebly continuous* (*finitely separately continuous*, *finitely continuous*).

2.14. Fact. *The functional H defined on $\mathfrak{W}_{FC}$ is a finitely continuous strongly regular hypoentropy.*

Proof. By straightforward verification of conditions (HE 1)–(HE 4), (SR), and of the condition from 2.13.

2.15. We now present two examples of hypoentropies defined on $\mathfrak{W}_F$ which coincide with H on $\mathfrak{W}_{FC}$ and possess some, though not all, properties introduced in 2.8, 2.9, 2.13, and an example of a finitely continuous strongly regular hypoentropy (on $\mathfrak{W}$) whose restriction to $\mathfrak{W}_{FC}$ is very far from coinciding with H.

2.15.1. (A) If $P \in \mathfrak{W}_F$, $P = \langle Q, \varrho, \mu \rangle$, put $\psi_1 P = H(\mu)\, d(P)$. — (B) If $P = \langle Q, \varrho, \mu \rangle \in \mathfrak{W}_F$, put $\psi_2 P = \hat{r}(P)\, H(\mu) / ((wP)^2 - \sum((\mu q)^2 : q \in Q))$. — (C) If $P \in \mathfrak{W}$, put $\varphi P = 2\, \hat{r}(P)/wP$.

2.15.2. Fact. *The functional ψ_1 coincides with H on $\mathfrak{W}_{FC}$ and is a finitely feebly continuous regular hypoentropy. It is not finitely separately continuous.*

Proof. The first assertion is obvious. It is easy to see (using 2.4.A) that ψ_1 is a hypoentropy, and it is clear that ψ_1 is regular. If $P_n = \langle Q, \varrho_n, \mu\rangle$, $P = \langle Q, \varrho, \mu\rangle$ are FW-spaces and $\varrho_n \to \varrho$ (i.e. $\varrho_n(x, y) \to \varrho(x, y)$ for all $(x, y) \in Q \times Q$), then, clearly, $\psi_1 P_n \to \psi_1 P$. If $S_n = \langle Q, \varrho, \mu_n\rangle$, $S = \langle Q, \varrho, \mu\rangle$ are FW-spaces and $\mu q > 0$ for all $q \in Q$, then it is easy to see that, for all sufficiently large n, $d(S_n) = d(S)$, hence $\psi_1 S_n \to \psi_1 S$. We have shown that ψ_1 is finitely feebly continuous. – Let $Q = \{1, 2, 3\}$. Let $P_n = \langle Q, \varrho, \mu_n\rangle$, $n \in N$, $P = \langle Q, \varrho, \mu\rangle$ be FW-spaces, $\varrho(1, 2) = \varrho(1, 3) = 0$, $\varrho(2, 3) = 1$, $\mu_n\{1\} = \mu\{1\} = \mu_n\{2\} = \mu\{2\} = 1$, $\mu_n\{3\} = 1/n$, $\mu\{3\} = 0$. Then $\psi_1(P_n) = H(1, 1, 1/n) > 2$, $\psi_1(P) = 0$. Hence ψ_1 is not finitely separately continuous.

2.15.3. Fact. *The functional ψ_2 coincides with H on $\mathfrak{W}_{FC}$ and is a finitely continuous regular hypoentropy.*

Proof. Clearly, ψ_2 is a regular hypoentropy. If $P = \langle Q, a, \mu\rangle \in \mathfrak{W}_{FC}$, then $\hat{r}(P) = a \sum(\mu x . \mu y : x \in Q,\ y \in Q,\ x \neq y) = a((\mu Q)^2 - \sum((\mu q)^2 : q \in Q))$, hence $\psi_2 P = a\, H(\mu) = H(P)$.

Let T be an arbitrary non-void finite set. Assume that $P = \langle T, \varrho, \mu\rangle$, $P_n = \langle T, \varrho_n, \mu_n\rangle$, $n = 1, 2, \ldots$, are FW-spaces and that $P_n \to P$ in $\mathfrak{W}_F(T)$. If $(\mu T)^2 > \sum((\mu t)^2 : t \in T)$, then it is easy to see that $\psi_2 P_n \to \psi_2 P$. Consider the case $(\mu T)^2 = \sum((\mu t)^2 : t \in T)$. Then, clearly, $\mu t_0 = \mu T$ for some $t_0 \in T$, $H(\mu) = 0$, hence $H(\mu_n) \to 0$. Clearly, for any $S = \langle T, \sigma, v\rangle \in \mathfrak{W}_F$, $\psi_2 S \leqq H(v) \max(|\sigma(x, y)| : x, y \in T)$. Since $\varrho_n(x, y) \to \varrho(x, y)$ for all $x, y \in T$, the set of all $\varrho_n(x, y)$, $\varrho(x, y)$, where $x, y \in T$, is bounded. Since $H(\mu_n) \to 0$, we get $\psi_2 P_n \to 0$. Obviously, $\psi_2 P = 0$.

2.15.4. The functionals ψ_1 and ψ_2 are not strongly regular. – Let $P = \langle\{1, 2\}, 1, \mu\rangle$ where $\mu\{1\} = 1$, $\mu\{2\} = 2$. Let $S = \langle\{1, 2, 3\}, \varrho, v\rangle$ where $\varrho(1, 2) = \varrho(1, 3) = 1$, $\varrho(2, 3) = 0$, $v\{1\} = v\{2\} = v\{3\} = 1$. Let $f(1) = 1$, $f(2) = f(3) = 2$. Then $f : S \to P$ is conservative. However, $\psi_1 P = \psi_2 P = H(1, 2) = \log(27/4)$, $\psi_1 S > \psi_2 S = (2/3) H(1, 1, 1) = \log 9$.

2.16. To prove that the functional φ from 2.15.1C is a hypoentropy, we shall need the following simple lemma.

2.16.1. Lemma. *Let x, y be positive reals. Then $H(x, y) \geqq 4xy/(x + y)$, and the equality holds if and only if $x = y$.*

Since I have not found this elementary fact in current textbooks, a proof is given, although it does not require more than elementary calculus. Clearly, it is sufficient to prove that putting $F(z) = H(1 + z, 1 - z) - 2(1 + z)(1 - z)$ for $0 \leqq z \leqq 1$, we have $F(z) > 0$ for $0 < z < 1$. If $0 \leqq z < 1$, we have $F'(z) = -\log(1 + z) +$

$+ \log(1 - z) + 4z$, $F''(z) = -2\log(e/(1 - z^2)) + 4$. Clearly, $F''(0) > 0$, F'' is decreasing in $[0, 1)$, $F''(z) \to -\infty$ for $z \to 1$. Hence there is exactly one number a such that $0 < a < 1$, $F''(a) = 0$. We have $F''(z) > 0$ if $0 \leqq z < a$, $F''(z) < 0$ if $a < z < 1$, and therefore F' is increasing in $[0, a)$, decreasing in $[a, 1)$. Since $F'(0) = 0$, $F'(z) \to -\infty$ for $z \to 1$, there exists exactly one b such that $0 < b < 1$, $F'(b) = 0$. Clearly, F' is positive in $(0, b)$, negative in $(b, 1)$. Since $F(0) = F(1) = 0$, we get $F(z) > 0$ for all z from the interval $(0, 1)$.

2.17. Fact. *The functional $\varphi = 2\hat{r}/w$ defined on $\mathfrak{W}$* (see 2.15.1C) *is a finitely continuous strongly regular hypoentropy. Its restriction to $\mathfrak{W}_{FC}$ does not coincide with H.*

Proof. Clearly, φ satisfies conditions (HE 1), (HE 2), (HE 4) from 2.5. If $P = \langle\{a, b\}, \varrho, \mu\rangle \in \mathfrak{W}_F$, then $\varphi P = 4\varrho(a, b) . \mu a . \mu b/(\mu a + \mu b)$, hence, by 2.16.1, $\varphi P \leqq \varrho(a, b) H(\mu a, \mu b)$ and therefore condition (HE 3) from 2.6 is also satisfied. Hence φ is a hypoentropy. – It is easy to see that φ is strongly regular. Let $P_n = \langle Q, \varrho_n, \mu_n\rangle$, $n \in N$, $P = \langle Q, \varrho, \mu\rangle$ be FW-spaces, $P_n \to P$ in $\mathfrak{W}_F(Q)$. Clearly, $\hat{r}(P_n) \to \hat{r}(P)$, $wP_n \to wP$, hence if $wP > 0$, then $\varphi P_n \to \varphi P$. If $wP = 0$, then $wP_n \to 0$ and since $\hat{r}(P_n) \leqq d(P_n)(wP_n)^2$, we get $\varphi(P_n) \leqq 2\, d(P_n)\, wP_n$ and therefore, the sequence $(d(P_n))$ being bounded, $\varphi P_n \to 0$. Hence φ is finitely continuous.

If $P = \langle\{a, b\}, 1, \mu\rangle \in \mathfrak{W}_F$, $wP = 1$, then $\varphi P = 4\mu a . \mu b$, $H(P) = H(\mu a, \mu b)$, hence, by 2.16.1, $H(P) > \varphi P$ except if $\mu a = \mu b$ or $\mu a = 0$ or $\mu b = 0$.

2.18. If we want to have a fairly broad concept of an "extended Shannon entropy" on $\mathfrak{W}_F$, it seems appropriate (in view of 2.15.2) to require, beside the natural condition that the functional in question should be a regular hypoentropy, only a weak version of continuity; an additional condition is the strong regularity.

2.19. Definition. A functional φ on $\mathfrak{W}_F$ will be called an *extended Shannon semientropy* (abbreviation: *e.S. semientropy*) on $\mathfrak{W}_F$ if the following conditions are satisfied:

(E1) φ is a regular hypoentropy,
(E2) $\varphi P = H(P)$ for any $P = \langle Q, a_Q, \mu\rangle \in \mathfrak{W}_F$,
(E3) φ is finitely feebly continuous.

If, in addition,

(E4) φ is strongly regular,

then φ will be called an *extended Shannon entropy* (abbreviation: *e.S. entropy*) on $\mathfrak{W}_F$.

2.20. Fact. *The functional ψ_1 from* 2.15.1A *is an extended Shannon semientropy on $\mathfrak{W}_F$. The functional ψ_2 from* 2.15.1B *is a finitely continuous extended Shannon semientropy on $\mathfrak{W}_F$.*

This is, in fact, a paraphrase of part of 2.15.2, 2.15.3.

2.21. Remark. There are exactly $\exp \aleph_0$ extended Shannon semientropies on $\mathfrak{W}_F$. We give only an outline of proof. If e.S. semientropies φ_1, φ_2 coincide on every $P \in \mathfrak{W}_F$ of the form $P = \langle K_n, \varrho, \mu \rangle$, where $n \in N$, $n > 0$, $K_n = \{i : i \in n\}$ and all $\varrho(i, j)$, $\mu\{i\}$ are rational, then $\varphi_1 = \varphi_2$; hence there are at most $\exp \aleph_0$ e.S. semientropies. On the other hand, every $a_1\psi_1 + a_2\psi_2$, where ψ_1, ψ_2 are functionals from 2.15.1A, B and $a_1 \geqq 0$, $a_2 \geqq 0$, $a_1 + a_2 = 1$, is an e.S. semientropy.

2.22. The question now arises whether there exist e.S. entropies or even finitely continuous e.S. entropies on $\mathfrak{W}_F$. An affirmative answer will be given later (in Section 5 and, for finitely continuous e.S. entropies, in Part II). However, two examples will be presented below (2.24) without proof. To prepare the ground for the examples we state and prove some simple propositions on the Shannon entropy (we shall need some of these propositions also later on).

2.23.1. Fact. *For* $x, y \in \boldsymbol{R}$, $x > 0$, $y > 0$, *put* $V(x, y) = H(x, y)/xy$. *Then* $V(x_1, y_1) < V(x_2, y_2)$ *whenever* $x_1 \geqq x_2 > 0$, $y_1 \geqq y_2 > 0$, $(x_1, y_1) \neq (x_2, y_2)$.

Proof. The partial derivatives of V,

$$\frac{\partial V}{\partial x}(x, y) = \frac{1}{x^2} \log \frac{y}{x + y}\,, \quad \frac{\partial V}{\partial y}(x, y) = \frac{1}{y^2} \log \frac{x}{x + y}\,,$$

are always negative.

2.23.2. Lemma. *Let* $(a_k : k \in K)$, $(b_k : k \in K)$ *be countable non-void families of non-negative numbers. Put* $a = \sum(a_k : k \in K)$, $b = \sum(b_k : k \in K)$. *Assume* $a < \infty$, $b < \infty$. *Then*

$$H((a_k + b_k) : k \in K) \leqq H(a_k : k \in K) + H(b_k : k \in K) + \\ + H(a, b)\,(1 - \sum(a_k b_k : k \in K)/ab)\,.$$

Proof. It is easy to see that it is sufficient to prove the assertion for the case of a finite K. By 2.23.1, $H(a, b) \,.\, a_k b_k/ab \leqq H(a_k, b_k)$ for each $k \in K$, hence $H(a, b) \,.\, (1 - \sum(a_k b_k : k \in K)/ab) \geqq H(a, b) - \sum(H(a_k, b_k) : k \in K)$. Since, clearly, $H((a_k + b_k) : k \in K) - H(a_k : k \in K) - H(b_k : k \in K) = H(a, b) - \sum(H(a_k, b_k) : k \in K)$, the assertion is proved.

2.23.3. Proposition. *If* $P = \langle Q, a_Q, \mu \rangle \in \mathfrak{W}_F$, $P_1 + P_2 = P$, *then* $H(P) \leqq H(P_1) + H(P_2) + H(wP_1, wP_2)\, r(P_1, P_2)$.

This follows at once from 2.23.2, since if $P_i = \langle Q, a_Q, \mu_i \rangle$, then clearly $r(P_1, P_2) = a - a \sum(\mu_1 q \,.\, \mu_2 q : q \in Q)/wP_1 \,.\, wP_2$.

2.23.4. Proposition. *If* $P = \langle Q, a_Q, \mu\rangle \in \mathfrak{W}_F$, $P_1 + P_2 = P$, *then* $H(P) \leqq H(P_1) + H(P_2) + H[(P_1, P_2)]$.

This is a paraphrase of 2.23.3.

2.23.5. The property of H expresssd by 2.23.4 seems to be fairly important, as well as the weaker one obtained by considering pure partitions (P_1, P_2) only. Properties of this type will be considered in Sections 3 and 4.

2.24. Let $\tau = r$ or $\tau = E$. For any $P \in \mathfrak{W}_F$, let $c_\tau(P)$ or $c_\tau^*(P)$ be the supremum of all ψP, where ψ is a regular hypoentropy on $\mathfrak{W}_F$ satisfying

$$\psi S \leqq \psi S_1 + \psi S_2 + \psi[(S_1, S_2)]_\tau$$

for every $S \in \mathfrak{W}_F$ and every τ-admissible partition or, respectively, pure partition (S_1, S_2) of S. It can be shown (this will be done in Sections 3 through 5 for fairly general case including that of $\tau = r$ and $\tau = E$) that c_τ^* are e.S. semientropies on $\mathfrak{W}_F$ and c_τ are e.S. entropies on $\mathfrak{W}_F$.

2.25. We now turn to concepts expressing the general idea of an extension of the Shannon entropy to the whole class $\mathfrak{W}$. Clearly, such concepts should be compatible with those introduced for functionals on $\mathfrak{W}_F$. Therefore, the following approach suggests itself (and, moreover, seems to be the simplest and the broadest one).

2.26. Definition. A hypoentropy φ on $\mathfrak{W}$ such that $\varphi \restriction \mathfrak{W}_F$ is an e.S. semientropy on $\mathfrak{W}_F$ will be called an *extended (in the broad sense) Shannon semientropy* on $\mathfrak{W}$, abbreviated *e. (b.s.) S. semientropy* on $\mathfrak{W}$. – If, in addition, $\varphi \restriction \mathfrak{W}_F$ is an e.S. entropy, then φ will be called an *extended (in the broad sense) Shannon entropy* on $\mathfrak{W}$, abbreviated *e. (b.s.) S. entropy* on $\mathfrak{W}$.

2.27. Convention. In the names and abbreviations introduced in 2.19, 2.26, expressions "on $\mathfrak{W}_F$", "on $\mathfrak{W}$", "(in the broad sense)", and "(b.s.)", will be often omitted provided there is no danger of confusion.

2.28. Fact. *Let* a_1, a_2 *be non-negative reals,* $a_1 + a_2 = 1$. *Then, for any e.S. semientropies* φ_1, φ_2 *on* $\mathfrak{W}_F$ (on $\mathfrak{W}$), $\varphi = a_1\varphi_1 + a_2\varphi_2$ *is also an e.S. semientropy on* $\mathfrak{W}_F$ (on $\mathfrak{W}$).

Proof. Follows at once from definitions.

2.29. Remark. The definition 2.26 is very broad. For instance, if ψ is any e.S. semientropy on $\mathfrak{W}_F$ and if we put (1) $\varphi P = \psi P$ if $P \in \mathfrak{W}_F$, (2) $\varphi P = 0$ if $P \in \mathfrak{W} \setminus \mathfrak{W}_F$, then φ is an e. (b.s.) S. semientropy. Therefore, it seems worth-while to look for concepts not so broad as those in 2.26 but still wide enough to include e.g. the functionals C_τ^* and C_τ examined in the following sections. We will return to this question in Section 5.

2.30. Having introduced a fairly broad concept of extended Shannon entropies, we now have to answer, in particular, the following questions, and to exhibit, whenever possible, specified functionals with properties required.

I) Do there exist (a) e.S. entropies on $\mathfrak{W}$, (b) finitely continuous e.S. entropies on $\mathfrak{W}$? – II) If so, how many functionals of this kind are there? – III) Do there exist (finitely continuous) e.S. entropies or semientropies φ on $\mathfrak{W}$ such that $\varphi P > 0$ except for the trivial cases and $\varphi P < \infty$ whenever P is bounded, in a sense to be specified? IV–VI) Questions I–III under additional conditions such as some kind of projectivity (see Section 3), etc. – VII) Is there, for some suitable e.S. semientropies φ, a clearly described procedure for calculating, at least approximately, φP for $P \in \mathfrak{W}_F$ or perhaps also for some infinite $P \in \mathfrak{W}$?

For IVa, hence also for Ia, and for VII (partially), this will be done in Sections 3–5. The remaining questions will be answered (some of them only partially) in Part II.

3

3.1. For $\tau = r$ or $\tau = E$, let c_τ^* be the functional defined in 2.24 and let C_τ^* and C_τ be functionals on $\mathfrak{W}$ defined as follows: $C_\tau^*(P) = \mathscr{F}^*_{\mathrm{Pt}}(P)\text{-}\underline{\lim}\ c_\tau^*[\mathscr{U}]_\tau$, $C_\tau(P) = \mathscr{F}_{\mathrm{Pt}}(P)\text{-}\underline{\lim}\ c_\tau^*[\mathscr{U}]_\tau$. We intend to show that C_τ^* and C_τ are extended Shannon semientropies (entropies) possessing, in addition, various convenient properties. However, (1) in this setting, we should have to perform all proofs twice, for $\tau = r$ and for $\tau = E$, (2) there are many other functionals (see below) with properties similar to those of r_τ and E, for instance, the functionals r_t and d defined as follows.

3.2. Notation. We denote by r_t, where $0 < t \leqq \infty$, and by d the following functionals defined on the class of all $(P_1, P_2) \in \mathfrak{W} \times \mathfrak{W}$ such that $P_1 \leqq P$, $P_2 \leqq P$ for some $P \in \mathfrak{W}$:

if $P_i = \langle Q, \varrho, \mu_i \rangle$, $i = 1, 2$, then

(1) $r_t(P_1, P_2) = (\int \varrho^t \,\mathrm{d}(\mu_1 \times \mu_2)/wP_1 \,.\, wP_2)^{1/t}$ for $0 < t < \infty$;

(2) $r_\infty(P_1, P_2) = d(P_1, P_2)$.

3.3. For reasons mentioned in 3.1, we are going to examine C_τ^* and C_τ not only for $\tau = r$, E, but for a fairly general case, namely, for arbitrary gauge functionals τ (see 3.4). – We note that the definition of a gauge functional, and even more that of a normal gauge functional (see 3.7), is rather involved. In a sense, this is natural: we extract those properties which r_t, E, etc., have in common and which are sufficient for the corresponding functionals C_τ^* and C_τ to behave reasonably. One can hardly expect such properties to be simple. On the other hand, we wish to stress that most proofs concerning C_τ^* and C_τ, where τ is any (normal) gauge functional, are only slightly more involved than those concerning only the case $\tau = r$.

3.4. Definition. A functional τ defined on the class of all $(P_1, P_2) \in \mathfrak{W} \times \mathfrak{W}$ such that $P_1 \leqq P$, $P_2 \leqq P$ for some $P \in \mathfrak{W}$ will be called a *gauge functional* (abbreviation: GF) if, for any $(P_1, P_2) \in \operatorname{dom} \tau$,

(GF1) $\tau(P_1, P_2) = \tau(P_2, P_1) \geqq 0$;

(GF2) $\tau(P_1, P_2) \leqq d(P_1 + P_2)$;

(GF3) if $wP_1 = 0$ or $wP_2 = 0$, then $\tau(P_1, P_2) = 0$;

(GF4) if $P_i = \langle Q, \varrho, \mu_i \rangle$, $i = 1, 2$, $a, b_1, b_2 \in \boldsymbol{R}_+$, $b_1 > 0$, $b_2 > 0$, then $\tau(\langle Q, a\varrho, b_1\mu_1 \rangle, \langle Q, a\varrho, b_2\mu_2 \rangle) = a\tau(P_1, P_2)$;

(GF5) if $P_i = \langle Q, \varrho, \mu_i \rangle$, $S_i = \langle Q, \sigma, \mu_i \rangle$, $i = 1, 2$, $(P_1, P_2) \in \operatorname{dom} \tau$, $(S_1, S_2) \in \operatorname{dom} \tau$, $\varrho \geqq \sigma$, then $\tau(P_1, P_2) \geqq \tau(S_1, S_2)$;

(GF6) if $d(P_1, P_2) = r(P_1, P_2)$, $d(P_i) \leqq d(P_1, P_2)$, $i = 1, 2$, then $\tau(P_1, P_2) = d(P_1, P_2)$;

(GF7) if P, S are FW-spaces, $f : P \to S$ is conservative, $P_i = \langle Q, \varrho, \mu_i \rangle \leqq P$, $S_i = \langle T, \sigma, \nu_i \rangle \leqq S$, $i = 1, 2$, and, for any $t \in T$, $\mu_i(f^{-1}t) = \nu_i t$, $i = 1, 2$, then $\tau(S_1, S_2) = \tau(P_1, P_2)$.

3.5. Fact. *The functionals r_t, $0 < t < \infty$, $r_\infty = d$, E are gauge functionals.*

We omit the proof since it consists in a straightforward, though lengthy verification of (GF1)–(GF7).

3.6. Definition. Let τ be a gauge functional. Let P be a W-space. A partition $\mathscr{U} = (U_k : k \in K)$ of P is called *τ-admissible* if, for any $i, j \in K$, $i \neq j$, we have $\tau(U_i, U_j) < \infty$. If $\mathscr{U} = (U_k : k \in K)$ is τ-admissible, then the FW-space $\langle K, \sigma, \nu \rangle$, where $\sigma(i, j) = \tau(U_i, U_j)$ for $i \neq j$, $\nu\{k\} = wU_k$ for every $k \in K$, will be denoted by $[\mathscr{U}]_\tau$ and will be called the *τ-quotient* of P *according to* $\mathscr{U}$. – We note that, due to 3.5, the definition 1.37.1 is a special case of 3.6.

3.7. Definition. A gauge functional τ will be called *normal* (abbreviation: NGF) if the following conditions are satisfied:

(NGF1) if $\mathscr{U} = (U_k : k \in K)$ is a τ-admissible partition of a W-space P, $S = \langle K, \sigma, \nu \rangle = [\mathscr{U}]_\tau$, $S_1 \leqq S$, $S_2 \leqq S$, $S_i = \langle K, \sigma, \nu_i \rangle$, $i = 1, 2$, and $\nu_1\{k\} \cdot \nu_2\{k\} = 0$ whenever $k \in K$, $d(U_k) > 0$, then $\tau(S_1, S_2) = \tau(\sum(a_{1k}U_k : k \in K), \sum(a_{2k}U_k : k \in K))$, where $a_{ik} = \nu_i\{k\}/\nu\{k\}$ for $i = 1, 2$, $k \in K$;

(NGF2) if $(P_1, P_2) \in \operatorname{dom} \tau$, $S_i \leqq P_i$, $i = 1, 2$, then $wS_1 \cdot wS_2 \cdot \tau(S_1, S_2) \leqq wP_1 \cdot wP_2 \cdot \tau(P_1, P_2)$;

(NGF3) if $S_i = \langle Q, \sigma, \mu_i \rangle$, $P_i = \langle Q, \varrho, \mu_i \rangle$, $i = 1, 2$, $(S_1, S_2) \in \operatorname{dom} \tau$, $(P_1, P_2) \in \operatorname{dom} \tau$, $a \in \boldsymbol{R}_+$, and $\sigma(x, y) \leqq \varrho(x, y) + a$ for all $x, y \in Q$, then $\tau(S_1, S_2) \leqq \tau(P_1, P_2) + a$.

3.8. Proposition. *The functionals* r_t, $1 \leqq t < \infty$, $r_\infty = d$, E *are normal gauge functionals.*

For similar reasons as in 3.5 we omit the proof.

3.9.1. Fact. *Let* P_1, P_2 *be W-spaces. Assume that there exists a W-space* P *such that* $P_1 \leqq P$, $P_2 \leqq P$. *Then* $1 \leqq t < u \leqq \infty$ *implies* $r_t(P_1, P_2) \leqq r_u(P_1, P_2)$, *and* $r_t(P_1, P_2) \to d(P_1, P_2)$ *if* $t \to \infty$.

This follows at once from the fact that $r_t(P_1, P_2)$, where $1 \leqq t \leqq \infty$, $P_i = \langle Q, \varrho, \mu_i \rangle$, is equal to the norm of the function ϱ in the space $L_t(T)$, where T is the measure space $\langle Q \times Q, \mu_1 \times \mu_2 \rangle$.

3.9.2. Fact. *If* τ *is a gauge functional, then* $\tau \leqq E$. — This is an obvious consequence of (GF2), (GF3).

3.10. Definition. Let τ be a gauge functional. Let φ be a non-negative functional, $\operatorname{dom} \varphi \subset \mathfrak{W}$. If, for any $P \in \operatorname{dom} \varphi$ and any τ-admissible partition or pure partition (P_1, P_2) of P such that $P_1 \in \operatorname{dom} \varphi$, $P_2 \in \operatorname{dom} \varphi$, we have $[(P_1, P_2)]_\tau \in \operatorname{dom} \varphi$, $\varphi P \leqq \varphi P_1 + \varphi P_2 + \varphi[(P_1, P_2)]_\tau$, then φ will be called *τ-projective* or, respectively, *τ-semiprojective*. — Cf. [3], 2.3.

3.11. Proposition. *Let* σ, τ *be gauge functionals,* $\sigma \leqq \tau$. *Then* (1) *any τ-admissible partition of a W-space is σ-admissible,* (2) *if a hypoentropy* φ *on a class* $\mathscr{X} \subset \mathfrak{W}$ *is σ-projective (σ-semiprojective), then* φ *is also τ-projective (τ-semiprojective).*

Proof. Assertion (1) is obvious. Let $\mathscr{U} = (P_1, P_2)$ be a τ-admissible partition of $P \in \mathscr{X}$ and let $P_1 \in \mathscr{X}$, $P_2 \in \mathscr{X}$. Then $\mathscr{U}$ is σ-admissible, hence $\varphi P \leqq \varphi P_1 + \varphi P_2 + \varphi[\mathscr{U}]_\sigma$. Clearly, $[\mathscr{U}]_\sigma = \langle \{1, 2\}, a, v \rangle$, $[\mathscr{U}]_\tau = \langle \{1, 2\}, b, v \rangle$ where $a = \sigma(P_1, P_2)$, $b = \tau(P_1, P_2)$. Since $\sigma \leqq \tau$, we have $a \leqq b$, hence, by (HE2), $\varphi[\mathscr{U}]_\sigma \leqq \varphi[\mathscr{U}]_\tau$, and therefore $\varphi P \leqq \varphi P_1 + \varphi P_2 + \varphi[\mathscr{U}]_\tau$.

3.12. Fact. *The functional* H *defined on* $\mathfrak{W}_{FC}$ *is r-projective.*

Proof. Follows from 2.23.4.

3.13.1. Proposition. *Let* τ *be a gauge functional. Let* $\mathscr{X} = \mathfrak{W}$ *or* $\mathscr{X} = \mathfrak{W}_F$. *Let* φ *be a τ-semiprojective or τ-projective hypoentropy on* $\mathscr{X}$. *Let* $P \in \mathscr{X}$ *and let* $\mathscr{U} = (U_k : k \in K)$ *be a pure partition or, respectively, any partition of* P. *If* $\varphi U_k = 0$ *for each* $k \in K$, *then* $\varphi P \leqq d(P)\, H(wU_k : k \in K)$.

Proof. We consider the case of a τ-projective φ and an arbitrary partition; the other case is completely analogous. If $\operatorname{card} K = 1$, then $\varphi P = 0$. Let $n > 1$ and assume that the inequality in question holds whenever $\operatorname{card} K < n$. Let $\mathscr{U} = (U_k : k \in K) \in \operatorname{Pt}(P)$, $\operatorname{card} K = n$. Choose $j \in K$ and put $K' = K \setminus \{j\}$, $S = \sum(U_k : k \in K')$. Clearly, $(U_k : k \in K') \in \operatorname{Pt}(S)$, hence (1) $\varphi S \leqq d(S)\, H(wU_k : k \in K')$. Since φ is a τ-projective hypoentropy, we have $\varphi P \leqq \varphi S + \varphi U_j + \varphi[(S, U_j)]_\tau$, hence, by 3.9.2 and (HE3), (2) $\varphi P \leqq \varphi S + \varphi U_j + d(P)\,H(wS, wU_j)$. From (1) and (2) we get, by 2.4D, $\varphi P \leqq d(P)\,H(wU_k : k \in K)$.

3.13.2. Fact. *If φ is a regular hypoentropy on $\mathfrak{W}_F$ and $P = \langle Q, \varrho, \mu\rangle$ is an FW-space, then $\varphi(q \,.\, P) = 0$ for any $q \in Q$.*

Proof. Let $S = \langle\{1\}, 0, \nu\rangle \in \mathfrak{W}_F$, $\nu\{1\} = \mu\{q\}$. By (HE3), $\varphi S = 0$. Put $f(1) = q$. Then $f : S \to q \,.\, P$ is conservative injective, hence $\varphi(q \,.\, P) = \varphi(S) = 0$.

3.14. Proposition. *Let τ be a gauge functional. For any $S \in \mathfrak{W}_F$, let $c_\tau^*(S)$ be the supremum of all ψS, where ψ is a τ-semiprojective regular hypoentropy on $\mathfrak{W}_F$. Then (1) the functional $S \mapsto c_\tau^*(S)$ is a τ-semiprojective regular hypoentropy on $\mathfrak{W}_F$, (2) for any FW-space $P = \langle Q, \varrho, \mu\rangle$, $c_\tau^*(P) \leqq d(P)\,H(\mu q : q \in Q)$.*

Proof. Clearly, the functional $S \mapsto c_\tau^*(S)$ is regular and satisfies conditions (HE1)–(HE3) from 2.6. By 3.13.1 and 3.13.2, assertion (2) holds; hence (HE4) is satisfied. If $\mathscr{U} = (P_0, P_1)$ is a pure partition of an *FW*-space P, then, for any τ-semiprojective HE ψ on $\mathfrak{W}_F$, we have $\psi P \leqq \psi P_0 + \psi P_1 + \psi[\mathscr{U}]_\tau$, hence $\varphi P \leqq$ $\leqq c_\tau^*(P_0) + c_\tau^*(P_1) + c_\tau^*[\mathscr{U}]_\tau$. This proves the τ-semiprojectivity of $S \mapsto c_\tau^*(S)$.

3.15. Notation. Let τ be a gauge functional. If $\psi : \mathfrak{W}_F \to \bar{\boldsymbol{R}}_+$, then, for any *W*-space P, we put $[\psi]_\tau^*(P) = \mathscr{F}_{\mathrm{Pt}}^*(P)\text{-}\underline{\lim}\,\psi[\mathscr{U}]_\tau$, $[\psi]_\tau(P) = \mathscr{F}_{\mathrm{Pt}}\text{-}\underline{\lim}\,\psi[\mathscr{U}]_\tau$; the functionals $P \mapsto [\psi]_\tau^*(P)$ and $P \mapsto [\psi]_\tau(P)$ will be denoted by $[\psi]_\tau^*$ and $[\psi]_\tau$, respectively.

3.16. Proposition. *Let τ be a gauge functional. If $\psi : \mathfrak{W}_F \to \bar{\boldsymbol{R}}_+$ is regular, then $[\psi]_\tau^* \restriction \mathfrak{W}_F = \psi$.*

Proof. Let $P = \langle Q, \varrho, \mu\rangle \in \mathfrak{W}_F$. If $wP > 0$, put $\mathscr{V} = (q \,.\, P : q \in Q,\ \mu q > 0)$; if $wP = 0$, put $\mathscr{V} = (P)$. It is easy to see that, for every $\mathscr{U} \in \Phi_{\mathrm{Pt}}^*(\mathscr{V})$, there exists a conservative injective mapping $f : [\mathscr{V}]_\tau \to [\mathscr{U}]_\tau$, and therefore $\psi[\mathscr{V}]_\tau = \psi[\mathscr{U}]_\tau$. Condition (GF6) implies that if $q_1 \in Q$, $q_2 \in Q$, $\mu q_1 > 0$, $\mu q_2 > 0$, $q_1 \neq q_2$, then $\tau(q_1 \,.\, P,\ q_2 \,.\, P) = \varrho(q_1, q_2)$. Therefore, if $wP > 0$, then the identity mapping of $[\mathscr{V}]_\tau$ into P is conservative. Obviously, if $wP = 0$, then there exists a conservative injective mapping of $[\mathscr{V}]_\tau$ into P. Hence $\psi[\mathscr{V}]_\tau = \psi P$, and therefore $\psi[\mathscr{U}]_\tau = \psi P$ for every $\mathscr{U} \in \Phi_{\mathrm{Pt}}^*(\mathscr{V})$. The definition of $[\psi]_\tau^*$ now implies $[\psi]_\tau^*(P) = \psi P$.

3.17. Definition. Let τ be a gauge functional. Let c_τ^* denote the functional $S \mapsto c_\tau^*(S)$ described in 3.14. Then the functionals $[c_\tau^*]_\tau^*$ and $[c_\tau^*]_\tau$ will be denoted by C_τ^* and C_τ, respectively. Instead of C_r^* and C_r we will often write C^* and C. We will call $C_\tau^*(P)$ the *C_τ^*-semientropy* of P, and $C_\tau(P)$ the *C_τ-entropy* of P.

Remark. The names C_τ^*-semientropy, C_τ-entropy anticipate the fact, to be proved later, that C_τ^* and C_τ are extended Shannon semientropies (entropies) provided τ is an NGF, $\tau \geqq r$.

3.18. Fact. *Let τ be a* GF. *Let $P, S \in \mathfrak{W}_F$ and let $f : P \to S$ be conservative injective. For each $U = \langle Q, \varrho, \mu \rangle \leqq P$, let $f_1 U = \langle T, \sigma, \nu \rangle \leqq S$, where $\nu t = \mu(f^{-1}t)$ for any $t \in T$. For any $\mathscr{U} = (U_k : k \in K) \in \mathrm{Pt}(P)$, put $f_2 \mathscr{U} = (f_1 U_k : k \in K)$. Then $f_2 : \mathrm{Pt}(P) \to \mathrm{Pt}(S)$ is bijective, $\mathscr{U}$ refines $\mathscr{V}$ iff $f_2\mathscr{U}$ refines $f_2\mathscr{V}$, and $[\mathscr{U}]_\tau = [f_2\mathscr{U}]_\tau$ for any $\mathscr{U} \in \mathrm{Pt}(P)$.*

Proof. By (GF7), $\tau(U, V) = \tau(f_1 U, f_1 V)$ for any $U, V \leqq P$. The rest is easy.

3.19. Proposition. *Let τ be a gauge functional. Then C_τ^* and C_τ are regular hypoentropies, and $C_\tau^* \restriction \mathfrak{W}_F$ is the greatest τ-semiprojective regular hypoentropy on $\mathfrak{W}_F$.*

Proof. The assertion concerning $C_\tau^* \restriction \mathfrak{W}_F$ and the regularity of C_τ^* follows directly from 3.14 and 3.16. It is easy to see (directly from the definition) that C_τ^* and C_τ satisfy (HE1)−(HE3). The regularity of C_τ follows easily from 3.18. We are going to show that C_τ satisfies (HE4); for C_τ^*, the proof is analogous. Let $P = \langle Q, \varrho, \mu \rangle$ be a finite W-space. Let $\mathscr{A}$ be the set of all atoms of the σ-algebra dom μ. Clearly, $\mathscr{U}_0 = (A \,.\, P : A \in \mathscr{A})$ is a partition of P. Let $\mathscr{U} = (U_k : k \in K) \in \mathrm{Pt}(P)$. By 1.36, there exists a $\mathscr{V} = (V_m : m \in M) \in \mathrm{Pt}(P)$ refining both $\mathscr{U}$ and $\mathscr{U}_0$. Since $\mathscr{V}$ refines $\mathscr{U}_0$, there exists a partition $(M_A : A \in \mathscr{A})$ of M such that $\sum(V_m : m \in M_A) = A \,.\, P$ for all $A \in \mathscr{A}$. Put $S = \langle M, \sigma, \nu \rangle = [\mathscr{V}]_\tau$. By (GF2), $d(S) \leqq d(P)$. For $A \in \mathscr{A}$, put $S_A = M_A \,.\, S$. By (GF2) and 1.29, $d(S_A) = 0$ for all $A \in \mathscr{A}$. Hence, by 3.16 and 3.14, $C_\tau^*(S_A) = 0$ for any $A \in \mathscr{A}$. By 3.13.1, we get $C_\tau^*[\mathscr{V}]_\tau = C_\tau^*(S) \leqq d(P)\, H(\mu A : A \in \mathscr{A})$. This implies $C_\tau(P) \leqq d(P)\, H(\mu A : A \in \mathscr{A})$.

3.20. Proposition. *Let σ, τ be gauge functionals, $\sigma \leqq \tau$. Then $C_\sigma^* \leqq C_\tau^*$, $C_\sigma \leqq C_\tau$.*

Proof. By 3.19 and 3.11, $C_\sigma^* \restriction \mathfrak{W}_F$ is τ-semiprojective, hence $C_\sigma^* \restriction \mathfrak{W}_F \leqq C_\tau^* \restriction \mathfrak{W}_F$. If $P \in \mathfrak{W}$ and $\mathscr{U} \in \mathrm{Pt}(P)$ is τ-admissible, then, by (HE2), $C_\sigma^*[\mathscr{U}]_\sigma \leqq C_\tau^*[\mathscr{U}]_\tau$. By 3.17 and 3.15, this proves the proposition. The proof of $C_\sigma \leqq C_\tau$ is analogous.

3.21. Proposition. *Let τ be a gauge functional. If P, S are FW-spaces and there exists a conservative mapping $f : S \to P$, then $C_\tau^*(S) \leqq C_\tau^*(P)$.*

Proof. I. For any $P \in \mathfrak{W}_F$, let $\Psi(P)$ be the class of all $T \in \mathfrak{W}_F$ such that there exists a conservative mapping $f : T \to P$. Put $\varphi P = \sup \{C_\tau^*(T) : T \in \Psi(P)\}$. Clearly, the functional φ, defined on $\mathfrak{W}_F$, satisfies (HE1)−(HE2). If $T \in \Psi(P)$ and $g : T \to P$ is conservative, then $d(g^{-1}q \,.\, P) = 0$ for each $q \in |P|$, and therefore, by 3.19, 3.14 and 3.13.1, we get $C_\tau^*(T) \leqq d(P)\, H(\mu q : q \in |P|)$. Hence φ satisfies (HE3) and (HE4). Clearly, φ is regular. We have shown that φ is a regular hypoentropy.

II. We are going to prove that φ is τ-semiprojective. Let $P \in \mathfrak{W}_F$ and let (P_0, P_1) be a τ-admissible pure partition of P. Let (B_0, B_1) be a partition of $|P|$ such that $P_i = B_i \,.\, P$, $i = 0, 1$. For an arbitrary $T \in \Psi(P)$ choose a conservative mapping $f : T \to P$. Put $A_i = f^{-1}B_i$, $T_i = A_i \,.\, T$. It is easy to see that, for $i = 0, 1, f : T \to P_i$

is conservative. Hence, $T_i \in \Psi(P_i)$, and therefore $C_\tau^*(T_i) \leqq \varphi P_i$. Since C_τ^* is τ-semi-projective, we have

$$C_\tau^*(T) \leqq \varphi P_0 + \varphi P_1 + C_\tau^*[T_0, T_1)]_\tau \, .$$

By (GF7), we get $\tau(T_0, T_1) = \tau(P_0, P_1)$, hence $C_\tau^*[(T_0, T_1)]_\tau = C_\tau^*[(P_0, P_1)]_\tau \leqq \leqq \varphi[(P_0, P_1)]_\tau$. Thus, we have, for any $T \in \Psi(P)$,

$$C_\tau^*(T) \leqq \varphi P_0 + \varphi P_1 + \varphi[(P_0, P_1)]_\tau \, ,$$

and therefore

$$\varphi P \leqq \varphi P_0 + \varphi P_1 + \varphi[(P_0, P_1)]_\tau \, .$$

We have shown that φ is τ-semiprojective.

III. By 3.19, we now have $\varphi P \leqq C_\tau^*(P)$ for any $P \in \mathfrak{W}_F$, hence $C_\tau^*(S) \leqq C_\tau^*(P)$ for any $S \in \Psi(P)$.

3.22. Proposition. *Let τ be a gauge functional. Then, for any FW-space $P = \langle Q, \varrho, \mu \rangle$, $C_\tau(P)$ is equal to the infimum of all $C_\tau^*[\mathscr{U}]_\tau$ where $\mathscr{U}$ is a partition of P finer than $(q \, . \, P : q \in Q)$.*

Proof. Let a denote the infimum in question. By the definition of $C_\tau(P)$, we have $a \leqq C_\tau(P)$. Let $\varepsilon > 0$. There exists a partition $\mathscr{V}_1$ finer than $\mathscr{V}_0 = (q \, . \, P : q \in Q)$ and such that $C_\tau^*[\mathscr{V}_1]_\tau < a + \varepsilon$. Let $\mathscr{U}$ be any partition of P. By 1.36, there exists a partition $\mathscr{V}_2$ finer than $\mathscr{V}_1$ and $\mathscr{U}$. It is easy to see that (since $\mathscr{V}_1$ refines $\mathscr{V}_0$, $\mathscr{V}_2$ refines $\mathscr{V}_1$) $\mathscr{V}_1$ and $\mathscr{V}_2$ are of the following form: $\mathscr{V}_1 = (V_k : k \in K)$, where $V_k = a_k(q_k \, . \, P)$, $a_k \in \boldsymbol{R}_+$, $q_k \in Q$, $\mathscr{V}_2 = (V'_m : m \in M)$, where $V'_m = b_m(q'_m \, . \, P)$, $b_m \in \boldsymbol{R}_+$, $q_m \in Q$, and there exists a partition $(M_k : k \in K)$ of M such that, for any $k \in K$, $m \in M_k$, we have $q'_m = q_k$, $\sum(b'_m : m \in M_k) = a_k$. Define $f : M \to K$ by the condition $m \in M_{f(m)}$. Then, by (GF2) and (GF4), $f : [\mathscr{V}_2]_\tau \to [\mathscr{V}_1]_\tau$ is a conservative maping. Hence, by 3.21, $C_\tau^*[\mathscr{V}_2]_\tau \leqq C_\tau^*[\mathscr{V}_1]_\tau$. We have shown that there exists a partition $\mathscr{V}_2$ finer than $\mathscr{U}$ and satisfying $C_\tau^*[\mathscr{V}_2]_\tau < a + \varepsilon$. This proves $C_\tau(P) \leqq a + \varepsilon$. Since $\varepsilon > 0$ was arbitrary, we get $C_\tau(P) \leqq a$.

3.23. Proposition. *Let τ be a gauge functional. Let P be an FW-space. Then $C_\tau(P)$ is equal to the infimum of all $C_\tau^*(S)$ where S is an FW-space such that there exists a conservative mapping $f : S \to P$.*

Proof. If $S \in \mathfrak{W}_F$, $f : S \to P$ is conservative, then, for any $t \in |S|$, put $A_t = f^{-1}(ft)$, $a_t = w(t \, . \, S)/w(A_t \, . \, S)$, $U_t = a_t(ft \, . \, P)$. It is easy to see that $\mathscr{U} = (U_t : t \in |S|)$ is a partition of P refining $\mathscr{U}_0 = (q \, . \, P : q \in |P|)$ and that the identity mapping $i : [\mathscr{U}]_\tau \to S$ is conservative injective. Hence, by the regularity of C_τ^*, $C_\tau^*[\mathscr{U}]_\tau = C_\tau^*(S)$, and $f \circ i : [\mathscr{U}]_\tau \to P$ is conservative. On the other hand, if $\mathscr{U}$ is a partition of P refining $\mathscr{U}_0$, then, clearly, there exists a conservative $g : [\mathscr{U}]_\tau \to P$. By 3.22, this proves the proposition.

3.24. Proposition. *Let τ be a gauge functional. If P, S are FW-spaces and there exists a conservative mapping $f : S \to P$, then $C_\tau(S) = C_\tau(P)$.*

Proof. From 3.23 it follows at once that $C_\tau(S) \geqq C_\tau(P)$. We shall suppose $C_\tau(S) > C_\tau(P)$ and derive a contradiction. – There exists (see the proof of 3.23) a partition $\mathscr{U}$ of P refining $(q \,.\, P\colon q \in |P|)$ and such that there is a conservative injective $f : [\mathscr{U}]_\tau \to S$. Now, if a partition $\mathscr{V}$ refines $\mathscr{U}$, then clearly (see the proof of 3.23) there exists a conservative mapping $g : [\mathscr{V}]_\tau \to [\mathscr{U}]_\tau$, hence, by 3.23, $C^*_\tau[\mathscr{V}]_\tau \geqq C_\tau(S) > C_\tau(P)$. This is a contradiction, for $C_\tau(P) = \mathscr{F}_{\mathrm{Pt}}(P) - \underline{\lim}\, C^*_\tau[\mathscr{V}]_\tau$.

3.25. Proposition. *If τ is a gauge functional, then C_τ is strongly regular.*

This is a paraphrase of 3.24.

3.26. Proposition. *Let τ be a gauge functional. Then, for any FW-space P, $C_\tau(P) \leqq C^*_\tau(P)$.*

Proof. Follows at once from 3.23.

4

In this section, we introduce (see 4.3) dyadic expansions $\mathscr{P}$ of W-spaces. For any gauge functional τ and any $\mathscr{P}$, we define the τ-value, $\Gamma_\tau(\mathscr{P})$, of $\mathscr{P}$. It turns out (see 4.29) that if τ is normal, then, for any W-space P, $C_\tau(P)$ and $C^*_\tau(P)$ can be obtained as certain limits (see 4.9) of $\Gamma_\tau(\mathscr{P})$. We recall that C and C^* (i.e., C_τ and C^*_τ with $\tau = r$) have been defined, in [3], in the way just mentioned. Thus, in particular, C^* and C as defined here coincide with C^* and C from [3].

Dyadic expansions have been defined in [3] (for FW-spaces, already in [2]). However, we re-state all relevant definitions.

4.1. Notation. Let $x = (x_i : i < m)$, $y = (y_j : j < n)$, where $m, n \in N$, be finite sequences. We write $x \prec y$ if $m < n$ and $x_i = y_i$ for $i < m$, and $x \precsim y$ if $x \prec y$ or $x = y$. We denote by $x \,.\, y$ the concatenation of x and y, i.e. the sequence $(z_k : k < m + n)$, where $z_i = x_i$ for $i < m$, $z_{m+j} = y_j$ for $j < n$. Instead of $x \,.\, y$ we sometimes write xy. We often write $x \,.\, b$ or xb instead of $x \,.\, (b)$, and $a \,.\, x$ or ax instead of $(a) \,.\, x$. The void sequence is denoted by $\emptyset$.

4.2. Notation. We denote by Δ the collection of all finite non-void sets $D \subset \bigcup(\{0, 1\}^n : n \in N)$ (i.e. of all finite non-void sets of finite sequences, including the void one, of elements 0 and 1) such that (1) if $x \in D$, $y \prec x$, then $y \in D$, (2) if $x \in D$, then either $\{x0, x1\} \subset D$ or $\{x0, x1\} \cap D = \emptyset$. If $D \in \Delta$, we put $D' = \{x \in D\colon x0 \in D\}$, $D'' = D \setminus D'$. If $D \in \Delta$, $X \subset D$, we put $D(X) = \{y \in D\colon x \precsim y \text{ for some } x \in X\}$.

4.3. Definition. Let P be a W-space. An indexed set $\mathscr{P} = (P_x : x \in D)$, where $D \in \Delta$, will be called a *(pure) dyadic expansion* of P if (1) every P_x, $x \in D$, is a (pure) subspace of P, (2) $P_\emptyset = P$, (3) $P_{x0} + P_{x1} = P_x$ for each $x \in D'$.

4.4. Notation. If $\mathscr{P} = (P_x : x \in D)$ is a dyadic expansion, then $\mathscr{P}''$ denotes the indexed set $(P_x : x \in D'')$.

4.5. Fact. *If $\mathscr{P}$ is a dyadic expansion of a W-space P, then $\mathscr{P}''$ is a partition of P, and $\mathscr{P}''$ is pure iff $\mathscr{P}$ is.*

4.6.1. Fact. *Let $n \in N$, $n > 0$. Let $(U_k : k < n)$ be a partition of a W-space P. Let $D_n \in \Delta$ be the set of all sequences $(a_i : i < j)$ such that (1) $j < n$, (2) $a_i = 1$ if $0 \leqq i < j - 1$. Put $P_\emptyset = P$. If $x \in D$, $x = (a_i : i < j)$ with $j > 0$, put $P_x = U_{j-1}$ if $a_{j-1} = 0$, and $P_x = \sum(U_i : j \leqq i < n)$ if $a_{j-1} = 1$. Then $\mathscr{P} = (P_x : x \in D_n)$ is a dyadic expansion of P and $\mathscr{P}''$ is equal to $\mathscr{U}$ re-indexed.*

Proof. Clearly, $\mathscr{P}$ is a dyadic expansion of P, $D''_n = \{(a_i : i < j) \in D_n : a_{j-1} = 0$ or $j = n - 1\}$. For any $x = (a_i : i < j) \in D''_n$, put $f(x) = \sum(a_i : i < j)$. Then $f : : D''_n \to \{0, \ldots, n - 1\}$ is bijective and $P_x = U_{f(x)}$ for each $x \in D''_n$.

4.6.2. Proposition. *If $\mathscr{U}$ is a partition of a W-space P, then there exists a dyadic expansion $\mathscr{P}$ of P such that $\mathscr{P}''$ is equal to $\mathscr{U}$ re-indexed.*

4.7. Notation. Let P be a W-space. Then the set of all dyadic (all pure dyadic) expansions of P will be denoted by $\mathrm{De}(P)$ (by $\mathrm{De}^*(P)$). If $\mathscr{U} \in \mathrm{Pt}(P)$, then the set of all $\mathscr{P} \in \mathrm{De}(P)$ (all $\mathscr{P} \in \mathrm{De}^*(P)$) such that $\mathscr{P}''$ refines $\mathscr{U}$ will be denoted by $\Phi_{\mathrm{De}}(\mathscr{U})$ (by $\Phi^*_{\mathrm{De}}(\mathscr{U})$). – Cf. [3], 2.13.

4.8. Fact. *Let P be a W-space. Then the collection of all $\Phi_{\mathrm{De}}(\mathscr{U})$, where $\mathscr{U} \in \mathrm{Pt}(P)$, is a filter base on* $\mathrm{De}(P)$, *and the collection of all $\Phi^*_{\mathrm{De}}(\mathscr{U})$, where $\mathscr{U} \in \mathrm{Pt}^*(P)$, is a filter base on* $\mathrm{De}^*(P)$.

Proof. Follows at once from 1.36 and 4.6.1.

4.9. Definition. Let P be a W-space. The filter on $\mathrm{De}(P)$ (on $\mathrm{De}^*(P)$) generated by all $\Phi_{\mathrm{De}}(\mathscr{U})$, where $\mathscr{U} \in \mathrm{Pt}(P)$ (respectively, by all $\Phi^*_{\mathrm{De}}(\mathscr{U})$, where $\mathscr{U} \in \mathrm{Pt}^*(P)$),will be denoted by $\mathscr{F}_{\mathrm{De}}(P)$ (by $\mathscr{F}^*_{\mathrm{De}}(P)$) and will be called the *projective (semiprojective) filter of dyadic expansions (pure dyadic expansions)* of P. The lower limit (see 1.43) of a function g with respect to $\mathscr{F}_{\mathrm{De}}(P)$ (to $\mathscr{F}^*_{\mathrm{De}}(P)$) will also be called the *projective (semiprojective) lower limit* of g. (Cf. 1.44. The danger of confusion is negligible.)

4.10. Definition. Let τ be a gauge functional. Then, for any $(P_0, P_1) \in \mathrm{dom}\,\tau$, we put $\Gamma_\tau(P_0, P_1) = H(wP_0, wP_1)\,\tau(P_0, P_1)$. If $\mathscr{P} = (P_x : x \in D)$ is a dyadic expansion of a W-space P, then we put $\Gamma_\tau(\mathscr{P}) = \sum(\Gamma_\tau(P_{x0}, P_{x1}) : x \in D')$ and call $\Gamma_\tau(\mathscr{P})$ the *τ-value of $\mathscr{P}$.*

4.11. Notation. For any gauge functional τ, the functionals $P \mapsto \mathscr{F}_{\mathrm{De}}(P)\text{-}\underline{\lim}\ \Gamma_\tau(\mathscr{P})$ and $P \mapsto \mathscr{F}^*_{\mathrm{De}}\text{-}\underline{\lim}\ \Gamma_\tau(\mathscr{P})$ will be denoted, respectively, by γ_τ and γ^*_τ. – Remark. It will be proved (4.29) that if τ is normal, then $\gamma_\tau = C_\tau$, $\gamma^*_\tau = C^*_\tau$. In view of this fact, the symbols γ_τ and γ^*_τ will seldom appear in the subsequent sections.

4.12. To prove (see 4.14) that $\gamma^*_\tau, \gamma_\tau$ are regular hypoentropies, we need some simple facts.

4.12.1. Fact. *Let $(K_m : m \in M)$ be a partition of a finite non-void set K. Let all K_m be non-void. For each $k \in K$, let $x_k \in \boldsymbol{R}_+$. Then $H(x_k : k \in K) = \sum(H(x_k : k \in K_m) : : m \in M) + H(\sum(x_k : k \in K_m) : m \in M)$.*

4.12.2. Fact. *Let τ be a gauge functional. Let $\mathscr{P} = (P_x : x \in D)$ be a dyadic expansion of a W-space $P = \langle Q, \varrho, \mu \rangle$. Then $\Gamma_\tau(\mathscr{P}) \leqq d(P)\, H(wP_x : x \in D'')$, and if P is a FW-space and $\mathscr{P}$ is pure, then $\Gamma_\tau(\mathscr{P}) \leqq d(P)\, H(\mu q : q \in Q)$.*

Proof. Clearly, $\Gamma_\tau(\mathscr{P}) = \sum(\Gamma_\tau(P_{x0}, P_{x1}) : x \in D') \leqq d(P) \sum(H(wP_{x0}, wP_{x1}) : x \in \in D')$. From the definition of H, it is easy to see that $\sum(H(wP_{x0}, wP_{x1}) : x \in D') = = H(wP_x : x \in D'')$. If $P \in \mathfrak{W}_F$, $\mathscr{P}$ is pure, then $\mathscr{P}'' = (B_x \,.\, P : x \in D'')$, where $(B_x : x \in D)$ is a partition of Q. We have $\Gamma_\tau(\mathscr{P}) \leqq d(P)\, H(\mu B_x : x \in D'', B_x \neq \emptyset) \leqq \leqq d(P)\, H(\mu q : q \in Q)$, by 4.12.1.

4.12.3. Fact. *Let $\mathscr{P} = (P_x : x \in D)$ be a dyadic expansion of a W-space P. For every $x \in D''$ let $D_x \in \Delta$, and let $\mathscr{T}_x = (T_{x,y} : y \in D_x)$ be a dyadic expansion of P_x. Let $\hat{D} = D \cup \{x \,.\, y : x \in D'', y \in D_x\}$. If $z \in D'$, let $S_z = P_z$; if $z = x \,.\, y$, $x \in D''$, $y \in D_x$, let $S_z = T_{x,y}$. Then $\mathscr{S} = (S_z : z \in \hat{D})$ is a dyadic expansion of P, and $\mathscr{S}$ is pure iff $\mathscr{P}$ and all $\mathscr{T}_x$ are pure. If τ is a gauge functional, then $\Gamma_\tau(\mathscr{S}) = \Gamma_\tau(\mathscr{P}) + + \sum(\Gamma_\tau(\mathscr{T}_x) : x \in D'')$.*

4.12.4. Fact. *Let τ be a gauge functional. Let P be a W-space. Let $\mathscr{P} = (P_x : x \in D)$ be a dyadic expansion of P such that $d(P_x) = 0$ for each $x \in D''$. Let $\mathscr{U}$ be a partition of P. Then there exists a dyadic expansion $\mathscr{S}$ of P such that (1) $\mathscr{S} = (S_x : x \in \hat{D})$, $D \subset \hat{D}$, $S_x = P_x$ for $x \in D$, (2) $\mathscr{S}''$ refines $\mathscr{U}$, (3) $\Gamma_\tau(\mathscr{S}) = \Gamma_\tau(\mathscr{P})$; (4) if $\mathscr{P}$ and $\mathscr{U}$ are pure, then $\mathscr{S}$ is pure.*

Proof. Let $P = \langle Q, \varrho, \mu \rangle$, $\mathscr{U} = (U_k : k \in K)$. By 1.34, there exist $\bar{\mu}$-measurable functions f_x such that $P_x = f_x \,.\, P$ for $x \in D$. For each $x \in D''$ there exists, by 4.6.2, a dyadic expansion $\mathscr{T}_x = (T_{x,y} : y \in D_x)$ of P_x such that $\mathscr{T}''_x$ is equal to $(f_x \,.\, U_k : : k \in K)$ re-indexed. Now let $\mathscr{S}$ be the dyadic expansion described in 4.12.3. Clearly, (1) is satisfied. It is easy to see that (2) is also satisfied. Since $d(P_x) = 0$ for $x \in D''$, we have, by 4.12.2 and 4.12.3, $\Gamma_\tau(\mathscr{S}) = \Gamma_\tau(\mathscr{P})$. Clearly, if $\mathscr{U}$ and $\mathscr{P}$ are pure, then so is $\mathscr{S}$.

4.12.5. Fact. *Let τ be a gauge functional. Let $P = \langle Q, \varrho, \mu \rangle$ be a finite W-space. Let $\mathscr{A}$ be the set of all atoms of the algebra* dom μ. *Let $\mathscr{U}$ be a partition of P. Then there exists a dyadic expansion $\mathscr{S}$ of P such that $\mathscr{S}''$ refines $\mathscr{U}$, $\Gamma_\tau(\mathscr{S}) \leqq d(P)\, H(\mu A : A \in \mathscr{A})$, and $\mathscr{S}$ is pure if $\mathscr{U}$ is pure.*

Proof. Clearly, $(A \,.\, P : A \in \mathscr{A}) \in \mathrm{Pt}^*(P)$. By 4.6.2, there exists a $\mathscr{P} \in \mathrm{De}^*(P)$ such that $\mathscr{P}''$ is equal to $(A \,.\, P : A \in \mathscr{A})$ re-indexed. Since, by 1.29, $d(A \,.\, P) = 0$ for all $A \in \mathscr{A}$, there exists a dyadic expansion $\mathscr{S} = (S_x : x \in D)$ with the properties described in 4.12.4. Since $\Gamma_\tau(\mathscr{S}) = \Gamma_\tau(\mathscr{P})$, we get, by 4.12.2, $\Gamma_\tau(\mathscr{S}) \leqq d(P)\, H(\mu A : A \in \mathscr{A})$.

4.13. To prove (see 4.14) that γ_τ^* is τ-semiprojective and γ_τ is τ-projective (provided $\tau \geqq r$), we need some further simple facts.

4.13.1. Fact. *If τ is a* GF, $P = \langle \{a, b\}, \varrho, \mu \rangle \in \mathfrak{W}_F$, *then* $\gamma_\tau^*(P) = H(\mu a, \mu b)\, \varrho(a, b)$.

4.13.2. Lemma. *Let τ be a* GF. *Let $\mathscr{X} \subset \mathfrak{W}$ and assume that $P \in \mathscr{X}$, $P_1 \leqq P$ imply $P_1 \in \mathscr{X}$. Let φ be a non-negative functional on $\mathscr{X}$. Assume that φ is τ-projective (τ-semiprojective), and $\varphi P \leqq H(\mu a, \mu b)\, \varrho(a, b)$ if $P = \langle \{a, b\}, \varrho, \mu \rangle \in \mathfrak{W}_F$. Then, for any dyadic expansion (pure dyadic expansion) $\mathscr{P} = (P_x : x \in D)$ of a W-space $P \in \mathscr{X}$, we have $\varphi P \leqq \Gamma_\tau(\mathscr{P}) + \sum(\varphi P_x : x \in D'') \leqq d(P)\, H(wP_x : x \in D'') + \sum(\varphi P_x : x \in D'')$. If, in addition, $\varphi S = 0$ for any $S \in \mathscr{X} \cap \mathfrak{W}_F$ such that $w(q \,.\, S) > 0$ for at most one $q \in |S|$, then $\varphi P \leqq \gamma_\tau(P)$ (respectively, $\varphi P \leqq \gamma_\tau^*(P)$) for any $P \in \mathscr{X} \cap \mathfrak{W}_F$.*

Proof. We consider only the case of a τ-projective φ and an arbitrary $\mathscr{P} \in \mathrm{De}(P)$, since the other case is analogous. Let (P_0, P_1) be a partition of P. Let us prove

$$(*) \quad \varphi P \leqq \varphi P_0 + \varphi P_1 + \Gamma_\tau(P_0, P_1)\,.$$

Indeed, if $\tau(P_0, P_1) = \infty$, then $(*)$ is obvious. If $\tau(P_0, P_1) < \infty$, then, since φ is τ-projective, $\varphi P \leqq \varphi P_0 + \varphi P_1 + \varphi[(P_0, P_1)]_\tau$, hence $(*)$ holds.

From $(*)$, the first inequality follows by induction. The second inequality follows from 4.12.2.

Now assume that $\varphi S = 0$ whenever $S \in \mathscr{X} \cap \mathfrak{W}_F$ and $w(q \,.\, S) > 0$ for at most one $q \in |S|$. Let $P \in \mathscr{X} \cap \mathfrak{W}_F$. If $\mathscr{P} = (P_x : x \in D)$ is a dyadic expansion of P such that $\mathscr{P}''$ refines $(q \,.\, P : q \in |P|)$, then $\varphi P \leqq \Gamma_\tau(\mathscr{P}) + \sum(\varphi P_x : x \in D'') = \Gamma_\tau(\mathscr{P})$. This implies $\varphi P \leqq \gamma_\tau(P)$.

4.13.3. Lemma. *Let τ be a gauge functional. Let φ be a regular hypoentropy and let P be an FW-space. If φ is τ-semiprojective, then $\varphi P \leqq \gamma_\tau^*(P)$. If φ is τ-projective, then $\varphi P \leqq \gamma_\tau(P)$.*

This is, in fact, a special case of 4.13.2 (since φ is regular); see also 3.13.2.

4.13.4. Fact. *For any* $P \in \mathfrak{W}_{FC}$, $H(P) \leqq \gamma_r^*(P)$, $H(P) \leqq \gamma_r(P)$.

Proof. Follows from 3.12, 2.14 and 4.13.2.

4.13.5. Fact. *If* σ *and* τ *are gauge functionals and* $\sigma \leqq \tau$, *then* $\gamma_\sigma^* \leqq \gamma_\tau^*$, $\gamma_\sigma \leqq \gamma_\tau$.

4.13.6. Fact. *If* τ *is a gauge functional,* $\tau \geqq r$, $P = \langle\{a, b\}, \varrho, \mu\rangle \in \mathfrak{W}_F$, *then* $\gamma_\tau(P) = H(\mu a, \mu b)\,\varrho(a, b)$.

Proof. Follows from 4.13.4, 4.13.5 and 4.12.5.

4.14. Proposition. *Let* τ *be a gauge functional. Then* γ_τ^* *and* γ_τ *are regular hypoentropies,* γ_τ^* *is* τ*-semiprojective, and if* $\tau \geqq r$, *then* γ_τ *is* τ*-projective.*

Proof. I. It is clear that γ_τ^* and γ_τ satisfy (HE1) and (HE2). By 4.12.5, they also satisfy (HE3) and (HE4). Hence γ_τ^* and γ_τ are hypoentropies. – II. We are going to prove that γ_τ^* and γ_τ are regular. Let P, S be FW-spaces and let $f : P \to S$ be a conservative injective mapping. For any $U \leqq P$, define $f_1 U$ as in 3.18. Clearly, $U \leqq P$ is pure iff $f_1 U \leqq S$ is pure, and $\mathscr{P} = (P_x : x \in D)$ is a dyadic expansion of P iff $(f_1 P_x : x \in D)$ is a dyadic expansion of S. By (GF7), we have $\tau(U, V) = \tau(f_1 U, f_1 V)$ for any $U, V \leqq P$. Hence, for any $\mathscr{P} = (P_x : x \in D) \in \mathrm{De}(P)$, we have $\Gamma_\tau(f_1 P_x : : x \in D) = \Gamma_\tau(\mathscr{P})$. This, together with 3.18, implies $\gamma_\tau^*(P) = \gamma_\tau^*(S)$, $\gamma_\tau(P) = \gamma_\tau(S)$. – III. We are going to prove that if $\tau \geqq r$, then γ_τ is τ-projective. Let $P \in \mathfrak{W}$ and let (P_0, P_1) be a τ-admissible partition of P. Let $b \in \boldsymbol{R}$, $\gamma_\tau(P_0) + \gamma_\tau(P_1) + \gamma_\tau[(P_0, P_1)]_\tau < b$. Then, by 4.13.6, we have $\gamma_\tau(P_0) + \gamma_\tau(P_1) + H(wP_0, wP_1) \cdot \tau(P_0, P_1) < b$. Let $\mathscr{V}$ be a partition of P. By 1.36, there exists a partition $\mathscr{U} = (U_k : k \in K)$ which refines both $\mathscr{V}$ and (P_0, P_1). Choose a partition (K_0, K_1) of K such that $\sum(U_k : k \in K_i) = P_i$, $i = 0, 1$. Since $(U_k : k \in K_i)$ is a partition of P_i, there exists, for $i = 0, 1$, a dyadic expansion $\mathscr{P}_i = (P_{i,x} : x \in D_i)$ of P_i such that $\mathscr{P}_i''$ refines $(U_k : k \in K_i)$ and $\Gamma_\tau(\mathscr{P}_0) + \Gamma_\tau(\mathscr{P}_1) + H(wP_0, wP_1)\,\tau(P_0, P_1) < b$. Let D consist of $\emptyset$ and of all $(i) . x$, where $i = 0, 1$ and $x \in D_i$. Clearly, $D \in \Delta$. Put $P_\emptyset = P$; if $i = 0, 1$, $x \in D_i$, put $P_{(i).x} = P_{i,x}$. Then $\mathscr{P} = (P_z : z \in D)$ is a dyadic expansion of P. Clearly, $\mathscr{P}''$ refines $\mathscr{U}$ and $\Gamma_\tau(\mathscr{P}) = H(wP_0, wP_1)\,\tau(P_0, P_1) + \Gamma_\tau(\mathscr{P}_0) + \Gamma_\tau(\mathscr{P}_1)$, hence $\Gamma_\tau(\mathscr{P}) < b$. This shows that $\gamma_\tau(P) \leqq b$. Hence, $\gamma_\tau(P) \leqq \gamma_\tau(P_0) + \Gamma_\tau(P_1) + \gamma_\tau[(P_0, P_1)]_\tau$. – IV. The proof of τ-semiprojectivity of γ_τ^* is analogous. However, instead of 4.13.5, we use 4.13.1 and therefore the assumption $\tau \geqq r$ can be omitted.

4.15. Proposition. *Let* τ *be a gauge functional. Then, for any* FW*-space* P, $C_\tau^*(P) = \gamma_\tau^*(P)$.

Proof. By 3.17, 3.16, 3.14 and 4.14, $\gamma_\tau^*(P) \leqq C_\tau^*(P)$ for any FW-space P. By 4.13.3, if P is an FW-space, then, for any τ-semiprojective regular hypoentropy φ, we have $\varphi P \leqq \gamma_\tau^*(P)$, which implies $C_\tau^*(P) \leqq \gamma_\tau^*(P)$.

4.16. Definition. Let B be a set. An indexed set $(B_x : x \in D)$ such that $D \in \Delta$, $B_\emptyset = B$ and, for any $x \in D'$, $B_x = B_{x0} \cup B_{x1}$, $B_{x0} \cap B_{x1} = \emptyset$ will be called a *dyadic expansion* of the set B. – The set of all dyadic expansions of a set B will be denoted by $\mathrm{De}^*(B)$.

4.17. Fact. *Let $P = \langle Q, \varrho, \mu \rangle$ be a W-space. If $(B_x : x \in D)$ is a dyadic expansion of Q and all B_x are $\bar{\mu}$-measurable, then $(B_x \,.\, P : x \in D)$ is a pure dyadic expansion of P. If $(P_x : x \in D)$ is a pure dyadic expansion of P, then there exists a dyadic expansion $(B_x : x \in D)$ of Q such that all B_x are $\bar{\mu}$-measurable and $P_x = B_x \,.\, P$ for all $x \in D$.*

4.18. Lemma. *Let P be a W-space, $wP > 0$. Let $\mathscr{P} = (P_x : x \in D)$ be a dyadic expansion of P. Then there exists a dyadic expansion $\mathscr{S} = (S_y : y \in \hat{D})$ of P such that (1) $wS_y > 0$ for each $y \in \hat{D}$, (2) $\mathscr{S}''$ is equal to $(P_x : x \in D'', wP_x > 0)$ re-indexed, (3) for any gauge functional τ, $\Gamma_\tau(\mathscr{S}) = \Gamma_\tau(\mathscr{P})$, (4) if $\mathscr{P}$ is pure, then $\mathscr{S}$ is also pure.*

Proof. Put $m = \operatorname{card}\{x \in D : wP_x = 0\}$. If $m = 0$, we put $\mathscr{S} = \mathscr{P}$. If $m > 0$, then $wP_u = 0$ for some $u \in D''$. Clearly, $u = v\,.\,(k)$, where $v \in D'$, $k = 0$ or $k = 1$. Put $\hat{D} = (D \setminus D(v)) \cup \{v \,.\, y : v \,.\, (\bar{k}) \,.\, y \in D\}$, where $\bar{k} = 1 - k$. It is easy to see that $\hat{D} \in \Delta$. For $x \in D \setminus D(v)$ put $\hat{P}_x = P_x$; for $x = v \,.\, y$, where $v \,.\, (\bar{k}) \,.\, y \in D$, put $\hat{P}_x = P_{v.(\bar{k}).y}$. It is easy to show that $\hat{\mathscr{P}} = (P_x : x \in \hat{D})$ is a dyadic expansion of P and that $\operatorname{card}\{x \in \hat{D} : w\hat{P}_x = 0\} < m$. Clearly, there exists a bijective $f : \{x \in D'' : : wP_x > 0\} \to \{y \in D'' : wP_y > 0\}$ such that $P_x = \hat{P}_{fx}$. It is also easy to see that, for any gauge functional τ, $\Gamma_\tau(\mathscr{P}) = \Gamma_\tau(\hat{\mathscr{P}})$.

Clearly, proceeding in this way we obtain, after at most m steps, a dyadic expansion $\mathscr{S}$ with properties (1)–(4).

4.19. Fact. *If τ is a gauge functional and P is an FW-space, then the set of all $\Gamma_\tau(\mathscr{P})$, where $\mathscr{P} \in \mathrm{De}^*(P)$, is finite.*

Proof. For any $\mathscr{P} \in \mathrm{De}^*(P)$ there exists an $\mathscr{S} = (S_x : x \in D) \in \mathrm{De}^*(P)$ with the properties described in 4.18; in particular, $\Gamma_\tau(\mathscr{S}) = \Gamma_\tau(\mathscr{P})$ and $wS_x > 0$ for each $x \in D$. By 4.17, there exists a dyadic expansion $(B_x : x \in D)$ of $|P|$ such that $S_x = B_x \,.$ $.\, P$ for each $x \in D$. Since $wS_x > 0$, we have $B_x \neq \emptyset$ for all $x \in D$. Since, for any finite set Q, the set of all dyadic expansions $(B_x : x \in D)$ of Q such that all B_x are non-void is finite, the assertion is proved.

4.20. Proposition. *Let τ be a gauge functional. Let $P = \langle Q, \varrho, \mu \rangle$ be an FW-space. Then $C^*_\tau(P)$ is equal to the least of all $\Gamma_\tau(\mathscr{P})$ where $\mathscr{P} = (B_x \,.\, P : x \in D)$, $(B_x : x \in D) \in \mathrm{De}^*(Q)$, $\operatorname{card} B_x = 1$ for all $x \in D''$. If $wP > 0$, then $C^*_\tau(P)$ is also equal to the least of all $\Gamma_\tau(\mathscr{P})$ where $\mathscr{P} = (B_x \,.\, P : x \in D)$, $(B_x : x \in D) \in \mathrm{De}^*\{q \in$ $\in Q : \mu q > 0\}$, $\operatorname{card} B_x = 1$ for all $x \in D''$.*

Proof. Put $\mathscr{V} = (q \, . \, P : q \in Q)$. Then $\Phi^*_{\mathrm{De}}(\mathscr{V})$ is the least set in the filter $\mathscr{F}^*_{\mathrm{De}}(P)$. Hence, by 4.15, $C^*_\tau(P)$ is equal to the infimum of all $\Gamma_\tau(\mathscr{P})$ where $\mathscr{P} \in \Phi^*_{\mathrm{De}}(\mathscr{V})$. In particular, $C^*_\tau(P) \leqq \Gamma_\tau(B_x \, . \, P : x \in D)$ for all $(B_x : x \in D) \in \mathrm{De}^*(Q)$ of the form described in the proposition. First we prove the second assertion. By 4.19, there exists a $\mathscr{P} \in \Phi^*_{\mathrm{De}}(\mathscr{V})$ such that $C^*_\tau(P) = \Gamma_\tau(\mathscr{P})$. Let $\mathscr{S} = (S_y : y \in \hat{D}) \in \mathrm{De}^*(P)$ possess the properties described in 4.18. We have $\Gamma_\tau(\mathscr{S}) = \Gamma_\tau(\mathscr{P}) = C^*_\tau(P)$. By 4.17, there is an $\mathscr{M} = (M_y : y \in \hat{D}) \in \mathrm{De}^*(Q)$ such that $S_y = M_y \, . \, P$ for each $y \in \hat{D}$. Put $B_y = \{q \in M : \mu q > 0\}$. Clearly, $S_y = B_y \, . \, P$ for each $y \in \hat{D}$, $(B_y : y \in \hat{D}) \in \mathrm{De}^*\{q \in Q : \mu q > 0\}$ and card $B_y = 1$ for each $y \in \hat{D}''$.

The first assertion is obvious if $wP = 0$. Let $wP > 0$. We have already shown that there is a $\mathscr{P} \in \mathrm{De}^*(P)$ such that $C^*_\tau(P) = \Gamma_\tau(\mathscr{P})$ and that $\mathscr{P}$ is of the form $(B_x \, . \, P : x \in D)$ described in the second assertion. Put $B = \{q \in Q : \mu q > 0\}$, $A = Q \setminus B$. We may assume $A \neq \emptyset$. Let $(A_z : z \in \tilde{D}) \in \mathrm{De}^*(A)$, card $A_z = 1$ for each $z \in \tilde{D}''$. Let $\hat{D}$ consist of $\emptyset$, of all $(0) \, . \, x$, $x \in D$, and all $(1) \, . \, z$, $z \in \tilde{D}$. Put $K_\emptyset = Q$, $K_{(0).z} = B_x$, $K_{(1).z} = A_z$. Then $(K_y : y \in \hat{D})$ has all the properties required.

4.21. By 4.20, for any gauge functional τ and any FW-space P, it is possible to calculate $C^*_\tau(P)$ in a finite number of steps. This is an important fact. Nonetheless, the number of steps can be quite large. A trivial estimate is, roughly, the number of dyadic expansions $(B_z : z \in D)$, with all B_z non-void, of a set of n elements. I do not know whether it is possible to give a substantially better estimate. The following question also remains open: given an $\varepsilon > 0$, to find an estimate for the number of steps necessary to find, for an FW-space P, a $\mathscr{P} \in \mathrm{De}^*(P)$ such that $|\Gamma_\tau(\mathscr{P}) - C^*_\tau(P)| < \varepsilon$.

4.22. Proposition. *Let τ be a gauge functional. Let P be an FW-space. Then $C_\tau(P) = \gamma_\tau(P)$.*

Proof. I. Let $b \in R$, $C_\tau(P) < b$. Let $\mathscr{U} \in \mathrm{Pt}(P)$. Then, by 1.36, there exists a $\mathscr{V} = (V_k : k \in K) \in \mathrm{Pt}(P)$ refining both $\mathscr{U}$ and $(q \, . \, P : q \in |P|)$ and such that $C^*_\tau[\mathscr{V}]_\tau < b$. Put $S = [\mathscr{V}]_\tau$. By 4.20, there exists an $\mathscr{S} = (S_x : x \in D) \in \mathrm{De}^*(S)$ such that $\mathscr{S}''$ is equal to $(k \, . \, S : k \in K)$ re-indexed and that $\Gamma_\tau(\mathscr{S}) = C^*_\tau(S)$. Let $f : D'' \to K$ be bijective, $S_x = f(x)$. S for all $x \in D''$. For $x \in D$, put $P_x = \sum(V_{fy} : y \in D(x) \cap D'')$. Since $\mathscr{V}$ refines $(q \, . \, P : q \in |P|)$, it is easy to see that there exists a conservative $g : S \to P$. From (GF7) it follows easily that $\Gamma_\tau(\mathscr{P}) = \Gamma_\tau(\mathscr{S})$, hence $\Gamma_\tau(\mathscr{P}) < b$. Clearly, $\mathscr{P}''$ refines $\mathscr{U}$. Since $\mathscr{U} \in \mathrm{Pt}(P)$ was arbitrary, this proves $\gamma_\tau(P) \leqq b$, hence $\gamma_\tau(P) \leqq C_\tau(P)$. — II. Let $b \in R$, $\gamma_\tau(P) < b$. Let $\mathscr{U} \in \mathrm{Pt}(P)$. Then there exists a $\mathscr{P} = (P_x : x \in D) \in \mathrm{De}(P)$ such that $\Gamma_\tau(\mathscr{P}) < b$ and $\mathscr{P}''$ refines both $\mathscr{U}$ and $(q \, . \, P : q \in |P|)$. Put $S = [\mathscr{P}'']_\tau$. For $x \in D$, put $S_x = (D(x) \cap D'') \, . \, S$. Then $\mathscr{S} = (S_x : x \in D)$ is a dyadic expansion of S and $\mathscr{S}'' = (t \, . \, S : t \in |S|)$. It is easy to show, using (GF7), that $\Gamma_\tau(\mathscr{P}) = \Gamma_\tau(\mathscr{S})$. Hence, $\Gamma_\tau(\mathscr{S}) < b$ and therefore, by 4.20, $C^*_\tau[\mathscr{P}'']_\tau = C^*_\tau(S) < b$. Since $\mathscr{U} \in \mathrm{Pt}(P)$ was arbitrary, this proves $C_\tau(P) \leqq b$, hence $C_\tau(P) \leqq \gamma_\tau(P)$.

4.23. Proposition. *Let τ be a gauge functional, $\tau \geqq r$. Then $C_\tau \upharpoonright \mathfrak{W}_F$ is the greatest τ-projective regular hypoentropy on $\mathfrak{W}_F$.*

Proof. Follows from 4.22, 4.14 and 4.13.3.

4.24. Proposition. *Let τ be a gauge functional, $\tau \geqq r$. Then, for any FW-space P, $C_\tau(P)$ is equal to the infimum of all $\Gamma_\tau(\mathscr{P})$, where $\mathscr{P}$ is a dyadic expansion of P such that $\mathscr{P}''$ refines $(q \,.\, P : q \in |P|)$.*

Proof. Let b denote the infimum in question. Clearly, $\gamma_\tau(P) \geqq b$, hence, by 4.22, $C_\tau(P) \geqq b$. By 4.23, $C_\tau \upharpoonright \mathfrak{W}_F$ is τ-projective. Since C_τ is a regular hypoentropy, $C_\tau(q \,.\, P) = 0$ for any $q \in |P|$. Hence, by 4.13.2, $C_\tau(P) \leqq \Gamma_\tau(\mathscr{P})$ for any $\mathscr{P} \in \mathrm{De}(P)$ such that $\mathscr{P}''$ refines $(q \,.\, P : q \in |P|)$, and therefore $C_\tau(P) \leqq b$.

4.25. Proposition. *Let τ be a gauge functional, $\tau \geqq r$. Then for any $P = \langle Q, a_Q, \mu \rangle \in \mathfrak{W}_{FC}$, $C^*_\tau(P)$ and $C_\tau(P)$ are equal to $H(P) = a\, H(\mu)$, the Shannon entropy of P.* — For $\tau = r$, cf. [3], 3.2 and 3.5.

Proof. By 3.12 and 4.13.2, $H(P) \leqq \Gamma_r(\mathscr{P})$ whenever $\mathscr{P} \in \mathrm{De}(P)$ and $\mathscr{P}''$ refines $(q \,.\, P : q \in |P|)$. Hence $H(P) \leqq \gamma^*_r(P)$, $H(P) \leqq \gamma_r(P)$. By 4.12.5, we get $\gamma^*_\tau(P) \leqq d(P)\, H(\mu)$, $\gamma_\tau(P) \leqq d(P)\, H(\mu)$. By 4.15, 4.22 and 3.20, this proves the proposition.

4.26. In the rest of this section, and also in Section 5, we shall need some assumptions on gauge functionals τ, namely some (or all) of the (NGF1)–(NGF3). To be precise: I do not know whether these assumptions are always necessary; however, the proofs presented here do not work unless some of the (NGFi), $i = 1, 2, 3$, are assumed. — To simplify the statements, we will assume, as a rule, that τ is an NGF, even if (GF1)–(GF7) plus some specified (NGFi) are sufficient.

4.27.1. Fact. *Let τ be a normal gauge functional. Let $\mathscr{P} = (P_x : x \in D)$ be a dyadic expasion of a W-space P. Assume that the partition $\mathscr{P}''$ is τ-admissible. Put $S = [\mathscr{P}'']_\tau$, $T_x = D'' \cap D(x)$, $S_x = T_x \,.\, S$ for any $x \in D$, $\mathscr{S} = (S_x : x \in D)$. Then $\mathscr{S}$ is a pure dyadic expansion of S, $\Gamma_\tau(\mathscr{S}) = \Gamma_\tau(\mathscr{P}) \geqq C^*_\tau[\mathscr{P}'']_\tau$.*

Proof. If $x \in D'$, then $T_{x0} \cap T_{x1} = \emptyset$. By (NGF1), we get $\tau(S_{x0}, S_{x1}) = \tau(P_{x0}, P_{x1})$, hence $\Gamma_\tau(S_{x0}, S_{x1}) = \Gamma_\tau(P_{x0}, P_{x1})$ and therefore $\Gamma_\tau(\mathscr{S}) = \Gamma_\tau(\mathscr{P})$. The inequality $\Gamma_\tau(\mathscr{S}) \geqq C^*_\tau(S)$ follows from 4.20.

4.27.2. Fact. *Let τ be a normal gauge functional. Let $\mathscr{P}$ be a dyadic expansion of a W-space P. If $\mathscr{P}''$ is not τ-admissible, then $\Gamma_\tau(\mathscr{P}) = \infty$.*

Proof. Let $\mathscr{P} = (P_x : x \in D)$. Since $\mathscr{P}''$ is not τ-admissible, there exist $x, y \in D''$ such that $x \neq y$, $\tau(P_x, P_y) = \infty$. Put $Z = \{u \in D : u \lesssim x, u \lesssim y\}$. Choose a maximal (with respect to $\lesssim$) element $z \in Z$. Clearly, we have $zi \lesssim x$, $zj \lesssim y$ for appropriate

$i, j = 0, 1,\ i \neq j$. Since $\tau(P_x, P_y) = \infty$, we have, by (GF3), $wP_x > 0$, $wP_y > 0$, hence $wP_{zi} > 0$, $wP_{zj} > 0$. By (NGF2), we get $wP_x \,.\, wP_y \,.\, \tau(P_x, P_y) \leqq wP_{zi} \,.\, wP_{zj} \,.\, \tau(P_{zi}, P_{zj})$, hence $\tau(P_{zi}, P_{zj}) = \infty$, $\Gamma_\tau(P_{zi}, P_{zj}) = \infty$, $\Gamma_\tau(\mathscr{P}) = \infty$.

4.28. Proposition. *Let τ be a normal gauge functional. Let $\mathscr{U}$ be a τ-admissible partition of a W-space P. Then there exists a dyadic expansion $\mathscr{P}$ of P such that $\mathscr{P}''$ is equal to $\mathscr{U}$ re-indexed, $\Gamma_\tau(\mathscr{P}) = C^*_\tau[\mathscr{U}]_\tau$.*

Proof. Let $\mathscr{U} = (U_k : k \in K)$. By the regularity of C^*_τ, we may assume that $wU_k > 0$ for all $k \in K$. By 4.20. there exists a pure dyadic expansion $\mathscr{S} = (S_x : x \in D)$ of $S = [\mathscr{U}]_\tau$ such that $\Gamma_\tau(\mathscr{S}) = C^*_\tau(S)$ and $\mathscr{S}''$ is equal to $(k \,.\, S : K \in K)$ re-indexed. Let $f : D'' \to K$ be bijective, $S_z = fz \,.\, S$ for each $z \in D''$. For each $x \in D$, put $P_x = \sum(U_{fy} : y \in D(x) \cap D'')$. Then $\mathscr{P} = (P_x : x \in D)$ is a dyadic expansion of P, $\mathscr{P}'' = (U_{fz} : z \in D'')$, hence $\mathscr{P}''$ is equal to $\mathscr{U}$ re-indexed. By (NGF1), we get $\Gamma_\tau(\mathscr{P}) = \Gamma_\tau(\mathscr{S}) = C^*_\tau(S) = C^*_\tau[\mathscr{U}]_\tau$.

4.29. Proposition. *Let τ be a normal gauge functional. Let P be a semimetrized measure space. Then $C^*_\tau(P) = \gamma^*_\tau(P)$, $C_\tau(P) = \gamma_\tau(P)$.*

Proof. Both assertions are proved in an analogous way. Therefore we prove only the latter. First, we will prove $C_\tau(P) \leqq \mathscr{F}\text{-}\underline{\lim}\, \Gamma_\tau(P)$, where $\mathscr{F}$ stands for $\mathscr{F}_{\mathrm{De}}(P)$. We can assume $\mathscr{F}\text{-}\underline{\lim}\, \Gamma_\tau(\mathscr{P}) < \infty$. Now, let $b \in \boldsymbol{R}$, $b > \mathscr{F}\text{-}\underline{\lim}\, \Gamma_\tau(\mathscr{P})$. Let $\mathscr{V}$ be an arbitrary partition of P. Then there exists a dyadic expansion $\mathscr{P}$ of P such that $\mathscr{P}''$ refines $\mathscr{V}$ and $\Gamma_\tau(\mathscr{P}) < b$. By 4.27.2, $\mathscr{P}''$ is τ-admissible; by 4.27.1 we have $\Gamma_\tau(\mathscr{P}) \geqq C_\tau[\mathscr{P}'']_\tau$. This implies $b \geqq C_\tau(P)$. Hence $\mathscr{F}\text{-}\underline{\lim}\, \Gamma_\tau(\mathscr{P}) \geqq C_\tau(P)$.

We now prove $\mathscr{F}\text{-}\underline{\lim}\, \Gamma_\tau(\mathscr{P}) \leqq C_\tau(P)$. We can assume $C_\tau(P) < \infty$. Choose $b \in \boldsymbol{R}$, $b > C_\tau(P)$. Let $\mathscr{V}$ be an arbitrary partition of P. Then there exists a τ-admissible partition $\mathscr{U}$ finer than $\mathscr{V}$ and such that $C^*_\tau[\mathscr{U}]_\tau < b$. By 4.28, there exists a dyadic expansion $\mathscr{P}$ of P such that $\mathscr{P}''$ is equal to $\mathscr{U}$ re-indexed and $\Gamma_\tau(\mathscr{P}) = C^*_\tau[\mathscr{U}]_\tau$, hence $\Gamma_\tau(\mathscr{P}) < b$. This implies $\mathscr{F}\text{-}\underline{\lim}\, \Gamma_\tau(\mathscr{P}) \leqq b$. Hence $\mathscr{F}\text{-}\underline{\lim}\, \Gamma_\tau(\mathscr{P}) \leqq C_\tau(P)$.

5

In this section we prove some propositions concerning continuity of C^*_τ and C_τ. These propositions and the results of Section 3 and 4 immediately yield the main theorems (see Section 6) of the present Part I. We add some observations concerning certain fairly mild conditions fulfilled by all C^*_τ and C_τ (where τ is a normal gauge functional and $\tau \geqq r$) but strong enough to exclude some "bad" e.S. semientropies, e.g. those equal to ∞ for every infinite W-space.

5.1. Proposition. *Let τ be a normal gauge functional. Let $P_i = \langle Q, \varrho_i, \mu \rangle \in \mathfrak{W}_F$, $i = 1, 2$. Then $|C^*_\tau(P_1) - C^*_\tau(P_2)| \leqq \mathrm{dist}_Q(P_1, P_2) \,.\, H(\mu q : q \in Q)$.*

Proof. Put $a = \operatorname{dist}(P_1, P_2)$. Clearly, $\varrho_1 \leqq \varrho_2 + a_Q$, $\varrho_2 \leqq \varrho + a_Q$, hence, by (NGF3), for any $v' \leqq \mu$, $v'' \leqq \mu$, we have $|\tau(\langle Q, \varrho_1, v' \rangle, \langle Q, \varrho_1, v'' \rangle) - \tau(\langle Q, \varrho_2, v' \rangle, \langle Q, \varrho_2, v'' \rangle)| \leqq a$. It is now easy to see that if $\mathscr{P} = (P_x : x \in D)$, $\mathscr{S} = (S_x : x \in D)$ are pure dyadic expansions of P_1 and P_2, respectively, if $\mathscr{P}''$ refines $(q \,.\, P_1 : q \in Q)$, $\mathscr{S}''$ refines $(q \,.\, P_2 : q \in Q)$, and if $P_x = \langle Q, \varrho_1, \mu_x \rangle$, $S_x = \langle Q, \varrho_2, \mu_x \rangle$ for all $x \in D$, then $|\Gamma_\tau(\mathscr{P}) - \Gamma_\tau(\mathscr{S})| \leqq a \sum(H(\mu_{x0}, Q, \mu_{x1} Q) : x \in D') = a\, H(\mu q : q \in Q)$. By 4.20, this proves the proposition.

5.2. Fact. *Let τ be an* NGF. *Let $P_n = \langle Q, \varrho_n, \mu \rangle$, $n \in N$, $P = \langle Q, \varrho, \mu \rangle$ be FW-spaces. If $P_n \to P$, then $C_\tau^*(P_n) \to C_\tau^*(P)$.*

Proof. Follows at once from 5.1.

5.3. Lemma. *Let τ be a normal gauge functional. Let $P = \langle Q, \varrho, \mu \rangle$, $S = \langle Q, \varrho, v \rangle$, $P_n = \langle Q, \varrho_n, \mu \rangle$, $S_n = \langle Q, \varrho_n, v \rangle$, $n \in N$, be FW-spaces. Let $P_n \to P$, $S_n \to S$. Then $\tau(P_n, S_n) \to \tau(P, S)$.*

Proof. Clearly, for any p, $0 < p < 1$, we have, for all sufficiently large n, $\varrho_n \geqq p\varrho$, hence $\tau(P_n, S_n) \geqq p\tau(P, S)$. On the other hand, if $a > 0$, then for all sufficiently large n we have $\varrho_n \leqq \varrho + a$, hence, by (NGF3), $\tau(P_n, S_n) \leqq \tau(P, S) + a$. This implies $\overline{\lim}\, \tau(P_n, S_n) \leqq \tau(P, S)$ and proves the lemma.

5.4. Proposition. *Let τ be a normal gauge functional. Let $P_n = \langle Q, \varrho_n, \mu \rangle$, $n \in N$, $P = \langle Q, \varrho, \mu \rangle$ be FW-spaces. If $P_n \to P$, then $C_\tau(P_n) \to C_\tau(P)$.*

Proof. For any t, $0 < t < 1$, we have, for all sufficiently large n, $\varrho_n \geqq t\varrho$, hence $C_\tau(P_n) \geqq t\, C_\tau(P)$. This implies $\underline{\lim}\, C_\tau(P_n) \geqq C_\tau(P)$. If $C_\tau(P) = \infty$, this proves $C_\tau(P_n) \to C_\tau(P)$. Hence we consider only the case $C_\tau(P) < \infty$. Let b be an arbitrary number greater than $C_\tau(P)$. By 3.22, there exists a partition $\mathscr{U} = (U_k : k \in K)$ of P such that $\mathscr{U}''$ refines $(q \,.\, P : q \in Q)$ and $C_\tau^*[\mathscr{U}]_\tau < b$. Put $U_k = \langle Q, \varrho, \mu_k \rangle$. For $n \in N$, $k \in K$, put $U_{n,k} = \langle Q, \varrho_n, \mu_k \rangle$, $\mathscr{U}_n = (U_{n,k} : k \in K)$, $[\mathscr{U}_n]_\tau = \langle K, \sigma_n, v_n \rangle$, $[\mathscr{U}]_\tau = \langle K, \sigma, v \rangle$. Clearly, $v_n = v$ for all $n \in N$. Since, for $i, j \in K$, $i \neq j$, we have $\sigma_n(i, j) = \tau(U_{n,i}, U_{n,j})$, $\sigma(i, j) = \tau(U_i, U_j)$, we get, by 5.3, $\sigma_n \to \sigma$. Hence, by 5.2, $C_\tau^*[\mathscr{U}_n]_\tau \to C_\tau^*[\mathscr{U}]_\tau$ and therefore $C_\tau^*[\mathscr{U}_n]_\tau < b$ for all sufficiently large n. Since, by 3.22, $C_\tau(P_n) \leqq C_\tau^*[\mathscr{U}_n]_\tau$, we have $\overline{\lim}\, C_\tau(P_n) \leqq b$, hence $\overline{\lim}\, C_\tau(P_n) \leqq C_\tau(P)$. This proves the proposition.

5.5. Lemma. *Let τ be a normal gauge functional. Let $P = \langle Q, \varrho, \mu \rangle$, $S = \langle Q, \varrho, v \rangle$, $P_n = \langle Q, \varrho, \mu_n \rangle$, $S_n = \langle Q, \varrho, v_n \rangle$, $n \in N$, be FW-spaces. If $P_n \to P$, $S_n \to S$, then $\underline{\lim}\, \tau(P_n, S_n) \geqq \tau(P, S)$. If, in addition, for any $q \in Q$ and any $\mu < 1$' $\mu_n q \leqq u \,.\, \mu q$, $v_n q \leqq u \,.\, vq$ for all sufficiently large n (in particular, if $\mu q > 0$, $vq > 0$ for all $q \in Q$), then $\tau(P_n, S_n) \to \tau(P, S)$.*

Proof. We can assume $wP > 0$, $wS > 0$. Let $0 < t < 1$. Then, for all sufficiently large n, $P_n \geqq tP$, $S_n \geqq tS$, hence, by (NGF2), $wP_n \,.\, wS_n \,.\, \tau(P_n, S_n) \geqq w(tP) \,.\, w(tS)$.

$. \tau(tP, tS) = t^2 wP . wS . \tau(P, S)$. Since $wP_n . wS_n \to wP . wS > 0$, the first assertion is proved. – To prove the second assertion, let $u > 1$. Then, for all sufficiently large n, we have $P_n \leqq uP$, $S_n \leqq uS$, hence, by (NGF2), $wP_n . wS_n . \tau(P_n, S_n)| \leqq w(uP)$, $w(uS) . \tau(uP, uS) = u^2 . wP . wS . . \tau(P, S)$. Since $wP_n . wS_n \to wS . wP > 0$, we get $\overline{\lim}\ \tau(P_n, S_n)| \leqq \tau(P, S)$.

5.6. Lemma. *Let τ be a normal gauge functional. Let $P = \langle Q, \varrho, \mu \rangle$, $P_n = \langle Q, \varrho, \mu_n \rangle$, where $n \in N$, be FW-spaces. If $P_n \to P$, then $\underline{\lim}\ C_\tau^*(P_n) \geqq C_\tau^*(P)$. If, in addition, all $\mu q > 0$ for all $q \in Q$, then $C_\tau^*(P_n) \to C_\tau^*(P)$.*

Proof. Suppose $\underline{\lim}\ C_\tau^*(P_n) = b < C_\tau^*(P)$. Then it is easy to see (using 4.20) that there exists a subsequence $(P_{n(m)} : m \in N)$ of (P_n) and a dyadic expansion $(B_z : z \in D)$ of Q such that card $B_z = 1$ whenever $z \in D''$ and $\Gamma_\tau(\mathscr{P}_m) \to b$ where $\mathscr{P}_m = (B_z . P_{n(m)} : z \in D)$. By 5.5, we now get $\Gamma_\tau(\mathscr{P}) \leqq b$, where $\mathscr{P} = (B_z . P : z \in D)$, which contradicts $b < C_\tau^*(P)$.

Let all $\mu q > 0$ for all $q \in Q$. By 4.20, there exists a dyadic expansion $(B_z : z \in D)$ of Q such that card $B_z = 1$ for $z \in D''$ and $\Gamma_\tau(B_z . P : z \in D) = C_\tau^*(P)$. Since $w(B_z . P) > 0$, $w(B_z . P_n) > 0$ for all $z \in D$ and all sufficiently large n, we get, by 5.5 (second assertion), $\Gamma_\tau(B_{z0} . P_n, B_{z1} . P_n) \to \Gamma_\tau(B_{z0} . P, B_{z1} . P)$ for each $z \in D'$. Now, $C_\tau^*(P_n) \leqq \Gamma_\tau(B_z . P_n : z \in D)$, which proves $\overline{\lim}\ C_\tau^*(P_n) \leqq C_\tau^*(P)$, hence $C_\tau^*(P_n) \to C_\tau^*(P)$.

5.7. Proposition. *Let τ be a normal gauge functional. Then C_τ^* is finitely continuous.*

Proof. Let Q be a non-void finite set. Let ϱ be a semimetric on Q. We are going to prove that C_τ^* is continuous on $\mathfrak{W}(Q, \varrho, \cdot) \cap \mathfrak{W}_F$. Let $P = \langle Q, \varrho, \mu \rangle$, $P_n = \langle Q, \varrho, \mu_n \rangle$ be FW-spaces and let $\mu_n \to \mu$. By 5.6, $\underline{\lim}\ C_\tau^*(P_n) \geqq C_\tau^*(P)$. Put $B = \{q \in Q : \mu q > 0\}$, $A = Q \setminus B$, $S_n = B . P_n$, $T_n = A . P_n$. Since C_τ^* is regular, we get, by 5.6, $C_\tau^*(S_n) \to C_\tau^*(P)$. Since C_τ^* is τ-semiprojective, $C_\tau^*(P_n) \leqq C_\tau^*(S_n) + C_\tau^*(T_n) + H(wS_n, wT_n)\, d(P_n)$. Since $wT_n \to 0$, we get $\overline{\lim}\ C_\tau^*(P_n) \leqq \lim C_\tau^*(S_n) = C_\tau^*(P)$.

Now let $P = \langle Q, \varrho, \mu \rangle$, $P_n = \langle Q, \varrho_n, \mu_n \rangle$, $n \in N$, be FW-spaces, and let $P_n \to P$. Clearly, there exists a number b such that $H(\mu q : q \in Q) \leqq b$, $H(\mu_n q : q \in Q) \leqq b$ for all $n \in N$. For $n \in N$, put $S_n = \langle Q, \varrho, \mu_n \rangle$. By 5.1, we have $|C_\tau^*(S_n) - C_\tau^*(P_n)| \leqq b . \operatorname{dist}(S_n, P_n)$. Clearly, $\operatorname{dist}(P_n, S_n) \leqq \operatorname{dist}(P_n, P)$, hence $|C_\tau^*(S_n) - C_\tau^*(P_n)| \to 0$ for $n \to \infty$. Since, as already shown, $C_\tau^*(S_n) \to C_\tau^*(P)$, we get $C_\tau^*(P_n) \to C_\tau^*(P)$.

5.8. Lemma. *Let τ be a gauge functional. Let φ be a τ-projective regular hypoentropy on $\mathfrak{W}_F$. Let $P_n = \langle Q, \varrho, \mu_n \rangle$, $P = \langle Q, \varrho, \mu \rangle$ be FW-spaces and let $\mu q > 0$ for all $q \in Q$. If $P_n \to P$, then $\varphi P_n \to \varphi P$.*

Proof. I. Let $t > 1$, $\bar{P} = \langle Q, \varrho, t\mu \rangle$. Then, for all sufficiently large n, we have $P_n \leqq \bar{P}$, hence, φ being τ-projective, $t\varphi P = \varphi \bar{P} \leqq \varphi P_n + \varphi(\bar{P} - P_n) + H(wP_n, w(\bar{P} - P_n))\, \tau(P_n, \bar{P} - P_n))$. By 3.13.1 and 3.13.2, $\varphi(\bar{P} - P_n) \leqq d(P)\, H(t . \mu q -$

$- \mu_n q : q \in Q)$. Clearly, $\tau(P_n, \bar{P} - P_n) \leqq d(P)$. If $n \to \infty$, then $H(t\mu q - \mu_n q : : q \in Q) \to (t - 1) H(\mu q : q \in Q)$, $H(wP_n, w(\bar{P} - P_n)) \to H(wP, (t - 1) wP) = wP . . H(1, t - 1)$. Hence we have $\underline{\lim}\ \varphi P_n \geqq t\varphi P - (t - 1) d(P) H(\mu q : q \in Q) - wP . . d(P) H(1, t - 1)$. Since $t > 1$ was arbitrary, we get $\underline{\lim}\ \varphi P_n \geqq \varphi P$.

II. Let $0 < u < 1$, $\hat{P} = \langle Q, \varrho, u\mu \rangle$. Then, for all sufficiently large n, $\hat{P} \leqq P_n$, hence $\varphi P_n \leqq \varphi \hat{P} + \varphi(P_n - \hat{P}) + H(w\hat{P}, w(P_n - \hat{P})) \tau(\hat{P}, P_n - \hat{P})$. By 3.13.1 and 3.13.2, $\varphi(P_n - \hat{P}) \leqq d(P) H(\mu_n q - u\mu q : q \in Q)$, hence $\overline{\lim}\ \varphi(P_n - \hat{P}) \leqq (1 - u) . . d(P) H(\mu q : q \in Q)$. Clearly, $\overline{\lim}\ H(w\hat{P}, w(P_n - \hat{P})) \tau(\hat{P}, P_n - P) \leqq wP/d(P) . . H(1, 1 - u)$. Thus, we get $\overline{\lim}\ \varphi P_n \leqq \varphi \hat{P} + (1 - u) d(P) H(\mu q : q \in Q) + wP . . d(P) H(1, 1 - u)$. Since $u < 1$ is arbitrary, we have $\overline{\lim}\ \varphi P_n \leqq \varphi P$. This proves the lemma.

5.9. Proposition. *Let τ be a normal gauge functional. Then C_τ is finitely feebly continuous.* — This follows from 5.4 and 5.8.

5.10. The functionals C_τ^* and C_τ, where τ is an NGF and $\tau \geqq r$, are, in fact, a rather special case of e.S. semientropies on $\mathfrak{W}$. There is a lot of other e.S. semientropies (and even e.S. entropies) on $\mathfrak{W}$. Some of these possess fairly reasonable properties, whereas some are not nice at all and do not seem to be useful. We now present some examples.

5.11.1. Notation. If f, g are functionals, $\operatorname{dom} f = \operatorname{dom} g$, then $\min(f, g)$ and $\max(f, g)$ denote the functionals $x \mapsto \min(fx, fg)$ and $x \mapsto \max(fx, gx)$, respectively.

5.11.2. Fact. *Let $\mathscr{X} = \mathfrak{W}_F$ or $\mathscr{X} = \mathfrak{W}$. Let ψ_1 and ψ_2 be e.S. semientropies (entropies) on $\mathscr{X}$. Then* $\min(\psi_1, \psi_2)$, $\max(\psi_1, \psi_2)$ *and $a_1\psi_1 + a_2\psi_2$, where $a_1, a_2 \in \in \mathbf{R}_+$, $a_1 + a_2 = 1$, are e.S. semientropies (entropies) on $\mathscr{X}$.*

5.11.3. Let $a > 0$, $b > 0$, $a + b = 1$. Then the e.S. semientropy (on $\mathfrak{W}$) $\varphi = = aC_r^* + bC_r$ is not of the sort described in 5.10. Moreover, if τ is a GF and $\tau \geqq r$, then there are FW-spaces P and S such that $\varphi P < C_\tau^*(P)$, $\varphi S \neq C_\tau(S)$. — This follows from the fact (which will be proved in Part II) that there exists an FW-space P such that $C_r(P) < C_r^*(P)$. Indeed, clearly, $a\, C_r^*(P) + b\, C_r(P) < C_r^*(P)$, hence, by 3.20, $\varphi P < C_\tau^*(P)$. By 3.23, for some FW-space S such that there exists a conservative mapping $f : S \to P$, we have $C_r^*(S) < C_r^*(P)$. Now, clearly, either $\varphi P > C_\tau(P)$, or $\varphi P \leqq C_\tau(P)$ and therefore, C_τ and C_r being strongly regular, $\varphi S = aC_r^*(S) + + b\, C_r(S) < C_\tau(S)$.

5.12.1. Let ψ be an e.S. semientropy or an entropy on $\mathfrak{W}_F$. We define η_ψ as follows: (1) if $P \in \mathfrak{W}$ is finite, then $\eta_\psi(P) = [\psi]_E^*(P)$, (2) if $P \in \mathfrak{W}$ is infinite, then $\eta_\psi(P) = \infty$. By 3.16, $\eta_\psi \restriction \mathfrak{W}_F = \psi$, and it is easy to prove that $[\psi]_E^*(P)$ is finite if P is finite. Hence η_ψ is an e.S. semientropy or, respectively, an entropy on $\mathfrak{W}$. — It is clear that the functionals η_ψ are of little interest (except as counterexamples):

for all ψ and all infinite $P \in \mathfrak{W}$, $\eta_\psi(P)$ has the same value, whereas η_ψ coincides with ψ on $\mathfrak{W}_F$.

5.12.2. The functionals η_ψ are not of the kind described in 5.10. — This assertion, intuitively fairly clear, is a consequence of 5.12.5 below.

5.12.3. Fact. *Let ψ be an e.S. semientropy on $\mathfrak{W}$ such that* (SR') *if $P \in \mathfrak{W}_F$, $S \in \mathfrak{W}_{FC}$ and there exists a conservative mapping $f : P \to S$, then $\psi P = \psi S$. Let* $P = \langle N, 1, \mu \rangle \in \mathfrak{W}$, $\operatorname{dom} \mu = \exp N$, $H(\mu) < \infty$. *Then. for any partition $\mathscr{U}$ of P and any real $a > H(\mu)$, there exists a partition $\mathscr{V}$ such that $\psi[\mathscr{V}]_E < a$ and $\mathscr{V}$ is a relatively pure refinement of $\mathscr{U}$.*

Proof. Choose $\varepsilon > 0$ such that $H(\mu) + \varepsilon < a$. Let $\mathscr{U} = (U_k : k \in K)$ be a partition of P. We can assume that $K \cap N = \emptyset$. Put $m = \operatorname{card} K$. Choose $\delta > 0$ such that $\delta \log m < \varepsilon/2$ and $H(wP, \delta) < \varepsilon/2$. Choose $p \in N$ such that $\mu\{p, p+1, \ldots\} < \delta$. Put $B = \{p, p+1, \ldots\}$. For $k \in K$, $i \in N$, $i < p$, put $V_{i,k} = \{i\} \,.\, U_k$; for $k \in K$, put $V_{p,k} = B \,.\, U_k$. Put $\mathscr{V} = (V_{i,k} : k \in K,\ i \leqq p)$. Clearly, $\mathscr{V}$ is a relatively pure refinement of $\mathscr{U}$. Put $[\mathscr{V}]_E = \langle Q, \sigma, v \rangle$. Clearly, $\sigma(x, y) \leqq 1$ for all $x, y \in Q$ and $\sigma(x, y) = 0$ whenever $x = (i, k_1)$, $y = (i, k_2)$, $i < p$. Define a semimetric $\hat{\sigma}$ on Q as follows: (1) $\hat{\sigma}(x, y) = 0$ if $x = y$ or $x = (i, k_1)$, $y = (i, k_2)$, $i < p$, (2) $\hat{\sigma}(x, y) = 1$ otherwise. Put $T = \{i : i < p\} \cup K$. Let v' be the measure on T defined as follows: $v'\{i\} = \mu\{i\}$ for $i < p$, $v'\{k\} = w(B \,.\, U_k)$ for $k \in K$. Clearly, $S = \langle T, 1, v' \rangle$ is an FW-space. For $(i, k) \in Q$, put $f(i, k) = i$ if $i < p$ and $f(p, k) = k$ for any $k \in K$. It is easy to show that f is a conservative mapping of $\langle Q, \hat{\sigma}, v \rangle$ onto $\langle T, 1, v' \rangle$. Since ψ satisfies (SR'), we have $\psi\langle Q, \hat{\sigma}, v \rangle = \psi S$, hence $\psi[\mathscr{V}]_E = \psi\langle Q, \sigma, v \rangle \leqq \leqq \psi S$. Now, since ψ is an e.S. semientropy, we get (see 4.12.1, 2.4) $\psi S = H(v't : : t \in T) \leqq H(\mu\{i\} : i < p) + H(w(B \,.\, U_k) : k \in K) + H(\sum(\mu\{i\} : i < p), \mu B) \leqq \leqq H(\mu) + \delta \log m + H(wP, \delta) < H(\mu) + \varepsilon$.

5.12.4. Fact. *If τ is a* GF, $\tau \geqq r$, *then C^*_τ satisfies* (SR') *from* 5.12.3.

Proof follows at once from 4.25, 3.21 and 3.23.

5.12.5. Fact. *Let ψ be an e.S. semientropy on $\mathfrak{W}_F$ satisfying* (SR') *from* 5.12.3. *Let* $P = \langle N, 1, \mu \rangle \in \mathfrak{W}$, $\operatorname{dom} \mu = \exp N$, $H(\mu) < \infty$. *Then, for any gauge functional* τ, $[\psi]^*_\tau (P) \leqq H(\mu)$, $[\psi]_\tau (P) \leqq H(\mu)$.

Proof. By 5.12.3, we have $[\psi]^*_E (P) \leqq H(\mu)$, $[\psi]_E (P) \leqq H(\mu)$.

5.13.1. As shown by the preceding considerations, the functionals C^*_τ and C_τ (where τ is a normal gauge functional, $\tau \geqq r$) represent a very important, but relatively special sort of e.S. semientropies. Therefore we are interested in finding a sort (or sorts) of e.S. semientropies defined in a not too involved way and (A) including all C^*_τ and C_τ, where τ is an NGF, $\tau \geqq r$, (B) not including "bad" semientropies such as η_ψ. In addition, the sort we are looking for should (C) include the functionals $[\psi]^*_\tau$, $[\psi]_\tau$, possibly subject to some conditions on ψ and τ, and (D) every e.S. semientropy φ of the sort we want to introduce should be obtainable from some e.S. semientropy ψ on $\mathfrak{W}_F$ in the sense that φ is either equal to some $[\psi]^*_\tau$ or $[\psi]_\tau$ or, at least, "encircled" by these, which means that, for any $P \in \mathfrak{W}$, φP belongs to the smallest interval containing all values $[\psi]^*_\tau (P)$, $[\psi]_\tau (P)$. Finally, the following requirement seems to be reasonable, though not indispensable: (E) if φ_1 and φ_2 are of the sort to be introduced, then so are $\min(\varphi_1, \varphi_2)$, $\max(\varphi_1, \varphi_2)$ and all convex combinations $a_1\varphi_1 + a_2\varphi_2$.

5.13.2. Consider e.g. all e.S. semientropies φ on $\mathfrak{W}$ such that for some e.S. semientropy ψ on $\mathfrak{W}_F$, φ is "encircled" (see 5.13.1, (D)) by $[\psi]^*_\tau$, $[\psi]_\tau$, where τ is a GF, $\tau \geqq r$. Then, (D), hence also (C) and (A), are satisfied. However, it can be shown that the functionals η_ψ are not excluded. Therefore, with regard to 5.12.5, we will admit only those ψ which satisfy (SR'). Thus, we introduce the following definitions.

5.14.1. Definition. An e.S. semientropy φ on $\mathfrak{W}$ will be called *simply germinated* if there exists an e.S. semientropy ψ on $\mathfrak{W}_F$ such that (1) if $P \in \mathfrak{W}_F$, $S \in \mathfrak{W}_{FC}$ and there exists a conservative $f : P \to S$, then $\psi P = \psi S$, (2) for any $P \in \mathfrak{W}$, φP belongs to the smallest interval containing the following values: $[\psi]^*_r (P)$, $[\psi]_r (P)$, $[\psi]^*_E (P)$ and $[\psi]_E (P)$.

5.14.2. Definition. An e.S. semientropy φ on $\mathfrak{W}$ will be called *finitely germinated* if it can be obtained from simply germinated e.S. semientropies by applying, finitely many times, the transitions from φ_1 and φ_2 to $\min(\varphi_1, \varphi_2)$ or to $\max(\varphi_1, \varphi_2)$ or to some $a_1\varphi_1 + a_2\varphi_2$, where $a_1, a_2 \in \boldsymbol{R}_+$, $a_1 + a_2 = 1$.

5.15.1. Fact. *If τ is an* NGF, *$\tau \geqq r$, then C^*_τ or C_τ is a simply germinated e.S. semientropy or, respectively, entropy.*

5.15.2. Fact. *If τ is a* GF *and ψ is an e.S. semientropy on $\mathfrak{W}_F$ satisfying* (SR'), *then* (1) *$[\psi]^*_\tau$ is a simply germinated e.S. semientropy on $\mathfrak{W}$,* (2) *if $[\psi]_\tau$ is an e.S. semientropy, then it is simply germinated.*

Proof. The first assertion is a consequence of 3.16 and of the following fact (the proof of which is similar to that of 3.16): for any finite W-space $P = \langle Q, \varrho, \mu \rangle$, $[\psi]^*_\tau (P)$ is equal to $\psi[A \,.\, P : A \in \mathscr{A}]_\tau$ where $\mathscr{A}$ is the set of atoms of dom μ. The second assertion is evident.

Remark. I do not know whether there exists an e.S. semientropy ψ on $\mathfrak{W}_F$ and a gauge functional τ such that $[\psi]_\tau$ is not finitely feebly continuous.

5.15.3. Fact. *The e.S. semientropies η_ψ described in* 5.12.1 *are not finitely germinated.*

5.15.4. Fact. *If φ_1 and φ_2 are finitely germinated e.S. semientropies (entropies),* *then* min (φ_1, φ_2), max (φ_1, φ_2) *and* $a_1\varphi_1 + a_2\varphi_2$, *where* $a_1, a_2 \in \boldsymbol{R}_+$, $a_1 + a_2 = 1$, *are finitely germinated e.S. semientropies (entropies).*

6

We now sum up the main results of the preceding section in the form of eight propositions, four of which (the more important ones) are labelled as theorems.

Proposition 6.1. *Let τ be a gauge functional. Then (1) C_τ^* and C_τ are regular hypoentropies, (2) C_τ^* restricted to the class $\mathfrak{W}_F$ of all finite separated W-spaces is the greatest τ-semiprojective regular hypoentropy on $\mathfrak{W}_F$, (3) if $\tau \geqq r$, then C_τ restricted to $\mathfrak{W}_F$ is the greatest τ-projective regular hypoentropy on $\mathfrak{W}_F$, (4) for any $P \in \mathfrak{W}_F$, $C_\tau(P) \leqq C_\tau^*(P)$.*

Proof. See 3.19, 3.26 and 4.23.

Proposition 6.2. *If σ and τ are gauge functionals, $\sigma \leqq \tau$, then $C_\sigma^* \leqq C_\tau^*$, $C_\sigma \leqq C_\tau$.*

Proof. See 3.20.

Proposition 6.3. *Let τ be a gauge functional, $\tau \geqq r$. Then, for any finite W-space P of the form $P = \langle Q, 1, \mu \rangle$, $C_\tau^*(P)$ and $C_\tau(P)$ are equal to the Shannon entropy of $\langle Q, \mu \rangle$.*

Proof. Follows from 4.25 and the obvious fact that any $\langle Q, 1, \mu \rangle \in \mathfrak{W}$ is a separated space.

Proposition 6.4. *Let τ be a normal gauge functional. Then C_τ^* is finitely continuous, and C_τ is finitely feebly continuous.*

Proof. See 5.7 and 5.9.

Theorem I. *If τ is a normal gauge functional, $\tau \geqq r$, then C_τ^* is a τ-semiprojective finitely continuous extended (in the broad sense) Shannon semientropy. In particular, $C^* = C_r^*$ is an r-semiprojective finitely continuous extended (in the broad sense) Shannon semientropy.*

Theorem II. *If τ is a normal gauge functional, $\tau \geqq r$, then C_τ is a τ-projective extended (b.s.) Shannon entropy. In particular, $C = C_r$ is an r-projective extended (b.s.) Shannon entropy.*

Proof. Both Theorem I and Theorem II follow from 4.29, 4.14, Proposition 6.3 and Proposition 6.4.

Remark. The fact that, for an NGF τ satisfying $\tau \geqq r$, C_τ^* is τ-semiprojective and C_τ is τ-projective can also be proved directly, without using 4.29.

Theorem III. *Let τ be a normal gauge functional. Then, for any W-space P, $C_\tau^*(P)$ is equal to the lower semiprojective limit, $\mathscr{F}_{\mathrm{De}}^*(P)$-$\underline{\lim}\ \Gamma_\tau(\mathscr{P})$, of the τ-values of pure daydic expansions of P, and $C_\tau(P)$ is equal to the lower projective limit, $\mathscr{F}_{\mathrm{De}}(P)$-$\underline{\lim}\ \Gamma_\tau(\mathscr{P})$, of the τ-values of dyadic expansions of P.*

Proof. See 4.29.

Theorem IV. *Let τ be a gauge functional. Let P be an FW-space. Then (1) $C_\tau^*(P)$ is equal to the minimum of the τ-values, $\Gamma_\tau(\mathscr{P})$, of all pure daydic expansions $\mathscr{P}$ of P such that $\mathscr{P}''$ refines the partition $(q \,.\, P : q \in |P|)$, (2) if $\tau \geqq r$, then $C_\tau(P)$ is equal to the infimum of the τ-values, $\Gamma_\tau(\mathscr{P})$, of all dyadic expansions $\mathscr{P}$ of P such that $\mathscr{P}''$ refines $(q \,.\, P : q \in |P|)$.*

Proof. See 4.20 and 4.24.

Remarks. – (1) In Part II it will be proved that the e.(b.s.)S. entropies C_τ, where τ is an NGF, $\tau \geqq r$, are finitely continuous. – (2) I do not know under what conditions weaker than $P \in \mathfrak{W}_{FC}$, $C_\tau^*(P) = C_\tau(P)$ holds for all normal gauge functionals satisfying $\tau \geqq r$ or, at least, for $\tau = r$. In particular, I do not know whether there exists an FW-space $\langle Q, \varrho, \mu \rangle$ such that ϱ is a metric and $C_r(P) < C_r^*(P)$. – (3) After the existence of e.S. entropies has been established and some e.S. entropies have been exhibited, various lines of investigation are open. We can examine properties common to some fairly broad kind of e.S. entropies (semientropies), e.g. to the functionals C_τ (respectively, C_τ^*) where τ is an NGF, $\tau \geqq r$. We can also investigate some particular e.S. entropies (or semientropies). Now the question arises which e.S. entropies are the most important and/or interesting ones. Intuitively, one would guess that $C = C_r$ and its counterpart, $C^* = C_r^*$, are fairly important (for various reasons, e.g. since r is a quite natural gauge functional). Thus, the problem appears whether C_r is a "privileged", in some precise sense, e.S. entropy, e.g. whether $C_r \leqq \varphi$ for all e.S. entropies φ, perhaps satisfying some additional conditions. Some other e.S. entropies may also be "privileged", in a different sense. For instance, I do not know whether $\varphi \leqq C_E$ of rall e.S. entropies φ.

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