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RELATIONS, COVERINGS, HYPERGRAPHS AND MATROIDS

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1. The purpose of this paper is to consider symmetric and reflexive k-ary relations. A binary symmetric and reflexive relation is called a tolerance, whence we call k-ary symmetric and reflexive relations *k-tolerances* and, in particular, a tolerance is a 2-tolerance. As in the case of 2-tolerances ,k-tolerances are induced by certain coverings of the set where they are defined. At first we will consider properties of coverings inducing k-tolerances and compatible k-tolerances. In the second part of this paper we will consider hypergraphs and matroids and their connection to k-tolerances.

2-tolerances and related covering are given by Chajda, Niederle and Zelinka in [3]. Unsymmetric binary relations, related coverings and an application is considered in [5]. As a basic reference for hypergraphs we have used the book [2] of Berge and for matroids the book [1] of Aigner.

2. A k-ary relation T_k on a set A is reflexive and symmetric i.e. a k-tolerance on A, if $(a, \ldots, a) \in T_k$ for every $a \in A$ and if $(a_1, \ldots, a_k) \in T_k$ implies that $(b_1, \ldots, b_k) \in T_k$ for all k elements b from $\{a_1, \ldots, a_k\}$. In [3] Chajda, Niederle and Zelinka show that a 2-tolerance T_2 on a set A corresponds to a family $\mathscr{M}$ of subsets of A called *τ-covering of A*. $\mathscr{M} = \{M_i \mid i \in I\}$ is a τ-covering of A if (1)–(3) below hold:

(1) $A = \bigcup\{M_i \mid i \in I\}$;

(2) if $j \in I$ and $S \subset I$, then $M_j \subset \bigcup\{M_s \mid s \in S\} \Rightarrow \bigcap\{M_s \mid s \in S\} \subset M_j$;

(3) if $N \subset A$ and N is not contained in any set from $\mathscr{M}$, then N contains a two-element subset of the same property.

In [3, Thm. 1] Chajda, Niederle and Zelinka show that there is a one-to-one correspondence between τ-coverings $\mathscr{M}$ and 2-tolerances T_2 such that if $\mathscr{M}$ is the τ-covering corresponding to T_2, then any two elements of A are in the relation T_2 if and only if there exists a set $M_i \in \mathscr{M}$ containing these two elements. Following [3] we call a family $\mathscr{M}_k = \{M_i \mid i \in I_k\}$ of subsets of A a τ_k-covering if the following conditions (4)–(6) hold:

(4) $A = \bigcup\{M_i \mid i \in I_k\}$;

(5) $M_i \not\subset M_j$ when $i \neq j$ and $i, j \in I_k$;

(6) if $N \subset A$ and N is not contained in any set of $\mathscr{M}_k$, then there is a k-sequence

$a_1, \ldots, a_k$ of elements from N (not necessarily disjoint) such that $a_1, \ldots, a_k$ is not contained in any set from $\mathcal{M}_k$.

A family $\mathcal{M} = \{M_i \mid i \in I\}$ of subsets of a set A is called a *covering of* A, if (1) holds for $\mathcal{M}$. We assume that $M_i \neq M_j$ whenever $i \neq j$ and $i, j \in I$.

At first we like to present a connection between τ_2-coverings and τ-coverings of Chajda, Niederle and Zelinka.

Theorem 1. *A τ-covering $\mathcal{M} = \{M_i \mid i \in I\}$ is a τ_2-covering of A and vice versa.*

Proof. By putting $|S| = 1$ in (2), one sees that a τ-covering $\mathcal{M}$ satisfies (5), and because (1) is equivalent to (4) and (3) to (6), $\mathcal{M}$ is a τ_2-covering. Conversely, let $\mathcal{M}_2 = \{M_i \mid i \in I_2\}$ be a τ_2-covering of A. $\mathcal{M}_2$ is a τ-covering if (2) holds for $\mathcal{M}_2$, and thus we assume that $j \in I_2$, $S \subset I_2$ and $M_j \subset \bigcup\{M_s \mid s \in S\}$. If now $\bigcap\{M_s \mid s \in S\} \not\subset M_j$, then $a \in \bigcap\{M_s \mid s \in S\}$ such that $a \notin M_j$. On the other hand, there is for every $b \in \bigcup\{M_s \mid s \in S\}$ some M_s containing a and b. In particular, this means that there is for every $c \in M_j$ some $M_{s(c)}$ containing a and c. Let us consider now $M_j \cup \{a\}$. It is contained in a set from $\mathcal{M}_2$ or not. If it is not, we obtain a contradiction with (6), and if it is contained in, then M_j is contained properly in a set from $\mathcal{M}_2$, which contradicts (5). Hence $\bigcap\{M_s \mid s \in S\} \subset M_j$.

Before proving an analogy to [3, Thm. 1], we like to show that there are τ_k-coverings of a set A that are not τ_m-coverings, $k, m \geqq 1$ and $k > m$. Let $A = \{a_1, \ldots, a_k\}$ and $\mathcal{M}_k$ consist of all disjoint $k - 1$-element subsets of A; as well known, there are k such subset M_i in A. Clearly (4) and (5) are satisfied in $\mathcal{M}_k$. The only subset N of A not contained in any set from $\mathcal{M}_k$ is the whole set A. A contains clearly a k-sequence $a_1, \ldots, a_k$ not contained in any set from $\mathcal{M}_k$, and thus $\mathcal{M}_k$ is a τ_k-covering of A. On the other hand, every $k - 1$-sequence of A is contained in some set M_i from $\mathcal{M}_k$ and thus $\mathcal{M}_k$ is not a τ_{k-1}-covering of A. Similarly one sees that $\mathcal{M}_k$ is not a τ_m-covering of A, $k > m$. Note that there is only one τ_1-covering of A: $\mathcal{M}_1 = \{A\}$.

Theorem 2. *Let A be a non-empty set. There exists a one-to-one correspondence between k-tolerances on A and τ_k-coverings of A such that if T_k is a k-tolerance on A and $\mathcal{M}_k$ is the τ_k-covering corresponding to T_k, then any k elements $a_1, \ldots, a_k$ of A are in the relation T_k if and only if there exists a set from $\mathcal{M}_k$ which contains $a_1, \ldots, a_k$.*

Proof. At first we show that every k-tolerance T_k on A determines a τ_k-covering $\mathcal{M}_k$ of A. Let $\mathcal{L} = \{L_j \mid j \in J\}$ be the family of all subsets of A such that every k elements of L_j are in the relation T_k, and let $\mathcal{M} = \{M_i \mid i \in I\}$ be the family of all maximal elements of $\mathcal{L}$, which exist by assuming Zorn's lemma. Because of the reflexivity of T_k, $\mathcal{L}$ and $\mathcal{M}$ are coverings of A and according to the maximality, (5) holds for $\mathcal{M}$. Let N be a subset of A not contained in any of the sets from $\mathcal{M}$. If every k-sequence of N is contained in some set from $\mathcal{M}$, then $N \in \mathcal{L}$, and according to the maximality of $\mathscr{M}$, N is contained in some $M_i \in \mathscr{M}$, which is a contradiction. Hence (6) holds for $\mathscr{M}$ and thus it is a τ_k-covering of A.

Obviously every τ_k-covering $\mathscr{M}_k$ of A determines uniquely a k-tolerance T_k, and further, $\mathscr{M}$ derived from T_k above determines the original T_k.

Let $\mathscr{M}_k$ be a given τ_k-covering of A, T_k the k-tolerance determined by $\mathscr{M}_k$ and $\mathscr{M}$ the τ_k-covering of A derived from T_k above. In the following we show that $\mathscr{M}_k \subset \mathscr{M}$ and $\mathscr{M} \subset \mathscr{M}_k$, whence $\mathscr{M} = \mathscr{M}_k$, which now implies the assertion of the theorem. $\mathscr{M}_k \subset \mathscr{M}$: Assume that $M_i \in \mathscr{M}_k$ and $M_i \notin \mathscr{M}$. Because of T there is a set $L \in \mathscr{M}$ containing M_i properly. But then, because T is determined by $\mathscr{M}_k$, for every k elements $a_1, \ldots, a_k \in L$ there is a set $M \in \mathscr{M}_k$ containing these elements. If L is not contained in a set from $\mathscr{M}_k$, we obtain now a contradiction with (6). Hence $L \subset M_j$ for some $M_j \in \mathscr{M}_k$. But then M_i is contained in M_j properly, which contradicts (5). Thus $\mathscr{M}_k \subset \mathscr{M}$. $\mathscr{M} \subset \mathscr{M}_k$: Let $L \in \mathscr{M} \setminus \mathscr{M}_k$. Because $\mathscr{M}_k \subset \mathscr{M}$, L is now a set N from (6) for τ_k-covering $\mathscr{M}_k$. Thus L contains a k-sequence $a_1, \ldots, a_k$ not in the relation T_k, which is a contradiction to $L \in \mathscr{M}$.

Accordingly, the investigation of k-tolerances on a set A is equivalent to the investigation of τ_k-coverings of A. As previously shown, a τ_k-covering need not be a τ_m-covering, $k > m$, whence k-ary tolerances need not be m-ary tolerances.

In the following we consider connections between different τ_k-coverings of a set A.

Theorem 3. *Let $\mathscr{M}_m$ be a τ_m-covering of a set A, then $\mathscr{M}_m$ is also a τ_k-covering of A for every finite $k \geqq m$.*

Proof. It is sufficient to show that (6) holds for $\mathscr{M}_m$ for every finite $k \geqq m$. If $N \subset A$ and N is not contained in any set from $\mathscr{M}_m$, there is an m-sequence $a_1, \ldots, a_m$ of elements of N not contained in any set from $\mathscr{M}_m$. But then the k-sequence $a_1, \ldots, a_m$, $a_{11}, \ldots, a_{1,k-m}$ of N, where $a_{11} = \ldots = a_{1,k-m} = a_1$, has the same property for every finite $k \geqq m$. Hence the theorem.

Theorem 4. *Let A be a finite non-empty set. Then the maximal sets of every covering $\mathscr{M}^* = \{M_i \mid i \subset I^*\}$ of A constitute a τ_k-covering of A for some $k \geqq 1$.*

Proof. Choose from $\mathscr{M}^*$ all maximal sets and let the family such obtained be $\mathscr{M} = \{M_i \mid i \in I \subset I^*\}$. Because of the maximality of the sets in $\mathscr{M}$, (5) holds for $\mathscr{M}$ as well as (4). By putting $k = |A|$, $\mathscr{M}$ satisfies also (6), because if $N \subset A$ is not contained in any set from $\mathscr{M}$, then by joining to the sequence $a_{1N}, \ldots, a_{rN}$ of all elements of N $|A| - |N|$ times a_{1N}, the desired $|A|$-sequence is obtained.

Theorem 4 can also be generalized for infinite sets A if $\mathscr{M}^*$ satisfies an additional condition. A covering $\mathscr{M}^*$ of A is called *element finite*, if every $a \in A$ is contained in a finite number of sets of $\mathscr{M}^*$. By assuming Zorn's lemma, every covering $\mathscr{M}^*$ of A can be reduced to a covering $\mathscr{M}$ of A satisfying (4) and (5). If $\mathscr{M}^*$ is element finite, then also $\mathscr{M}$ is, but the converse need not hold. Assume that $\mathscr{M}$ is an element finite covering of A satisfying (4) and (5), and let $k = \max \{k_a \mid a$ belongs to k_a disjoint sets in $\mathscr{M}$, $a \in A\}$. We show that $\mathscr{M}$ is then a τ_{k+1}-covering of A. Let $N \subset A$ be a set not contained in any set from $\mathscr{M}$, a_1 an element of N and let $a_1 \in M_{i1}$, $i1 \in I$. Because of the property of N, there is an element $a_2 \in N \setminus M_{i1}$. If $a_1, a_2 \in M_{i2}$ for some $i2 \in I$, then according to the property of N, there is an element $a_3 \in N \setminus M_{i2}$. According to the choices of a_1 and a_2, $M_{i1} \neq M_{i2}$. If $a_1, a_2, a_3 \in M_{i3}$ for some $i3 \in I$, then there is an element $a_4 \in N \setminus M_{i3}$. Because $a_2 \in N \setminus M_{i1}$, $M_{i1} \neq M_{i3}$, and because $a_3 \in \in N \setminus M_{i2}$, $M_{i2} \neq M_{i3}$. By continuing this process we will find a set of m disjoint elements $a_1, \ldots, a_m$ from N not contained in any set from $\mathscr{M}$, $m \leqq k$, or a set of k disjoint elements $a_1, \ldots, a_k$ of N contained in a set M_{ik} from $\mathscr{M}$. In the first case, by joining the element a_1 $1 + k - m$ times to $a_1, \ldots, a_m$, a desired $k + 1$-sequence is obtained. In the second case, because N is not contained in any set from $\mathscr{M}$, $a_{k+1} \in N \setminus M_{ik}$. As above, the sets $M_{i1}, \ldots, M_{ik}$ are pairwise disjoint. The $k + 1$-sequence $a_1, \ldots, a_{k+1}$ is a desired subset of N because otherwise a_1 belongs to $k + 1$ disjoint sets from $\mathscr{M}$, which contradicts the definition of k. Thus we can write

Theorem 4′. *Let $\mathscr{M}$ be an element finite covering of A satisfying (5). Then $\mathscr{M}$ is a τ_{k+1}-covering of A for $k = \max \{k_a \mid a$ belongs to k_a disjoint sets from $\mathscr{M}$, $a \in A\}$.*

Let $k > m$ and $\mathscr{M}_k$ be a τ_k-covering of A without being simultaneously a τ_m-covering of A. In the following we look for a rule to determine the least τ_m-covering of A containing $\mathscr{M}_k$, i.e. the τ_m-hull of $\mathscr{M}_k$. For that reason we determine at first the family $\mathscr{N}_{km} = \{N \mid N \not\subset M_i$ for any $M_i \in \mathscr{M}_k$ and there is no m-sequence $a_1, \ldots, a_m$ in N having the same property as $N\}$. Moreover, let $\mathscr{K} = \{K \mid K$ is maximal among the sets of $\mathscr{M}_k$ and $\mathscr{N}_{km}$ and K is either from $\mathscr{M}_k$ or from $\mathscr{N}_{km}\}$; such $\mathscr{K}$ exists by assuming Zorn's lemma. Now we can prove

Theorem 5. *Let $\mathscr{M}_k$ be a τ_k-covering of a non-empty set A without being a τ_m-covering of A, $k > m$. Then $\mathscr{K}$ is a τ_m-covering of A and it is the least τ_m-covering containing $\mathscr{M}_k$.*

Proof. Obviously $\mathscr{K}$ is a covering of A, and (5) holds because of the definition of $\mathscr{K}$. Let $N \subset A$ such that N is not contained in any set from $\mathscr{K}$ and assume that there is no m-sequence $a_1, \ldots, a_m$ of N having the same property as N. But then N is also not contained in any M_i from $\mathscr{M}_k$ without containing an m-sequence with the same property. Hence $N \in \mathscr{N}_{km}$ and thus N is contained in some K_j from $\mathscr{K}$, which is a contradiction. Thus (6) holds for $\mathscr{K}$ and it is a τ_m-covering of A.

It remains to show that $\mathscr{K}$ is the least τ_m-covering of A containg $\mathscr{M}_k$, i.e. there is for every $M_i \in \mathscr{M}_k$ at least one $K_j \in \mathscr{K}$ containing M_i. Assume that $\mathscr{D}$ is a τ_m-covering of A containing $\mathscr{M}_k$ and $\mathscr{D}$ is contained in $\mathscr{K}$, i.e. for every $D_s \in \mathscr{D}$ there is a $K_j \in \mathscr{K}$ containing D_s. $\mathscr{D} \subset \mathscr{K}$ properly only if 1) some D_s is contained in some K_j properly or 2) there is a $K_j \in \mathscr{K}$ for which there exists no $D_s \in \mathscr{D}$ such that $D_s \subset K_j$.

1) Let $D_s \subset K_j$ propeıly and let $x \in K_j \setminus D_s$. Because $\mathscr{D}$ is a τ_m-covering of A, $D_s \cup \{x\} \not\subset D_h$ for any $D_h \in \mathscr{D}$. Thus there is an m-sequence $a_1, \ldots, a_m$ in $D_s \cup \{x\}$ not contained in any set frcm $\mathscr{D}$. On the other hand, this m-sequence is contained in

some $K_t \in \mathscr{K}$. Note that every m-sequence from a $K_j \in \mathscr{K}$ is contained in some $M_i \in \in \mathscr{M}_k$ according to the definition of $\mathscr{N}_{km}$. Hence the m-sequence $a_1, \ldots, a_m$ is contained in some M_i which is contained in some $D_h \in \mathscr{D}$, which is a contradiction. Hence 1) cannot hold.

2) Let $K_j \in \mathscr{K}$ be a set such that no $D_s \in \mathscr{D}$ is contained in K_j. Because $\mathscr{D} \subset \mathscr{K}$, K_j is not contained in any $D_s \in \mathscr{D}$ and because $\mathscr{D}$ is a τ_m-covering of A, there is an m-sequence $a_1, \ldots, a_m$ from K_j not contained in any $D_s \in \mathscr{D}$. This is absurd from the same reason as in 1), and hence 2) cannot hold.

Thus $\mathscr{D}$ is not contained in $\mathscr{K}$ properly. If there is $\mathscr{D}$ containing $\mathscr{M}_k$, then K and D have a common lower bound $\mathscr{H}$ (which can be constructed by means of the intersection of m-tolerances determined by $\mathscr{D}$ and $\mathscr{K}$) containing $\mathscr{M}_k$ and contained in $\mathscr{K}$. As the proof before shows, $\mathscr{H} = \mathscr{K}$. Hence $\mathscr{K}$ is the least τ_m-covering of A containing $\mathscr{M}_k$.

We will make some remarks about τ_m-hulls when considering hypergraphs related to a τ_k-covering $\mathscr{M}_k$.

Following Chajda [4], we call a k-tolerance T_k defined on the support A of an algebra $\boldsymbol{A} = (A, F)$ compatible with respect to $\boldsymbol{A}$ if and only if the corresponding τ_k-covering $\mathscr{M}_k$ of T_k has the following property

(7) for every n-ary operation $f \in F$ of $\boldsymbol{A}$ and for every n-tuple $M_1, \ldots, M_n \in \mathscr{M}_k$ (where $M_1, \ldots, M_n$ need not be disjoint) there exists at least one $M_0 \in \mathscr{M}_k$ such that $f(M_1, \ldots, M_n) = \{f(a_1, \ldots, a_n) \mid a_j \in M_j \text{ and } j = 1, \ldots, n\} \subset M_0$.

As easily seen, the definition above is equivalent with the following: every n-ary $f \in F$ and every n k-ary relations $(a_{11}, \ldots, a_{k1}), (a_{12}, \ldots, a_{k2}), \ldots, (a_{1n}, \ldots a_{kn}) \in T_k$ imply that $(f(a_{11}, a_{12}, \ldots, a_{1n}), \ldots, f(a_{k1}, a_{k2}, \ldots, a_{kn})) \in T_k$.

One can now prove an analogy of [3, Thm. 3]; the proof is similar to that of [3, Thm. 3], whence we omit it.

Theorem 6. *Let $\boldsymbol{A} = (A, F)$ be an algebra, T_k a k-tolerance on A, and $\mathscr{M}_k$ the corresponding τ_k-covering of A. T_k is compatible with respect to $\boldsymbol{A}$ if and only if there exists an algebra $\boldsymbol{B} = (B, G)$ with the following properties:*

(i) *there exists a one-to-one mapping $\varphi : F \to G$ such that for any positive integer n and for each $f \in F$ the operation φf is n-ary if and only if f is n-ary;*

(ii) *there exists a one-to-one mapping $\chi : \mathscr{M}_k \to B$ such that for every n-ary operation $f \in F$, where n is a positive integer, and for any $n + 1$ elements $M_0, M_1, \ldots \ldots, M_n$ of $\mathscr{M}_k$ the equality $\varphi f(\chi(M_1), \ldots, \chi(M_n)) = \chi(M_0)$ implies that for any n elements $a_1, \ldots, a_n$ of A such that $a_i \in M_i$, $i = 1, \ldots, n$, the element $f(a_1, \ldots, a_n) \in \in M_0$.*

A family $\mathscr{M} = \{M_i \mid i \in I \text{ and } M_i \subset A\}$ is called a compatible covering of an algebra $\boldsymbol{A} = (A, F)$, if $\mathscr{M}$ is a covering of A and (7) holds for $\mathscr{M}$. The maximal elements of $\mathscr{M}$ have the same properties and hence we can write a compatible analogy of Theorem 4 as a corollary. If A is infinite but the maximal elements of $\mathcal{M}$ constitute an element finite compatible covering of A, we obtain a compatible analogy of Theorem 4'. Because every covering of a finite set A with maximal subsets is element finite, we can write

Corollary. *Let $A = (A, F)$ be an algebra and $\mathcal{M}$ a compatible element finite covering of A satisfying (5). Then $\mathcal{M}$ is a τ_{k+1}-covering of a compatible $k + 1$-tolerance on A for $k = \max \{k_a \mid a$ belongs to k_a disjoint sets from $\mathcal{M}, a \in A\}$.*

3. Let A be a finite set and $\mathcal{E}$ a family of subsets of A. The couple $(A, \mathcal{E}) = H$ is called a *hypergraph*, if $\emptyset \notin \mathcal{E}$ and $\mathcal{E}$ is a covering of A. Its vertices are the elements of A and its edges the sets in $\mathcal{E}$. By $(H)_2$ is meant a graph (A, E) without loops, where two vertices a_1 and a_2 are adjacent whenever a_1 and a_2 are contained in an edge $E_i \in \mathcal{E}$ in H. In [2, Chpt. 17 : 3] a hypergraph is called conformal, if $\mathcal{E}_{\max}$ of all maximal edges of H is the set of all maximal cliques of the graph $(H)_2$.

Theorem 7. *A k-tolerance T_k on a finite set A is a 2-tolerance on A if and only if the hypergraph $(A, \mathcal{M}_k)$, where $\mathcal{M}_k$ is the τ_k-covering corresponding T_k, is conformal.*

Proof. Let T_k be a 2-tolerance on A, i.e. $k = 2$. In the graph $(H)_2$ vertices a and b are adjacent if and only if $(a, b) \in T_2$. According to the maximality of sets M_i, every M_i corresponds then to a maximal clique of $(H)_2$ and every clique of $(H)_2$ is contained in a set $M_i \in \mathcal{M}_2$. Hence $(A, \mathcal{M}_2)$ is conformal. Conversely, if $(A, \mathcal{M}_k)$ is conformal and N is not contained in any set from $\mathcal{M}_k$, then N contains at least one pair a, b of vertices not adjacent in $(H)_2$. Hence every N contains a two-element set with the same property as N has, and thus $\mathcal{M}_k$ is a 2-covering of A and the corresponding k-tolerance a 2-tolerance on A.

We will say that a hypergraph $H = (A, \mathcal{E})$ is *h-conformal*, $h \geqq 3$, if for every clique of $(H)_2$ not contained in an edge of H there is a number $s \leqq h$ such that every subset of $s - 1$ vertices is contained in some edge of H but some subset of s vertices not. Moreover, there exists at least one clique of $(H)_2$ with $s = h$.

Theorem 8. *Let $H = (A, \mathcal{E})$ be a hypergraph. $\mathcal{E}_{\max}$ is a τ_h-covering and not a τ_{h-1}-covering of A if and only if H is h-conformal, $h \geqq 3$.*

Proof. The theorem implies that T_k is a h-tolerance and not a $h - 1$-tolerance on A if and only if $(A, \mathcal{M}_k)$ is h-conformal. Now let H be h-conformal and N a set not contained in any set from $\mathcal{E}_{\max}$. The elements of N constitute a clique of $(H)_2$ or not. If not, then N contains at least one pair a, b of vertices not adjacent in $(H)_2$, whence N contains a h-sequence $a, b, \ldots, b$ not contained in any set from $\mathcal{E}_{\max}$. If the points of N constitute a clique of $(H)_2$, then the existence of an h-sequence not contained in any set from $\mathcal{E}_{\max}$ follows from h-conformality. Thus (6) holds for $\mathcal{E}_{\max}$, for which (4) and (5) hold obviously. Hence $\mathcal{E}_{\max}$ is a τ_h-covering of A and it is not a τ_{h-1}-covering of A because of the last sentence in the definition of h-conformality. The converse proof is now obvious, whence we omit it.

Let $\mathscr{M}_k$ be a τ_k-covering of a finite set and $H_k = (A, \mathscr{M}_k)$ in the least 2-covering $\mathscr{M}_2$ containing $\mathscr{M}_k$, two elements a and b belong to a set from $\mathscr{M}_2$ at least then when they belong to a set from $\mathscr{M}_k$. In particular, this means that a and b are adjacent in $(H_k)_2$, and on the other hand, every two vertices c and d adjacent in $(H_k)_2$ belong to at least one M_i from $\mathscr{M}_k$ simultaneously. Thus every maximal clique of $(H_k)_2$ is a set from $\mathscr{M}_2$, and because the maximal cliques of $(H_k)_2$ constitute a τ_2-covering of A containing $\mathscr{M}_k$, the maximal cliques of $(H_k)_2$ constitute the τ_2-hull of $\mathscr{M}_k$. As seen above, every τ_2-covering of A is also a τ_m-covering, $2 \leqq m \leqq k$, whence τ_m-hulls of $\mathscr{M}_k$ are contained in $\mathscr{M}_2$. These observations and Theorem 8 imply together

Theorem 9. *A τ_m-covering $\mathscr{M}_m$ of a finite set A is the τ_m-hull of a τ_k-covering $\mathscr{M}_k$ of A if and only if the graphs $(H_m)_2$ and $(H_k)_2$ derived from $H_m = (A, \mathscr{M}_m)$ and $H_k = (A, \mathscr{M}_k)$, respectively, are isomorphic and H_m is m-conformal, $k \geqq m \geqq 3$.*

We give next a few remarks on the connection between the Helly property and τ_k-coverings. A family $\mathscr{B} = \{B_i \mid i \in I\}$ of subsets of a finite set A satisfies the Helly property if $J \subset I$ and $B_i \cap B_j \neq \emptyset$ for all $i, j \in J$ implies that $\bigcap\{B_j \mid j \in J\} \neq \emptyset$. Let $H = (A, \mathscr{E})$ be a hypergraph, where $A = \{a_1, \ldots, a_t\}$ and $\mathscr{E} = \{E_1, \ldots, E_s\}$. In the dual hypergraph $H^d = (E^d, \mathscr{A}^d)$ of H the vertices in $E^d = \{e_1, \ldots, e_s\}$ represent the edges of H and the edges in $\mathscr{A}^d = \{A_1, \ldots, A_t\}$ the vertices of H such that $A_j = \{e_i \mid i \leqq s, a_j \in E_i\}$. Because a hypregraph H is conformal if and only if (the edge set of) its dual satisfies the Helly property [2, Chpt. 17 : 3], we can write

Theorem 10. *A τ_k-covering $\mathscr{M}_k$ of a finite set A is a τ_2-covering of A if and only if the dual of $(A, \mathscr{M}_k)$ satisfies the Helly property.*

Let $H = (A, \mathscr{E})$ be a hypergraph with s edges $E_1, \ldots, E_s$. The representative graph of H is a simple graph G of order s whose vertices $e_1, \ldots, e_s$ respectively represent the edges $E_1, \ldots, E_s$ of H and with vertices e_i and e_j joined by an edge if and only if $E_i \cap E_j \neq \emptyset$.

Theorem 11. *Every graph is the representative graph of a τ_k-covering $\mathscr{M}_k$ of a finite set A.*

Proof. Let $G' = (V', E')$ be a given graph. We will show that it repesents a τ_k-covering $\mathscr{M}_k$ of a finite set A. We add first to every pendant vertex v' of G' a vertex v adjacent only to v'; the graph thus obtained is $G = (V, E)$. Let $\mathscr{Q} = \{Q_1, \ldots, Q_h\}$ be the family of all maximal cliques of G and let Q_i contain the vertices $v_{i1}, \ldots, v_{it}$, $t \geqq 3$. There are t disjoint sets, each of which contains $t - 1$ vertices of Q_i and constitutes a clique of G; we denote these sets by $E_{i1}, \ldots, E_{it}$. Let $\mathscr{E}$ be the family of all such maximal sets and two-element maximal cliques Q of G. Every set from $\mathscr{E}$ is a clique of G and each vertex and each edge of G is covered by at least one set from $\mathscr{E}$. According to [2, Chpt. 17 : 4, Proposition 1] G is the representative graph of the dual hypergraph $H^d = (E^d, \mathscr{V}^d)$ of the hypergraph $H = (V, \mathscr{E})$. Because $\mathscr{V}^d_{\max}$

is a covering of the finite set E^d satisfying (5), it is a τ_k-covering of E^d for some finite k. Thus the assertion follows by showing that G' is the representative graph of $(E^d, \mathscr{V}^d_{\max})$; this is done by considering when $V_1 \subset V_2$ is possible in $\mathscr{V}^d$. Assume that $V_1 \subset V_2$, $V_1 \neq V_2$. According to the definition, $V_s = \{e_i \mid v_s \in E_i, E_i \in \mathscr{E}\}$ when $V_s \in \mathscr{V}^d$. If $V_1 \subset V_2$, then for every $e_i \in V_1$, the clique E_i of G contains v_1 as well as v_2, and because $V_1 \neq V_2$, there is an $e_j \in V_2$ such that the clique E_j of G contains v_2 but v_1 not. This shows that v_1 and v_2 are adjacent in G, and then $V_1 \subset V_2$ properly only when v_1 is a pendant vertex of G. Thus, when choosing $\mathscr{V}^d_{\max}$ from $\mathscr{V}^d$ only the sets corresponding to pendant vertices of G are dropped out. But then the sets of $\mathscr{V}^d_{\max}$ correspond to the vertices of the original graph G', and the theorem follows.

Previous result can be sharpened for τ_2-coverings of a finite set A. The sets of a τ_2-covering $\mathscr{M}_2$ of A are the maximal cliques of the graph $(H)_2$ derived from $(A, \mathscr{M}_2)$, and hence the graph representing a τ_2-covering is also the representative graph of the maximal cliques of $(H)_2$. According to the result concerning the representative graphs of maximal cliques of some graph [2, Chpt. 17 : 4, Proposition 5], we can write

Theorem 12. *A graph G is the representative graph of a τ_2-covering $\mathscr{M}_2$ of a finite set A if and only if there exists in G a family $\{Q_i \mid i \in I\}$ of cliques such that*

(i) *each edge of G is covered by a Q_i;*
(ii) *$\{Q_i \mid i \in I\}$ satisfies the Helly property.*

Finally we will characterize finite matroids by means of k-ary relations. A matroid on a finite set A is a couple $(A, \mathscr{C})$, where $\mathscr{C} = \{C_i \mid i \in I\}$ is a family of subsets of A having the properties

(8) $\emptyset \notin \mathscr{C}$ and if $C_i, C_j \in \mathscr{C}$, $C_i \neq C_j$, then $C_i \not\subset C_j$ for every pair $i, j \in I$;

(9) if $C_i, C_j \in \mathscr{C}$, $C_i \neq C_j$, $b \in C_i \cap C_j$ and $a \in C_i \setminus C_j$, then there exists $C_s \in \mathscr{C}$ such that $a \in C_s \subset (C_i \cup C_j) \setminus \{b\}$.

The sets from $\mathscr{C}$ are called circuits of the matroid $(A, \mathscr{C})$. Note that $\mathscr{C}$ need not be a covering of A, but because $\emptyset \notin \mathscr{C}$, it is the covering of a subset $A' = \{a \mid a \in C_i \in \mathscr{C}\}$ of A. According to (8) and Theorem 4, $\mathscr{C}$ is a τ_k-covering of A' for some finite k. Thus the characterization of a matroid $(A, \mathscr{C})$ as a k-ary relation reduces to the characterization of $(A', \mathscr{C})$ as a k-tolerance T_k having $\mathscr{C}$ as the corresponding τ_k-covering of A', and, in particular, to the characterization of (9) as a special property of T_k. (9) means the *transitivity* of T_k corresponding to $\mathscr{C}$ such that if $(a, b, \ldots, b)$, $(c, b, \ldots, b) \in T_k$, then $(a, c, \ldots, c) \in T_k$. In the case $k = 1$, $\mathscr{C} = \{A'\}$, and in the case $k = 2$, there is no pair $C_i \neq C_j$ in $\mathscr{C}$ such that $b \in C_i \cap C_j$, and hence the cases $k \geqq 3$ remain. When $k \geqq 3$, the transitivity does not ensure the existence of a set C_s containing a such that $C_s \in (C_i \cup C_j) \setminus \{b\}$, and thus something more is needed.

Let B be a finite set, $\mathscr{M}_k$ a τ_k-covering of B and T_k the corresponding k-tolerance on B. If $\mathscr{M}_k \neq \{B\}$, then $\mathscr{M}_k^c = \{B \setminus M_i \mid M_i \in \mathscr{M}_k\}$ is a family of non-empty subsets of B satisfying (5). Clearly $\mathscr{M}_k^c$ is a τ_m-covering of $B^c = B \setminus \bigcap\{M_i \mid M_i \in \mathscr{M}_k\}$ and it determines an m-tolerance T_m on B^c. We call this relation the co-k-tolerance of T_k in B and denote it by T_k^c. By using co-k-tolerance we can characterize matroids as k-tolerances as follows

Theorem 13. *A non-empty family* $\mathscr{C} = \{C_i \mid i \in I\}$ *of subsets of a finite set* A *is the family of circuits of the matroid* $(A, \mathscr{C})$ *if and only if* $\mathscr{C}$ *is a* τ_k*-covering of* A *or of* $A' = \{a \mid a \in C \in \mathscr{C}\}$ *determining a transitive* k*-tolerance* T_k *such that* (10) *holds if* $k \geqq 3$:

(10) *Let* $(a, b, \ldots, b)$, $(c, b, \ldots, b) \in T_k$ *and* $(a, c, b, \ldots, b) \notin T_k$. *Then for every two points* a' *and* c', *for which* $(a, b, a', \ldots, a')$, $(c, b, c', \ldots, c') \in T_k$ *it holds:* $(b, x_1, \ldots, x_{k-1}) \in T_k^c$ *for every* $k - 1$ *elements* $x_t \in A'$ *for which* $(a, b, a', x_t, \ldots \ldots, x_t)$, $(c, b, c', x_t, \ldots, x_t) \notin T_k$.

Proof. Let T_k be a transitive k-tolerance on a set A or on its proper subset A', and let $\mathscr{C}$ be the τ_k-covering corresponding to T_k. As noted before, (8) holds for $\mathscr{C}$. The cases $k = 1, 2$ are clear because of the considerations before. Hence, let $k \geqq 3$, $C_i \neq C_j$, $b \in C_i \cap C_j$ and $a \in C_i \setminus C_j$. Because $C_j \not\subset C_i$, there is in $C_j \setminus C_i$ an element c for which $(a, c, b, \ldots, b) \notin T_k$ but $(a, b, \ldots, b)$, $(c, b, \ldots, b) \in T_k$.According to the transitivity of T_k, a and c are in the relation T_k, but the set C_s containing a and c need not be from $(C_i \cup C_j) \setminus \{b\}$. Let us choose $a' \in C_i \setminus C_j$ such that $a' \neq a$, and if there is not, a' from $C_i \cap C_j$ such that $a' \neq b$, and if there is not, we put $a' = a$. The element c' is choosen analogously. All the elements $x_t \in A'$, for which $(a, b, a', x_t, \ldots, x_t)$, $(c, b, c', x_t, \ldots, x_t) \in T_k$, are then outside from $C_i \cup C_j$. Because of (10), these elements constitute in common with b a class of C^c of T_k^c, the complement $C = A' \setminus C^c$ of which belongs to $\mathscr{C}$. Thus $a \in C \subset (C_i \cup C_j) \setminus \{b\}$, whence $\mathscr{C}$ is the family of circuits in the set A (and in A', too), and $(A, \mathscr{C})$ is a finite matroid. If $k = 3$, T_3 can also be represented as a 4-tolerance, and the proof above is then certainly applicable.

Conversely, let $(A, \mathscr{C})$ be a matroid and $\mathscr{C}$ its family of circuits. As pointed out before, $\mathscr{C}$ is a τ_k-covering of A' for some finite k. Let $k \geqq 3$. If $(a, b, \ldots, b)$, $(c, b, \ldots, b) \in T_k$ but $(a, c, b, \ldots, b) \notin T_k$, there are at least two disjoint sets C_a and C_c in $\mathscr{C}$ containing b such that $a \in C_a \setminus C_c$ and $c \in C_c \setminus C_a$. If $(a, b, a', \ldots, a')$, $(c, b, c', \ldots, c') \in T_k$, we may assume that $a' \in C_a$ and $c' \in C_c$. According to (9), there is in $\mathscr{C}$ a set C such that $a \in C \in (C_a \cup C_c) \setminus \{b\}$, and analogously $C' \in \mathscr{C}$ such that $c \in C' \subset \subset (C_a \cup C_c) \setminus \{b\}$. Then, in particular, every $x_t \notin (C_a \cup C_c) \setminus \{b\}$ has the property $(a, b, a', x_t, \ldots, x_t)$, $(c, b, c', x_t, \ldots, x_t) \notin T_k$. $x_t \in A' \setminus C$, $A' \setminus C'$ and thus every $k - 1$ such elements x_t has the property $(b, x_1, x_2, \ldots, x_{k-1}) \in T_k^c$. If $k = 3$, then $\mathscr{C}$ is also τ_4-covering, and the proof above is then applicable.

The transitivity of T_k on a set A defined above does not imply non-intersecting sets in $\mathscr{M}_k$ of T_k. The following transitivity, where $(b_1, a_2, a_3, \ldots, a_k)$, $(a_2, b_2, a_3, \ldots \ldots, a_k) \in T_k$ imply $(b_1, b_2, a_3, \ldots, a_k) \in T_k$ gives non-intersecting sets in the τ_k-covering $\mathscr{M}_k$ of T_k, and hence such a k-tolerance is a k-equivalence on A.

In the book [6] Pogonowski presents applications of 2-tolerances to linguistics. Some applications of [6] can be developed further by using k-tolerances given in this paper.

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