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Patrick Oswald

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ON BESOV-HARDY-SOBOLEV SPACES OF ANALYTIC FUNCTIONS IN THE UNIT DISC

P. OSWALD, Dresden

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In the last years a general theory of function spaces $B^s_{p,q}$ and $F^s_{p,q}$ of Besov-Hardy-Sobolev type on R^n and on domains has been developed in the work of J. Peetre, H. Triebel et al.. Their methods which are mainly based on multiplier criteria of Mihlin-Hörmander type, decomposition, maximal function, and interpolation techniques allow to consider the full range of parameters $0 < p, q \leqq \infty, -\infty < < s < \infty$. Most of known function spaces such as Sobolev spaces W^m_p, Besov spaces $B^s_{p,q}$, Hardy space H_p, Hölder-Zygmund spaces $\mathscr{C}^s$ and others are included in these scales. For further and deeper information we refer to [10], [21–23].

Concerning function spaces defined on the n-Torus T^n there exists a lot of classical and more recent results which were stimulated, especially, by various applications to trigonometric and power series (we refer to the monographs of A. Zygmund [25], E. M. Stein [14], E. M. Stein, G. Weiss [15], and the papers of M. H. Taibleson [19], T. M. Flett [4–6] et al.). Recently, H. Triebel [24] has shown how to extend the R^n-theory of $B^s_{p,q}$- and $F^s_{p,q}$-spaces to the periodic case (for $1 < p < \infty$ see also [8], [14]).

In the present paper we deal with distributions on T^1

$$f = f(e^{it}) = \sum_{n \geqq 0} c_n e^{int}, \quad t \in (-\pi, \pi],$$

of power series type and give (in a special situation) an alternative approach to periodic function spaces of Besov-Hardy-Sobolev type. The basic tool is the use of properties of the Cesaro means of the power series of f and related maximal functions. This makes it possible to obtain a substantial theory for parameters $0 < p, q \leqq \infty$, $-\infty < s < \infty$ in a rather uncomplicated manner. Since the classical Hardy spaces H_p as well as the Lipschitz spaces $H\Lambda(s, p, q)$ introduced in [6] are included in the scales $B^s_{p,q,+}$ and $F^s_{p,q,+}$ of function spaces under consideration our approach yields a number of well known but also new results, at least, for $0 < p < 1$. It should be mentioned that an extension of our investigatiosn to the n-dimensional case as well as to non-analytic periodic spaces is possible. This will be carried out elsewhere.

1. CESARO MEANS AND MAXIMAL FUNCTIONS

Let

$$D'_+ = \{f = \sum_{n \geqq 0} c_n e^{int} : |c_n| = O(n^\beta),\ n \to \infty,\ \beta = \beta(f) < \infty\}$$

be the subspace of $D'(T^1)$ consisting of periodic distributions of power series type. To any $f \in D'_+$ there corresponds an analytic function

$$f(z) = \sum_{n \geqq 0} c_n z^n, \quad |z| < 1 \quad \left(\text{or } f(re^{it}) = \sum_{n \geqq 0} c_n r^n e^{int},\ r < 1\right).$$

For given $f \in D'_+$ and $\alpha \geqq 0$ we introduce the Cesaro (C, α)-means

$$\sigma_n^\alpha(f) = \sigma_n^\alpha(f)(e^{it}) = (A_n^\alpha)^{-1} \sum_{k=0}^{n} A_{n-k}^\alpha c_k e^{ikt}, \quad n \geqq 0,$$

where

$$A_n^\alpha = \binom{n+\alpha}{n} = \frac{(n+\alpha)(n-1+\alpha)\dots(1+\alpha)}{n!} \left(\approx \frac{n^\alpha}{\Gamma(1+\alpha)}\right), \quad n \geqq 0, \quad \alpha \geqq 0.$$

In the forthcoming considerations the maximal functions

$$(1.1) \qquad \sigma_*^\alpha(f)(e^{it}) = \sup_{n \geqq 0} |\sigma_n^\alpha(f)(e^{it})|, \quad t \in (-\pi, \pi], \quad f \in D'_+,$$

play an important role. For functions from H_p various estimates of (1.1) were previously given by G. H. Hardy, J. E. Littlewood, A. Zygmund, G. Sunouchi, T. M. Flett, E. M. Stein et al. (cf. [4], [13]). Here

$$H_p = \{f \in D'_+ : \|f\|_{H_p} = \lim_{r \to 1} M_p(f, r) < \infty\}, \quad 0 < p \leqq \infty,$$

where

$$M_p(f, r) = \|f(re^{it})\|_{L_p} = \begin{cases} \left(\dfrac{1}{2\pi} \displaystyle\int_{-\pi}^{\pi} |f(re^{it})|^p \, dt\right)^{1/p}, & 0 < p < \infty \\ \operatorname*{supvrai}_{t \in (-\pi, \pi]} |f(re^{it})|, & p = \infty, \end{cases}$$

denotes the classical Hardy space of analytic functions in the unit disc. If $f \in H_p$ then there exist a.e. on $(-\pi, \pi]$ boundary values $f(e^{it}) = \lim_{r \to 1} f(re^{it}) \in L_p(-\pi, \pi)$ with the property $\|f\|_{H_p} = \|f(e^{it})\|_{L_p}$, $0 < p \leqq \infty$.

Furthermore, we need the following quasinorms for a sequence $\{f_j = f_j(e^{it})\}$, $j \geqq 0$, of functions belonging to $L_p(-\pi, \pi)$, $0 < p \leqq \infty$:

$$\|\{f_j\}\|_{l_q(L_p)} = \begin{cases} \{\sum_{j \geqq 0} \|f_j\|_{L_p}^q\}^{1/q}, & 0 < q < \infty \\ \sup_{j \geqq 0} \|f_j\|_{L_p}, & q = \infty \end{cases}$$

$$\|\{f_j\}\|_{L_p(l_q)} = \begin{cases} \|(\sum_{j \geqq 0} |f_j(e^{it})|^q)^{1/q}\|_{L_p}, & 0 < q < \infty \\ \|\sup_{j \geqq 0} |f_j(e^{it})|\,\|_{L_p}, & q = \infty. \end{cases}$$

For any sequence of H_p-functions $\{f_j\}$, $j \geqq 0$, we write $\{f_j\} \in H_p(l_q)$ $(\in l_q(H_p))$ if the related sequence of boundary value distributions belongs to $L_p(l_q)$ $(l_q(L_p))$, this means that the corresponding quasi norm is finite.

First we state a simple (may be, already known) estimate for (1.1) in terms of the usual real maximal function

$$M(g)(e^{it}) = \sup_{s>0} \frac{1}{2s} \int_{t-s}^{t+s} |g(e^{i\xi})| \, d\xi \,, \quad t \in (-\pi, \pi) \,, \quad g \in L_1 \,. \tag{1.2}$$

Theorem 1.1. *Suppose $0 < p \leqq 1$ and $f \in H_p$. Then we have for $\alpha > 1/p - 1$*

$$\sigma_*^\alpha(f)(e^{it}) \leqq C_{p,\alpha}\{M(|f|^p)(e^{it})\}^{1/p} \,, \quad t \in (-\pi, \pi] \,. \tag{1.3}$$

Proof. For given $\alpha \geqq 0$ and $f \in D'_+$ we can use the integral representation (see [16, p. 616])

$$\sigma_n^\alpha(f)(e^{it}) = (2\pi A_n^\alpha)^{-1} \int_{-\pi}^{\pi} f(re^{i(\xi+t)}) (re^{i\xi})^{-n} \left(\frac{1-(re^{i\xi})^{n+1}}{1-re^{i\xi}}\right)^{\alpha+1} d\xi \,,$$

where $0 < r < 1$, $n \geqq 0$. Putting $r = r_n = 1 - (n+1)^{-1}$ we obtain

$$|\sigma_n^\alpha(f)(e^{it})| \leqq C_\alpha(n+1)^{-\alpha} \int_{-\pi}^{\pi} |f(r_n e^{i(\xi+t)})| \left|\frac{1-(r_n e^{i\xi})^{n+1}}{1-r_n e^{i\xi}}\right|^{\alpha+1} d\xi \,, \quad n \geqq 0 \,.$$

Since

$$\varphi(z) = f(e^{it}z) \left(\frac{1-z^{n+1}}{1-z}\right)^{\alpha+1}$$

belongs to H_p a well known inequality of Hardy-Littlewood

$$M_1(\varphi, r) \leqq C_p(1-r)^{1-1/p} \|\varphi\|_{H_p} \,, \quad 0 < p \leqq 1 \,, \quad 0 < r < 1 \,, \tag{1.4}$$

(for a proof of (1.4) cf. [17]) immediately yields

$$|\sigma_n^\alpha(f)(e^{it})| \leqq C_{p,\alpha}(n+1)^{1/p-1-\alpha} \left\{\int_{-\pi}^{\pi} |f(e^{i(\xi+t)})|^p \left|\frac{\sin((n+1)\xi/2)}{\sin(\xi/2)}\right|^{(\alpha+1)p} d\xi\right\}^{1/p} =$$

$$= C_{p,\alpha} \left\{(n+1) \int_{-\pi/(n+1)}^{\pi/(n+1)} |f(e^{i(\xi+t)})|^p \, d\xi \, + \right.$$

$$\left. + (n+1)^{1-p(\alpha+1)} \int_{\pi/(n+1)}^{\pi} (|f(e^{i(\xi+t)})|^p + |f(e^{i(t-\xi)})|^p) \, \xi^{-p(\alpha+1)} \, d\xi\right\}^{1/p} .$$

Obviously, the first term on the right hand side can be majorized by $M(|f|^p)(e^{it})$ while integrating the second integral by parts we obtain the upper bound

$$C_{p,\alpha} n^{1-p(\alpha+1)} \left\{\|f\|_{L_p}^p + \int_{\pi/(n+1)}^{\pi} \left(\int_{-s}^{s} |f(e^{i(t+\xi)})|^p \, d\xi\right) s^{1-p(\alpha+1)} \, ds\right\} .$$

Again according to (1.2) this yields

$$|\sigma_n^\alpha(f)(e^{it})| \leqq C_{p,\alpha}(M(|f|^p)(e^{it}))^{1/p} \, .$$

$$\cdot \left\{1 + (n+1)^{1-p(\alpha+1)}\left(1 + \int_{\pi/(n+1)}^{\pi} s^{2-p(\alpha+1)}\, ds\right)\right\}^{1/p} \leqq C_{p,\alpha}(M(|f|^p)(e^{it}))^{1/p} ,$$

$$\alpha > 1/p - 1 , \quad n \geqq 0 , \quad t \in (-\pi, \pi] \, .$$

This proves (1.3).

Theorem 1.2. a) *Let* $f \in H_p$, $0 < p \leqq \infty$. *Then we have*

$$(1.5) \qquad \|\sigma_*^\alpha(f)\|_{L_p} \leqq C_{p,\alpha}\|f\|_{H_p} , \quad \alpha > \max(0, 1/p - 1) \, .$$

b) *If* $\{f_j\} \in l_q(H_p)$, $0 < p$, $q \leqq \infty$, *then*

$$(1.6) \qquad \|\{\sigma_*^\alpha(f_j)\}\|_{l_q(L_p)} \leqq C_{p,\alpha}\|\{f_j\}\|_{l_q(L_p)}, \quad \alpha > \max(0, 1/p - 1) \, .$$

c) *If* $\{f_j\} \in H_p(l_q)$, $0 < p < \infty$, $0 < q \leqq \infty$, *then*

$$(1.7) \qquad \|\{\sigma_*^\alpha(f_j)\}\|_{L_p(l_q)} \leqq C_{p,q,\alpha}\|\{f_j\}\|_{L_p(l_q)}$$

for $\alpha > \max(0, 1/p - 1, 1/q - 1)$.

The proof of the inequalities (1.5)–(1.7) is an immediate consequence of Theorem 1.1 and known estimates for the real maximal function (1.2). Part a) follows from the inequality

$$(1.8) \qquad \|M(f)\|_{L_r} \leqq C_r\|f\|_{L_r} , \quad f \in L_r , \quad 1 < r \leqq \infty ,$$

due to Hardy-Littlewood (see [25, v. 1, ch. 1]). For $1 < p \leqq \infty$ this is obvious (put $p = 1$ in (1.3) and $r = p$ in (1.8)). Considering the case $0 < p \leqq 1$ we choose some $p' < p$ satisfying $\alpha > 1/p' - 1$. According to (1.3), (1.8) we obtain

$$\|\sigma_*^\alpha(f)\|_{L_p} \leqq C_{p,\alpha}\|\{M(|f|^{p'})(e^{it})\}^{1/p'}\|_{L_p} = C_{p,\alpha}\|M(|f|^{p'})\|_{L_{p/p'}}^{1/p'} \leqq$$
$$\leqq C_{p,\alpha}\| \, |f|^{p'}\|_{L_{p/p'}}^{1/p'} = C_{p,\alpha}\|f\|_{H_p} \, .$$

Part b) easily follows by (1.5). Finally, inequality (1.7) can be deduced in analogy from the vector valued variant of (1.8)

$$(1.9) \qquad \|\{M(f_j)\}\|_{L_p(l)_q} \leqq C_{p,q}\|\{f_j\}\|_{L_p(l_q)} , \quad \{f_j\} \in L_p(l_q) ,$$

$1 < p$, $q < \infty$, due to C. Fefferman, E. M. Stein [2] (since $\sup_{j \geqq 0} M(f_j)(e^{it}) \leqq \leqq M(\sup_{j \geqq 0} |f_j|)(e^{it})$, $t \in (-\pi, \pi]$, (1.9) remains valid for $q = \infty$, too).

Remark 1.1. The statements of Theorem 1.2 are not new ones, except the case $0 < p \leqq 1$ in *c*). E.g., inequality (1.5) represents the above mentioned classical result on (1.1). The proof of (1.5)–(1.7) seems to be more elegant and compact than previously given ones.

Remark 1.2. If $1 < p < \infty$ one can admit the case $\alpha = 0$, too. Although this can not be shown by our methods it follows from the results on a.e. convergence of Fourier series in L_p due to L. Carleson, R. Hunt (see [11]). Therefore, most of the following statements remains valid in this case.

Now we introduce multiplier operators and some further properties of the Cesaro means. Any sequence of complex numbers $\lambda = \{\lambda_n\}$, $n \geqq 0$, with $\sum_{n\geqq 0} \lambda_n e^{int} \in D'_+$ defines in D'_+ a multiplier operator T_λ given by

$$T_\lambda : f = \sum_{n\geqq 0} c_n e^{int} \to \lambda f = \sum_{n\geqq 0} \lambda_n c_n e^{int} .$$

If $X_1, X_2 \subset D'_+$ are quasi Banach spaces and $T_\lambda : X_1 \to X_2$ is bounded we call the sequence λ *coefficient multiplier* (or *Fourier multiplier*) *for the pair* X_1, X_2, in the case $X_1 = X_2 = X$ we simply say *multiplier for* X.

In order to state special multiplier criteria for the spaces we are interested in one needs the following definitions. For given $\lambda = \{\lambda_j\}$, $j \geqq 0$, and real $\beta \geqq 0$ the sequence $\Delta^\beta \lambda = \{\Delta^\beta \lambda_k\} = \{\sum_{m\geqq 0} A_m^{-\beta-1} \lambda_{k+m}\}$, $k \geqq 0$, is called *difference sequence of order* β *corresponding to* λ. For integer β we obtain the usual differences of higher order (e.g. $\Delta^0 \lambda_k = \lambda_k$, $\Delta^1 \lambda_k = \lambda_k - \lambda_{k+1}$, $\Delta^2 \lambda_k = \lambda_k - 2\lambda_{k+1} + \lambda_{k+2}$, and so on). We set

$$bv_{\beta+1} = \{\lambda \in l_\infty : \sum_{k\geqq 0} A_k^\beta |\Delta^{\beta+1} \lambda_k| < \infty\} , \quad \beta \geqq 0 .$$

Supposing $\lambda \in bv_{\beta+1}$ there exists $\lim_{n\to\infty} \lambda_n = \lambda_\infty$, and $bv_{\beta+1}$, $\beta \geqq 0$, becomes a Banach space equipped with the norm $\|\lambda\|_{bv_{\beta+1}} = |\lambda_\infty| + \sum_{k\geqq 0} A_k^\beta |\Delta^{\beta+1} \lambda_k|$.

If $0 \leqq \beta < \alpha$ then we have $bv_{\alpha+1} \hookrightarrow bv_{\beta+1}$ ($\hookrightarrow$ means that the imbedding is continuous). These and further elementary properties of $bv_{\beta+1}$ are proved in the paper [20] of W. Trebels from which we have taken other suggestions, too. In [20], p. 20–22, it is stated that

$$\lambda f = \lambda_\infty f + \sum_{k\geqq 0} A_k \, \Delta^{\alpha+1} \lambda_k \, \sigma_k^\alpha(f) , \quad \alpha \geqq 0 , \tag{1.10}$$

at least for any polynomial $f \in D'_+$. Thus we obtain

$$|\lambda f(e^{it})| \leqq \|\lambda\|_{bv_{\alpha+1}} \sigma_*^\alpha(f) (e^{it}) , \quad t \in (-\pi, \pi] , \tag{1.11}$$

if $\lambda \in bv_{\alpha+1}$, $\alpha \geqq 0$, and f is a polynomial. This simple relation (together with theorem 1.2) makes it possible to deal with multiplier operators by using Cesaro summability properties. This idea will be systematically explored in the following sections (cf. [20]).

Furthermore, we need the following identities

$$\Delta^\alpha \eta_k - \Delta^\alpha \eta_n = \sum_{j=k}^{n-1} \Delta^{\alpha+1} \eta_j , \quad 0 \leqq k < n , \quad \alpha \geqq 0 , \tag{1.12}$$

and

$$\Delta^{\alpha+1}(\lambda\eta)_k = \sum_{l=0}^{\alpha+1} A_l^{\alpha+1-l} \Delta^{\alpha+1-1} \eta_{k+1} \Delta^1 \lambda_k, \quad k \geqq 0, \tag{1.13}$$

where $\alpha = 0, 1, \ldots$ is an integer and $(\lambda\eta)_k = \lambda_k \eta_k$, $k \geqq 0$.

Remark 1.3. Instead of (1.1) we can use with the same success the more general maximal function

$$\lambda_*^\alpha(f)(\mathrm{e}^{\mathrm{i}t}) = \sup_{\|\lambda\|_{bv_{\alpha+1}} \leqq 1} \left|\sum_{n \geqq 0} \lambda_n c_n \mathrm{e}^{\mathrm{i}nt}\right|, \quad \alpha \geqq 0, \tag{1.14}$$

because it immediately follows by (1.11) and the elementary property $\Delta^{\alpha+1} A_{n-k}^\alpha = A_{n-k}^{-1} = \delta_{n,k}$, $0 \leqq k \leqq n$, $\alpha \geqq 0$, that $\lambda_*^\alpha(f)(\mathrm{e}^{\mathrm{i}t}) = \sigma_*^\alpha(f)(\mathrm{e}^{\mathrm{i}t})$, $t \in (-\pi, \pi]$.

Remark 1.4. There exists a simple connection with other maximal functions corresponding to an analytic function $f(z)$. Let (see [25, v. 1, ch. 7], [3])

$$N_*^\alpha(f)(\mathrm{e}^{\mathrm{i}t}) = \sup_{z = r\mathrm{e}^{\mathrm{i}\xi}, r<1} |f(z)| \left(1 + \frac{|\xi - t|}{1 - r}\right)^{-\alpha-1}, \quad \alpha \geqq 0,$$

$$N_*(f)(\mathrm{e}^{\mathrm{i}t}) = \sup_{\substack{z = r\mathrm{e}^{\mathrm{i}\xi} \\ |\xi - t| < (1-r)\varrho}} |f(z)|, \quad 0 < \varrho < \infty,$$

and

$$N_+(f)(\mathrm{e}^{\mathrm{i}t}) = \sup_{r<1} |f(r\mathrm{e}^{\mathrm{i}t})|, \quad t \in (-\pi, \pi],$$

be the tangential, non-tangential, and radial complex maximal function of $f \in L'_+$, respectively. For arbitrary $t \in (-\pi, \pi]$ we have

$$N_+(f)(\mathrm{e}^{\mathrm{i}t}) \leqq N_*(f)(\mathrm{e}^{\mathrm{i}t}) \leqq C_{\alpha,\varrho} N_*^\alpha(f)(\mathrm{e}^{\mathrm{i}t}), \quad 0 < \varrho < \infty, \quad \alpha \geqq 0.$$

We can prove (at least for polynomials f)

$$N_*^\alpha(f)(\mathrm{e}^{\mathrm{i}t}) \leqq \sigma_*^\alpha(f)(\mathrm{e}^{\mathrm{i}t}), \quad \alpha \geqq 0, \quad t \in (-\pi, \pi]. \tag{1.15}$$

Indeed, according to (1.11) we obtain

$$|f(r\mathrm{e}^{\mathrm{i}\xi})| \leqq \|\{r^k \mathrm{e}^{\mathrm{i}k(\xi - t)}\}\|_{bv_{\alpha+1}} \sigma_*^\alpha(f)(\mathrm{e}^{\mathrm{i}t}).$$

But for arbitrary complex τ, $|\tau| < 1$, we have

$$\|\{\tau^k\}\|_{bv_{\alpha+1}} = \sum_{k \geqq 0} A_k^\alpha \left|\sum_{l \geqq 0} A_l^{-\alpha-2} \tau^{l+k}\right| = |1 - \tau|^{\alpha+1} \sum_{k \geqq 0} A_k^\alpha |\tau|^k =$$

$$= |1 - \tau|^{\alpha+1} (1 - |\tau|)^{-\alpha-1}.$$

Thus, we can estimate

$$N_*^\alpha(f)(\mathrm{e}^{\mathrm{i}t}) \leqq \sigma_*^\alpha(f)(\mathrm{e}^{\mathrm{i}t}) \sup_{|z|<1} \frac{|1 - z|^{\alpha+1}}{(1 - |z|)^{\alpha+1}} \cdot \frac{(1 - |z|)^{\alpha+1}}{(1 - |z| + |\arg z|)^{\alpha+1}}$$

which yields (1.15).

Using (1.15), (1.3) one can deduce various complex maximal inequalities directly from their real counterparts (1.8), (1.9).

2. THE SPACES $B^s_{p,q,+}$, $F^s_{p,q,+}$-DEFINITION AND BASIC PROPERTIES

For the definition below we need appropriate decompositions of the power series of $f \in D'_+$. Let $\Lambda = \{\lambda^{(j)}\}$ be a set of finite sequences $\lambda^{(j)} = \{\lambda^{(j)}_n\}$, $j \geqq 0$. We write $\Lambda \in A_\alpha$, $\alpha \geqq 0$, if

$$(2.1) \qquad \lambda^{(j)} \in bv_{\alpha+1}\,, \quad \|\Lambda\|_\alpha = \sup_{j \geqq 0} \|\lambda^{(j)}\|_{bv_{\alpha+1}} < \infty$$

and

$$(2.2) \qquad \operatorname{supp} \lambda^{(j)} = \{n \geqq 0 : \lambda^{(j)}_n \neq 0\} \subset \begin{cases} \{2^{j-r}+1, \ldots, 2^{j+r}\}\,, & j \geqq r \\ \{0, \ldots, 2^{j+r}\}\,, & j = 0, \ldots, r-1 \end{cases}$$

for an integer $r \geqq 1$. Furthermore, if $\Lambda \in A_\alpha$ with $r = 1$ and, in addition,

$$(2.3) \qquad \sum_{j \geqq 0} \lambda^{(j)}_n = 1\,, \quad n = 0, 1, \ldots,$$

we set $\Lambda \in A_{\alpha,0}$. Obviously, $A_{\alpha,0} \subset A_\alpha$, $\alpha \geqq 0$, and $A_{\alpha,0} \subset A_{\beta,0}$, $A_\alpha \subset A_\beta$, $0 \leqq \beta < \alpha$. To see that $A_{\infty,0} = \bigcap_{\alpha>0} A_{\alpha,0} \neq \emptyset$ we consider any function $\varphi(x) \in C^\infty(0, \infty)$ with $\varphi(x) = 1$ for $x \in [0, 1]$ and $\operatorname{supp} \varphi(x) \subset [0, 2]$. Then $\Lambda_\varphi = \{\lambda^{(j)}_\varphi\}$ defined by the relations

$$(2.4) \qquad \lambda^{(j)}_{\varphi,n} = \begin{cases} \varphi(n2^{-j}) - \varphi(n2^{-j+1})\,, & j \geqq 1\,, \quad n \geqq 0 \\ \varphi(n)\,, & j = 0\,, \end{cases}$$

belongs to $A_{\infty,0}$. Indeed, (2.2) with $r = 1$ and (2.3) are automatically fulfilled, and (2.1) can be deduced from

$$|\Delta^{\beta+1} \lambda^{(j)}_{\varphi,n}| \begin{cases} = 0\,, & k \geqq 2^{j+1}\,, \\ \leqq C \|\varphi^{(\beta+1)}\|_{L\infty(0,2)}\, 2^{-j(\beta+1)}\,, & k < 2^{j+1}\,, \quad j \geqq 0\,. \end{cases}$$

Every $\Lambda \in A_{\alpha,0}$ generates a decomposition $f = \sum_{j \geqq 0} \lambda^{(j)} f$ of the power series of $f \in D'_+$ into polynomials. Now we can introduce the quasi norms of Littlewood-Paley type

$$(2.5) \qquad \|f\|_{B^{s,\Lambda}_{p,q+}} = \|\{2^{js} \lambda^{(j)} f\}\|_{l_q(L_p)}\,, \quad 0 < p \leqq \infty\,,$$

$$(2.6) \qquad \|f\|_{F^{s,\Lambda}_{p,q+}} = \|\{2^{js} \lambda^{(j)} f\}\|_{L_p(l_q)}\,, \quad 0 < p < \infty\,,$$

where in both cases we admit $0 < q \leqq \infty$, $-\infty < s < \infty$. In the following we shall deal with these ranges of parameters for the $B^s_{p,q,+}$- and $F^s_{p,q,+}$-spaces, respectively, if not stated otherwise. Furthermore, we fix

$$(2.7) \qquad \alpha_0 = \begin{cases} \max(1/p, 1) - 1 & \text{for the spaces } B^s_{p,q,+}\,, \\ \max(1/p, 1/q, 1) - 1 & \text{for the spaces } F^s_{p,q,+}\,. \end{cases}$$

Lemma 2.1. *Let X be any one of the spaces $l_q(L_p)$ or $L_p(l_q)$ and suppose $\bar\Lambda \in A_{\bar\alpha}$, $\Lambda \in A_{\alpha,0}$ with $\alpha, \bar\alpha > \alpha_0$ where α_0 is defined in (2.7) with respect to X. Then for the above* *described range of parameters p, q, and s we have*

$$(2.8)\qquad \|\{2^{js}\bar\lambda^{(j)}f\}\|_X \leqq C\|\bar\Lambda\|_{\bar\alpha}\,\|\{2^{js}\lambda^{(j)}f\}\|_X\,,\quad f\in D'_+\,,$$

whenever the right-hand side of this inequality is finite.

Proof. By the assumption on $\Lambda, \bar\Lambda$ we get $(j \geqq 0)$

$$2^{js}\bar\lambda^{(j)}f = 2^{js}\bar\lambda^{(j)}\left(\sum{}'\lambda^{(i)}f\right) = \sum{}'2^{(j-i)s}\bar\lambda^{(j)}\left(2^{is}\lambda^{(i)}f\right)$$

and, according to (1.11), it follows that

$$\left|2^{js}\bar\lambda^{(j)}f(e^{it})\right| \leqq 2^{(r+1)|s|}\sum{}'\sigma_*^{\bar\alpha}\left(2^{is}\lambda^{(i)}f\right)(e^{it})\,\|\bar\lambda^{(j)}\|_{bv_{\bar\alpha}+1}$$

(here $\sum'$ stands for $\sum\limits_{i=\max(0,j-r-1)}^{j+r+1}$ where r depends on $\bar\Lambda$, see (2.2)). Therefore, inequality (2.8) is a consequence of theorem 1.2. The constant C in (2.8) depends, in general, on $p, q, s, \alpha, \bar\alpha$, and r. In the following we do not examine the dependence of the constants on the parameters if this is not necessary.

Lemma 2.1 makes the next definitions clear. We set for $0 < p,\ q \leqq \infty,\ -\infty < s < \infty$

$$(2.9)\qquad B^s_{p,q,+} = \{f\in D'_+ : \|f\|_{B^s{}_{p,q,+}} = \inf_{\substack{\Lambda\in A_{\alpha,0}\\ \alpha>\alpha_0}} \|f\|_{B^s{}^\Lambda{}_{p,q,+}} < \infty\}$$

$$(2.10)\qquad F^s_{p,q,+} = \{f\in D'_+ : \|f\|_{F^s{}_{p,\alpha,+}} = \inf_{\substack{\Lambda\in A_{\alpha,0}\\ \alpha>\alpha_0}} \|f\|_{F^s{}^\Lambda{}_{p,q,+}} < \infty\}$$

(in (2.10) we again exclude $p = \infty$). These are the spaces considered in the present paper.

Theorem 2.2. *Let X be any one of the spaces (2.9), (2.10). Then X is a quasi Banach space with the quasi norm $\|\cdot\|_X$ (a Banach space iff $\gamma = \min(1, p, q) = 1$) and $d_X(f, g) = \|f - g\|^\gamma_X$, $f, g \in X$, defines a translation invariant complete metric on X. Furthermore, $\|\cdot\|_{X^{\bar\Lambda}}$ is an equivalent quasi norm in X whenever $\bar\Lambda \in A_{\bar\alpha,0}$, $\bar\alpha > \alpha_0$.*

Proof. The latter statement follows by lemma 2.1 and the definitions (2.9), (2.10). We have $\|f\|_X \leqq \|f\|_{X^{\bar\Lambda}} \leqq C\|\bar\Lambda\|_{\bar\alpha}\,\|f\|_{X^\Lambda}$ and it remains to take the infimum over all $\Lambda \in A_{\alpha,0}$, $\alpha > \alpha_0$.

The other assertions are elementary, we give an outline of the proof of the completeness, only. Let $\{f_n\} \subset X$ be a Cauchy sequence and $\Lambda \in A_{\alpha,0}$, $\alpha > \alpha_0$. Then, for arbitrary q, s, the sequences $\{\lambda^{(j)}f_n\}$ consisting of polynomials of degree $\leqq 2^{j+1}$ are fundamental in H_p for any fixed $j \geqq 0$. Therefore, there exists a $f \in D'_+$ for which $\{\lambda^{(j)}f_n(e^{it})\}$ converges in $C(-\pi, \pi)$ to $\lambda^{(j)}f(e^{it})$ for every $j \geqq 0$. Now the proof of the convergence of $\{f_n\}$ in X (to f) is obvious.

Theorem 2.3. *Let X be any one of the spaces (2.9), (2.10) and $\gamma = \{\gamma_n\}$, $n \geqq 0$, a sequence of complex numbers.*

a) *If there exists some $\Lambda \in A_{\alpha,0}$, $\alpha > \alpha_0$, for which $\gamma\Lambda = \{\gamma\lambda^{(j)}\}$ belongs to some $A_{\bar{\alpha}}$, $\bar{\alpha} > \alpha_0$, then γ is a multiplier for X. More precisely,*

$$\|\gamma f\|_X \leqq C\|\gamma\Lambda\|_{\bar{\alpha}}\, \|f\|_X\,, \quad f \in X\,. \tag{2.11}$$

b) *If $m > \alpha_0$ is an integer and γ satisfies the condition*

$$\|\gamma\|_{l\infty} + \sup_{j \geqq 0} \sum_{i=2^j}^{2^{j+1}} \left|\Delta^{m+1}\gamma_i\right| 2^{mj} < \infty\,, \tag{2.12}$$

then γ is a multiplier for X. In particular, (2.12) is valid whenever $\gamma \in bv_{\alpha+1}$. $\alpha \geqq m$.

Proof. (2.11) is a direct consequence of lemma 2.1. The assertion from b) follows by a) and the properties (1.12), (1.13) of differences. We fix some Λ_φ (for definition, see (2.4)). Because $\operatorname{supp} \lambda_\varphi^{(j)} \subset \{2^{j-1} + 1, \ldots, 2^{j+1}\}$, $j \geqq 1$, we have according to (1.13)

$$\|\gamma\lambda_\varphi^{(j)}\|_{bv_{m+1}} \leqq C\, 2^{jm} \sum_{k=2^{j-1}-m}^{2^{j+1}} \left|\Delta^{m+1}(\gamma\lambda_\varphi^{(j)})_k\right| \leqq$$

$$\leqq C\, 2^{jm} \sum_{l=0}^{m+1} \sum_{k=2^{j-1}-m}^{2^{j+1}} \left|\Delta^{m+1-l}\lambda_{\varphi,k+l}^{(j)}\right| \left|\Delta^l\gamma_k\right|.$$

The above mentioned properties of $\lambda_\varphi^{(j)}$ yield

$$\|\gamma\Lambda_\varphi\|_m \leqq C\Big(\sup_{\substack{j=0,1,\ldots \\ l=1,\ldots,m+1}} \{2^{j(l-1)} \sum_{k=2^j}^{2^{j+1}} \left|\Delta^l\gamma_k\right|\} + \|\gamma\|_{l_\infty}\Big).$$

To see that the quantity in the right-hand side is finite we use (1.12) and (2.12). For $l = m$ we have

$$\left|\Delta^m\gamma_k - \Delta^m\gamma_n\right| \leqq \sum_{i=k}^{n-1} \left|\Delta^{m+1}\gamma_i\right| \leqq \sum_{j:2^{j+1}>k} \sum_{i=2^j}^{2^{j+1}} \left|\Delta^{m+1}\gamma_i\right| \leqq$$

$$\leqq C \sum_{j:2^{j+1}>k} 2^{-jm} \leqq C(k+1)^{-m}\,, \quad n > k \geqq 0\,.$$

Hence $\{\Delta^m\gamma_k\}$ is fundamental. It is obvious that $\lim_{n\to\infty} \Delta^m\gamma_n = 0$ (otherwise we should obtain a contradiction with $\gamma \in l_\infty$). This yields

$$2^{j(m-1)} \sum_{k=2^j}^{2^{j+1}} \left|\Delta^m\gamma_k\right| \leqq C\, 2^{j(m-1)} \sum_{k=2^1}^{2^{j+1}} (k+1)^{-m} \leqq C < \infty\,, \quad j \geqq 0\,,$$

analogously we estimate for $l = m - 1, \ldots, 1$. The case $l = m + 1$ immediately follows from (2.12). Thus, we have $\|\gamma\Lambda_\varphi\|_m < \infty$ under the assumption that (2.12) is valid. On the other hand, if $\gamma \in bv_{\alpha+1}$, $\alpha \geqq m$, then $\gamma \in bv_{m+1}$. This obviously yields (2.12).

Remark 2.1. Part b) represents a multiplier theorem of Marcinkiewicz type (for the classical result, see [25, v. 2, ch. 15] or [1]). It is not clear whether there exists a suitable extension of (2.12) to non-integer $\alpha > \alpha_0$. However, in this case we can show that γ is a multiplier for X if $\gamma \in bv_{\alpha+1}$. Since for our purposes it is sufficient to have theorem 2.3 we omit the more technical proof of this assertion.

Now we introduce the special multiplier operator

$$J^\beta f(e^{it}) = \sum_{n \geqq 0} (n+1)^{-\beta} c_n e^{int}, \quad -\infty < \beta < \infty, \quad f \in D'_+, \tag{2.13}$$

which is closely related to fractional integration ($\beta > 0$) and differentiation ($\beta < 0$).

Theorem 2.4. *Let X^s be any one of the spaces $B^s_{p,q,+}, F^s_{p,q,+}$ from (2.9), (2.10). Then the operator J^β yields an isomorphism from X^s onto $X^{s+\beta}$, more precisely, the restriction of J^β to X^s is a one-to-one mapping onto $X^{s+\beta}$, and $\|J^\beta f\|_{X^{s+\beta}}$ is an equivalent quasinorm in X^s, $-\infty < \beta < \infty$.*

Proof. This is a consequence of (2.11) if we can show that

$$\|\lambda^{(j)} J^\beta\|_{bv_{m+1}} \leqq C\, 2^{-j\beta}, \quad -\infty < \beta < \infty, \quad \Lambda \in A_{m,0}, \quad m, j \geqq 0.$$

Observing that $|\Delta^l\{(n+1)^{-\beta}\}| \leqq C(n+1)^{-\beta-l}$, $n \geqq 0$, and using (1.13) we obtain

$$\|\lambda^{(j)} J^\beta\|_{bv_{m+1}} \leqq C\, 2^{jm} \sum_{k=2^{j-1}-m}^{2^{j+1}} |\Delta^{m+1}(\lambda^{(j)}_k (k+1)^{-\beta})| \leqq$$

$$\leqq C\, 2^{jm} \sum_{l=1}^{m+1} \sum_{k=2^{j-1}-m}^{2^{j+1}} 2^{-j(\beta+m+1-l)} |\Delta^l \lambda^{(j)}_k| \leqq$$

$$\leqq C\, 2^{-j\beta} \sum_{l=1}^{m+1} \|\lambda^{(j)}\|_{bv_l} \leqq C\, 2^{-j\beta} \|\Lambda\|_m, \quad m, j \geqq 0.$$

Thus, the theorem is proved.

Finally, it should be mentioned that further properties such as duality, interpolation, imbeddings, etc. of the spaces $B^s_{p,q,+}, F^s_{p,q,+}$ can be established analogously to the case of spaces defined on R^n (cf. [10], [21–23]). These topics as well as an extension of our considerations to the n-dimensional torus will be considered elsewhere.

3. EQUIVALENT QUASI NORMS (MEAN VALUE PROPERTIES)

In the remaining sections of this paper we deal with various equivalent quasi norms and representations of the spaces defined above in (2.9), (2.10). Let us introduce the spaces

$$B^{s,\beta}_{p,q,+} = \Big\{ f \in D'_+ : \|f\|_{B^{s,\beta}_{p,q,+}} = \tag{3.1}$$

$$= \left\{ \int_0^1 (1-r)^{-q(\beta+s)-1} M_p(J^\beta f, r)^q \, dr \right\}^{1/q} < \infty \Big\}, \quad 0 < p \leqq \infty$$

(3.2) $$F^{s,\beta}_{p,q,+} = \Big\{ f \in D'_+ : \|f\|_{F^{s,\beta}_{p,q,+}} =$$

$$= \left\| \left\{ \int_0^1 (1-r)^{-q(\beta+s)-1} \left| J^\beta f(re^{it}) \right|^q dr \right\}^{1/q} \right\|_{L_p} < \infty \Big\}, \quad 0 < p < \infty,$$

where $-\infty < s < \infty$, $\beta + s < 0$, and $0 < q < \infty$. For $q = \infty$ we have to modify the definition of the quasinorms by setting $\|f\|_{B^{s,\beta}_{p,\infty,+}} = \sup_{0<r<1} (1-r)^{-(\beta+s)} \cdot M_p(J^\beta f, r)$, and $\|f\|_{F^{s,\beta}_{p,\infty,+}} = \| \sup_{0<r<1} (1-r)^{-(\beta+s)} |J^\beta f(re^{it})| \|_{L_p}$, respectively.

(3.1) coincides with the definition of the Lipschitz spaces $H\Lambda(s, p, q)$ investigated by T. M. Flett [6] if $\beta = -s-1$. The functions

(3.3) $$g_q^{s,\beta}(e^{it}) = \left\{ \int_0^1 (1-r)^{-q(\beta+s)-1} \left| J^\beta f(re^{it}) \right|^q dt \right\}^{1/q}, \quad \beta + s < 0,$$

appearing in (3.2) are appropriate generalizations of the Littlewood-Paley function (see [25, v. 2, ch. 14])

(3.4) $$g^*(e^{it}) = \left\{ \int_0^1 (1-r) \left| f'(re^{it}) \right|^2 dr \right\}^{1/2} (\triangleq g_2^{0,-1}(e^{it})).$$

Theorem 3.1. *For the above described ranges of parameters p, q, s and $\beta + s < 0$ we have*

(3.5) $$B^s_{p,q,+} = B^{s,\beta}_{p,q,+}, \quad F^s_{p,q,+} = F^{s,\beta}_{p,q,+}$$

with equivalent quasi norms.

Proof. We fix an integer $\alpha > \alpha_0$, and $\Lambda \in A_{\alpha,0}$. Furthermore, we choose $\bar{\Lambda} \in A_\alpha$ satisfying $\bar{\lambda}_n^{(j)} = 1$ if $n \in \operatorname{supp} \lambda^{(j)}$, $j \geqq 0$ (e.g., one can take $\bar{\lambda}^{(j)} = \lambda^{(j-1)} + \lambda^{(j)} + \lambda^{(j+1)}$).

First we show the imbeddings

(3.6) $$B^{s,\beta}_{p,q,+} \hookrightarrow B^{s,\Lambda}_{p,q,+}, \quad F^{s,\beta}_{p,q,+} \hookrightarrow F^{s,\Lambda}_{p,q,+}.$$

According to (1.11) for $f \in D'_+$ and $j \geqq 0$ it follows that

$$|\lambda^{(j)} f(e^{it})| = \Big| \sum_{n=2^{j-1}+1}^{2^{j+1}}{}' (\lambda_n^{(j)} r^{-n} (n+1)^\beta) ((n+1)^{-\beta} c_n r^n e^{int}) \Big| \leqq$$

$$\leqq \| \{ \lambda_n^{(j)} r^{-n} (n+1)^\beta \} \|_{bv_{\alpha+1}} \sigma_*^\alpha (J^\beta f(re^{it}))$$

($\sum'$ means that we have to write $\sum_{n=0}^{2}$ for $j = 0$).

The $bv_{\alpha+1}$-norm of $\{\lambda_n^{(j)} r^{-n}(n+1)^\beta\}$ can be estimated by using (1.13), this gives

$$\|\{\lambda_n^{(j)} r^{-n}(n+1)^\beta\}\|_{bv_{\alpha+1}} \leqq C r^{-2^{j+1}} 2^{j\beta} \|\lambda^{(j)}\|_{bv_{\alpha+1}}$$

(the details are omitted). Therefore, for $r \in \Delta_j = (1 - 2^{-j}, 1 - 2^{-j-1}]$, we obtain

$$2^{js}|\lambda^{(j)} f(e^{it})| \leqq C\, 2^{j(\beta+s)}\, \sigma_*^\alpha(J^\beta f(re^{it}))\,, \quad j \geqq 0\,. \tag{3.7}$$

For the $B^s_{p,q+}$-spaces (3.7) together with theorem 1.2 yields

$$\|f\|_{B^{s,A}_{p,q,+}} \leqq C\|\{2^{j(\beta+s)}\, \sigma_*^\alpha(J^\beta f(r_j e^{it}))\}\|_{l_q(L_p)} \leqq$$
$$\leqq C\|2^{j(\beta+s)}\, M_p(J^\beta f, r_j)\|_{l_q} \leqq C\|(1-r_j)^{-(\beta+s)}\, M_p(J^\beta f, r_j)\|_{l_q}$$

where $r_j \in \Delta_j$, $j \geqq 0$, are arbitrary. If $q = \infty$ this is already the desired result. For $0 < q < \infty$ we choose $r_j \in \Delta_j$ in such a way that

$$M_p(J^\beta f, r_j)^q \leqq 2^{j+1} \int_{\Delta_j} M_p(J^\beta f, r)^q\, dr\,, \quad j \geqq 0\,.$$

Thus it follows that

$$\|f\|_{B^{s,A}_{p,q,+}} \leqq C\left\{\sum_{j\geqq 0} 2^{j(\beta+s)q+1} \int_{\Delta_j} M_p(J^\beta f, r)^q\, dr\right\}^{q/1} \leqq C\|f\|_{B^{s,\beta}_{p,q,+}}\,.$$

In the case of the $F^s_{p,q,+}$-spaces we use besides of (3.7) and theorem 1.2 an additional consideration: We have

$$\|f\|_{F^{s,A}_{p,q,+}} = \|\{\sum_{j\geqq 0} 2^{jsq} N^{-1} \sum_{n=1}^{N} |\lambda^{(j)} f(e^{it})|^q\}^{1/q}\|_{L_p} \leqq$$
$$\leqq C\|\{\sum_{j\geqq 0} 2^{j(\beta+s)q}\, N^{-1} \sum_{n=1}^{N} \sigma_*^\alpha(J^\beta f(r_{j,n} e^{it}))^q\}^{1/q}\|_{L_p} \leqq$$
$$\leqq C\|\{\sum_{j\geqq 0} 2^{j(\beta+s)q}\, N^{-1} \sum_{n=1}^{N} |J^\beta f(r_{j,n} e^{it})|^q\}^{1/q}\|_{L_p}\,,$$

where $r_{j,n} = 1 - 2^{-j} + 2^{-j-1} n N^{-1}$, $n = 0, 1, \ldots, N$, forms an uniform partition of Δ_j into N parts, $N \geqq 1$, $j \geqq 0$. As one can easily see for $N \to \infty$ this yields

$$\|f\|_{F^{s,A}_{p,q,+}} \leqq C\left\|\left\{\sum_{j\geqq 0} 2^{j(q(\beta+s)+1)} \int_{\Delta_j} |J^\beta f(re^{it})|^q\, dr\right\}^{1/q}\right\|_{L_p} \leqq$$
$$\leqq C\|f\|_{F^{s,\beta}_{p,q,+}}\,, \quad 0 < q < \infty\,.$$

The changes in the case $q = \infty$ are obvious. Thus, (3.6) is established.

The proof of the inverse imbeddings

$$B^{s,A}_{p,q,+} \hookrightarrow B^{s,\beta}_{p,q,+}\,, \quad F^{s,A}_{p,q,+} \hookrightarrow F^{s,\beta}_{p,q,+} \tag{3.8}$$

is based on the estimate (see (1.11), (1.13))

$$(3.9)\qquad |J^\beta f(re^{it})| = \Big|\sum_{j\geq 0}\ \sideset{}{'}\sum_{n=2^{j-1}+1}^{2^{j+1}} (\bar\lambda_n^{(j)} r^n (n+1)^{-\beta})(\lambda_n^{(j)} c_n e^{int})\Big| \leqq$$

$$\leqq \sum_{j\geq 0} \|\{\bar\lambda^{(j)} r^n (n+1)^{-\beta}\}\|_{bv_{\alpha+1}}\, \sigma_*^\alpha(\lambda^{(j)} f)(e^{it}) \leqq$$

$$\leqq C\sum_{j\geq 0} r^{2^{j-2}}\, 2^{-\beta j}\sigma_*^\alpha(\lambda^{(j)} f)(e^{it}), \quad 0 < r \leqq 1, \quad -\infty < \beta < \infty.$$

Considering the $B^s_{p,q,+}$-spaces it follows from (3.9) and Theorem 1.2 that

$$(3.10)\qquad M_p(J^\beta f, r)^\gamma \leqq C\sum_{j\geq 0} r^{2^{j-2}\gamma}\, 2^{-\beta\gamma j}\|\sigma_*^\alpha(\lambda^{(j)} f)\|_{L_p}^\gamma \leqq$$

$$\leqq C\sum_{j\geq 0} r^{2^{j-2}\gamma}\, 2^{-\beta\gamma j}\|\lambda^{(j)} f\|_{L_p}^\gamma, \quad \gamma = \min(1, p)$$

and

$$\|f\|_{B_{p,q,+}{}^{s,\beta}} \leqq C\left\{\int_0^1 (1-r)^{-q(\beta+s)-1}\Big(\sum_{j\geq 0} r^{2^{j-2}\gamma} 2^{-\beta\gamma j}\|\lambda^{(j)} f\|_{L_p}^\gamma\Big)^{q/\gamma}\, dr\right\}^{1/q}$$

$$C \leqq \begin{cases} \left(\displaystyle\int_0^1 (1-r)^{-q(\beta+s)-1}\Big(\sum_{j\geq 0} r^{2^{j-2}q} 2^{-\beta q j}\|\lambda^{(j)} f\|_{L_p}^q\Big)\, dr\right)^{1/q}, & 0 < q \leqq \gamma \\ \left(\displaystyle\int_0^1 (1-r)^{-q(\beta+s)-1}\Big(\sum_{j\geq 0} r^{2^{j-2}\gamma} 2^{-\gamma(\beta+s)j}\Big)^{q/\gamma-1} \cdot \right. & \\ \left.\cdot \Big(\sum_{j\geq 0} r^{2^{j-2}\gamma} 2^{(sq-(\beta+s)\gamma)j}\|\lambda^{(j)} f\|_{L_p}^q\Big)\, dr\right)^{1/q}, & \gamma < q < \infty. \end{cases}$$

In the next steps we use the elementary inequalities ($\delta, \varepsilon > 0$)

$$(3.11)\qquad \int_0^1 (1-r)^{\delta-1}\, r^{2^j\varepsilon}\, dr \leqq C\, 2^{-j\delta}, \quad j \geqq 0,$$

$$\sum_{j\geq 0} r^{2^j\varepsilon}\, 2^{\delta j} \leqq C(1-r)^{-\delta}, \quad 0 < r < 1.$$

Hence, for $0 < q \leqq \gamma$ we continue with (3.11)

$$\|f\|_{B^{s,\beta}{}_{p,q,+}} \leqq C\Big(\sum_{j\geq 0} 2^{j(\beta+s)q}\, 2^{-j\beta q}\|\lambda^{(j)} f\|^q\Big)^{1/q} \leqq C\|f\|_{B^{s,\Lambda}{}_{p,q,+}},$$

and in the case $\gamma < q < \infty$ we have

$$\|f\|_{B^{s,\beta}{}_{p,q,+}} \leqq C\left(\int_0^1 (1-r)^{-\gamma(\beta+s)-1}\sum_{j\geq 0} r^{2^{j-2}}\, 2^{j(sq-(\beta+s)\gamma)}\|\lambda^{(j)} f\|_{L_p}^q\, dr\right)^{1/q} \leqq$$

$$\leqq C\Big(\sum_{j\geq 0} 2^{j\gamma(\beta+s)}\, 2^{j(sq-(\beta+s)\gamma)}\|\lambda^{(j)} f\|_{L_p}^q\Big)^{1/q} \leqq C\|f\|_{B^{s,\Lambda}{}_{p,q+}}.$$

If $q = \infty$ then from (3.10) we obtain

$$\|f\|_{B^{s,\beta}{}_{p,\infty,\pm}}^\gamma \leqq C\sup_{0<r<1}(1-r)^{-(\beta+s)\gamma}\sum_{j\geq 0} r^{2^{j-2}\gamma}\, 2^{-(\beta+s)\gamma j}\|f\|_{B^{s,\Lambda}{}_{p,\infty,+}}.$$

Now it remains to make use of (3.11). Thus the first imbedding in (3.8) is established in full detail.

For the $F^s_{p,q,+}$-spaces the considerations are analogous. We restrict ourselves to the case $0 < q \leqq 1$. According to (3.9), theorem 1.2, and (3.11) it follows that

$$\|f\|_{F^{s,\beta}_{p,q,+}} \leqq C \left\| \left\{ \int_0^1 (1-r)^{-q(\beta+s)-1} \left(\sum_{j\geqq 0} r^{2^j-2}\, 2^{-j\beta} \sigma^\alpha_*(\lambda^{(j)}f)(\mathrm{e}^{it}) \right)^q \mathrm{d}r \right\}^{1/q} \right\|_{L_p} \leqq$$

$$\leqq C \left\| \left\{ \sum_{j\geqq 0} 2^{-\beta q j} (\sigma^\alpha_*(\lambda^{(j)}f)(\mathrm{e}^{it}))^q \int_0^1 (1-r)^{-q(\beta+s)-1}\, r^{2^j-2q}\, \mathrm{d}r \right\}^{1/q} \right\|_{L_p} \leqq$$

$$\leqq C \|\{2^{js} \sigma^\alpha_*(\lambda^{(j)}f)(\mathrm{e}^{it})\}\|_{L_p(l_q)} \leqq C\|f\|_{F^{s,\Lambda}_{p,q,+}}\,.$$

(3.6) and (3.8) show the assertions of theorem 3.1 (see theorem 2.2).

Remark 3.1. Instead of $J^\beta f$ it is sometimes more useful to deal with the usual derivatives of f. For example, in the same way as above we can show that for $k \geqq 0$ and $s < k$ the quantities

$$(3.12)\qquad \|f\|_{B^{s,k}_{p,q,+}} = \sum_{l=0}^{k-1} |f^{(l)}(0)| + \left\{ \int_0^1 (1-r)^{q(k-s)-1} M_p(f^{(k)}, r)^q \,\mathrm{d}r \right\}^{1/q}$$

and

$$(3.13)\qquad \|f\|_{F^{s,k}_{p,q,+}} = \sum_{l=0}^{k-1} |f^{(l)}(0)| + \left\| \left(\int_0^1 (1-r)^{q(k-s)-1} |f^{(k)}(r\mathrm{e}^{it})|^q \,\mathrm{d}r \right)^{1/q} \right\|_{L_p}$$

($0 < q < \infty$, modification if $q = \infty$) are equivalent quasi norms in $B^s_{p,q,+}$ and $F^s_{p,q,+}$, respectively.

The most interesting case in (3.13) is $q = 2$, $s = 0$, $k = 1$. Taking into consideration (3.4) we obtain for $0 < p < \infty$

$$(3.14)\qquad C\|f\|_{F^0_{p,2,+}} \leqq \|g^*\|_{L_p} + |f(0)| \leqq C'\|f\|_{F^0_{p,2,+}}\,, \quad f \in F^0_{p,2,+}\,.$$

From (3.14) and known results concerning the Littlewood-Paley function $g^*(\mathrm{e}^{it})$ (see [25, v. 2, ch. 14], the inequality $\|f\|_{H_p} \leqq C(\|g^*\|_{L_p} + |f(0)|)$, $0 < p \leqq 1$, seems to be established only in the case of analytic functions $f(z)$ without zeros in the unit disc (cf. [4]) but can be obtained by the methods developed in [3] for Hardy spaces on R^n) it follows that

$$(3.15)\qquad F^0_{p,2,+} = H_p\,, \quad 0 < p < \infty\,,$$

with equivalent quasi norms. (3.15) is an extension of classical Littlewood-Paley results on equivalent norms in L_p and H_p, $1 < p < \infty$ (cf. [25, v. 2, ch. 15]) to the case $0 < p \leqq 1$.

Analogously, we have (see also theorem 2.4)

$$(3.16)\qquad F^s_{p,2,+} = H^s_p = \{f \in D'_+ : \|f\|_{H_p{}^s} = \|J^{-s}f\|_{H_p} < \infty\}$$

for the Hardy-Sobolev spaces ($0 < p < \infty$, $-\infty < s < \infty$).

4. EQUIVALENT QUASI NORMS (DIFFERENCES, APPROXIMATION, STRONG SUMMABILITY)

Let $f \in H_p$, $0 < p \leqq \infty$, and denote by $f(e^{it})$ the corresponding boundary value distribution. The moduli of smoothness of order $m \geqq 1$ of f are defined by

$$(4.1) \qquad \omega_m(f, t)_p = \sup_{h \leqq t} \|\Delta_h^m f(e^{ix})\|_{L_p}, \quad 0 < t \leqq \pi,$$

where $\Delta_h^m f(e^{ix})$ represents the usual m-th difference of $f(e^{it})$ at $t = x$ with step h. We introduce the best approximations

$$(4.2) \qquad E_n(f)_p = \inf_{P_n} \|f - P_n\|_{H_p}, \quad P_n(z) = \sum_{k=0}^{n} a_k z^k, \quad n \geqq 0,$$

by polynomials of $f \in H_p$, $0 < p \leqq \infty$.

The spaces $\boldsymbol{B}^{s,m}_{p,q,+}$ and $\mathscr{B}^s_{p,q,+}$ are defined as subspaces of H_p, $0 < p \leqq \infty$, for which the quasi norms

$$(4.3) \qquad \|f\|_{\boldsymbol{B}^{s,m}_{p,q,+}} = \|f\|_{H_p} + \left\{\int_0^\pi \left(\frac{\omega_m(f, t)_p}{t^s}\right)^q t^{-1}\, dt\right\}^{1/q}, \quad 0 < s < m,$$

and

$$(4.4) \qquad \|f\|_{\mathscr{B}^s_{p,q,+}} = \|f\|_{H_p} + \{\sum_{k \geqq 0} 2^{qsk} E_{2^k}(f)^q\}^{1/q}, \quad 0 < s < \infty,$$

$(0 < q < \infty$, modification if $q = \infty)$ are finite.

Theorem 4.1. *Let $0 < p, q \leqq \infty$, $m \geqq 1$. Then the representations*

$$(4.5) \qquad B^s_{p,q,+} = \boldsymbol{B}^{s,m}_{p,q,+}, \quad 0 < s < m, \quad B^s_{p,q,+} = \mathscr{B}^s_{p,q,+}, \quad 0 < s < \infty,$$

hold with equivalent quasi norms.

Proof. We are concentrated upon the case $0 < p < 1$ (if $1 \leqq p \leqq \infty$ one can work in analogy, see also [8]). The relation $\boldsymbol{B}^{s,m}_{p,q,+} = \mathscr{B}^s_{p,q,+}$, $0 < s < m$, with equivalent quasi norms, follows by standard methcds from the recently obtained Jackson type inequality for H_p due to E. A. Storoženko [16, 17] and known inverse inequalities for trigonometrical approximation in L_p, $0 < p < 1$ (see [7, 9, 18]). Thus, we only have to show the second equality in (4.5). Let P_n^* be the polynomials of best approximation in H_p, i.e.

$$\|f - P_n^*\|_{H_p} = \|f(e^{it}) - P_n^*(e^{it})\|_{L_p} = E_n(f)_p, \quad n \geqq 0,$$

for given $f \in \mathscr{B}^s_{p,q,+}(\subset H_p)$, $s > 0$. According to (1.11) and theorem 1.2 we obtain for $\Lambda \in A_{\alpha,0}$, $\alpha > \alpha_0$, that

$$\|f\|_{B^s_{p,q,+}} \leqq \|\{2^{js}\lambda^{(j)}(f - P^*_{2^{j-1}})\}\|_{l_q(L_p)} \leqq C\|\Lambda\|_\alpha \|\{2^{js}\sigma_*^\alpha(f - P^*_{2^{j-1}})\}\|_{l_q(L_p)} \leqq$$
$$\leqq C\|\Lambda\|_\alpha \|\{2^{js}(f - P^*_{2^{j-1}})\}\|_{l_q(L_p)} \leqq C\|f\|_{\mathscr{B}^s_{p,q,+}}$$

where $P^*_{2^{j-1}} = 0$ for $j = 0$.

On the other hand, for $0 < q < \infty$ we have

$$\|f\|_{\mathscr{B}^s_{p,q,+}} \leqq \|\sum_{j\geqq 0} \lambda^{(j)} f\|_{L_p} + \{\sum_{k\geqq 0} 2^{kqs} \|\sum_{j\geqq k} \lambda^{(j)} f\|^q_{L_p}\}^{1/q} \leqq$$

$$\leqq C\{\sum_{k\geqq 0} 2^{kqs} (\sum_{j\geqq k} \|\lambda^{(j)} f\|^p_{L_p})^{q/p}\}^{1/q} \leqq$$

$$\leqq C \begin{cases} (\sum_{k\geqq 0} 2^{kqs} \sum_{j\geqq k} \|\lambda^{(j)} f\|^q_{L_p})^{1/q}, & q \leqq p \\ (\sum_{k\geqq 0} 2^{kqs} (\sum_{j\geqq k} 2^{-j\gamma})^{q/p-1} (\sum_{j\geqq k} 2^{j\gamma(q/p-1)} \|\lambda^{(j)} f\|^q_{L_p}))^{1/q}, & q > p, \end{cases}$$

where $\gamma > 0$ is an appropriate real satisfying $qs - \gamma(q/p - 1) > 0$. After changing the order of summation we obtain the required estimate $\leqq C\|f\|_{B^{s,\Lambda}_{p,q,+}}$. The proof is analogous if $q = \infty$. Thus, theorem 4.1 is completely proved.

In the last part of this paper we deal with some properties of strong summability of power series. Recently, the following result, among others, has been established by H.-J. Schmeisser, W. Sickel [12]: Let $S_n f(e^{it}) = \sigma^0_n(f)(e^{it})$, $n \geqq 0$, be the partial sums of the power series related to $f \in D'_+$, then the quantity

$$\|\{(j+1)^{s-1/q} (f(e^{it}) - S_j f(e^{it}))\}\|_{L_p(l_q)} + \|f\|_{L_p} \tag{4.6}$$

yields an equivalent quasi norm in $F^s_{p,q,+}$ if $1 < p < \infty$, $0 < q \leqq \infty$, and $s > 0$. An analogous statement holds for $B^s_{p,q,+}$, as well.

Naturally, there arises the question whether this result can be extended to the case $0 < p \leqq 1$. Using summation methods of Vallee Poussin type we give an affirmative answer to this problem.

First we consider a special case. Let

$$V_n f(e^{it}) = n^{-1} \sum_{k=n}^{2n-1} S_k f(e^{it}) = \sum_{m=0}^{2n-1} \eta^n_m c_m e^{imt}, \quad n \geqq 1, \tag{4.7}$$

where

$$\eta^n_m = \begin{cases} 1, & m = 0, \dots, n-1 \\ 2 - m/n, & m = n, \dots, 2n \\ 0, & m = 2n+1, \dots \end{cases}$$

is the classical Vallee Poussin means of a function $f \in D'_+$ for the sequences $\eta^n = \{\eta^n_m\}$, $m \geqq 0$, we have

$$\|\eta^n\|_{bv_2} = n|\Delta^2 \eta^n_{n-1}| + 2n|\Delta^2 \eta^n_{2n-1}| = 3, \quad n \geqq 1. \tag{4.8}$$

Thus, defining $\Lambda = \{\lambda^{(j)}\}$ by $\lambda^{(0)} = \eta^1$ and $\lambda^{(j)} = \eta^{2^j} - \eta^{2^{j-1}}$ for $j \geqq 1$ we obtain $\Lambda \in A_{1,0}$. Theorem 2.2 shows that

$$\|\{2^{js}(V_{2^j} f - V_{2^{j-1}} f)\}\|_{L_p(l_q)}, \quad f \in D'_+ \tag{4.9}$$

(for $j = 0$ we should set $V_1 f$ in this expression) is an equivalent quasinorm in $F^s_{p,q,+}$ if $1/2 < p < \infty$, $1/2 < q \leqq \infty$, and $-\infty < s < \infty$.

In the following we restrict ourselves to $s > 0$ and $f \in H_p$ (it should be noted that $F^s_{p,q,+} \subsetneq H_p$ for $s > 0$). Then a standard consideration yields that

$$\|\{2^{js}(f(e^{it}) - V_{2^{j-1}} f(e^{it}))\}\|_{L_p(l_q)}, \quad f \in H_p, \tag{4.10}$$

is an equivalent quasinorm to that in (4.9) (and, therefore, to the $F^s_{p,q,+}$ quasinorm) for the above described ranges of parameters (for $j = 0$ we set $f(e^{it})$ in the expression (4.10)).

In order to obtain from (4.10) the desired quasinorms we can use the estimate

$$|f(e^{it}) - V_n f(e^{it})| = |(1 - \eta^n)(f - V_{2^{j-1}} f)(e^{it})| \leqq 3\sigma^1_*(f - V_{2^{j-1}} f)(e^{it})$$

where $n = 2^j, \ldots, 2^{j+1} - 1$, $j \geqq 0$ (modification if $j = 0$). This follows from (1.11) and (4.8). Now Theorem 1.2 gives

$$\|\{(j + 1)^{s - 1/q}(f(e^{it}) - V_j f(e^{it}))\}\|_{L_p(l_q)} \leqq$$
$$\leqq C\|\{2^{js}\sigma^1_*(f - V_{2^{j-1}} f)\}\|_{L_p(l_q)} \leqq C\|\{2^{js}(f - V_{2^{j-1}} f)\}\|_{L_p(l_q)}$$

for the parameters as above. The inverse inequality can similarly be shown by using

$$|f(e^{it}) - V_{2^j} f(e^{it})| = |(1 - \eta^{2^j})(f - V_n f)(e^{it})| \leqq 3\sigma^1_*(f - V_n f)(e^{it}),$$
$$n = 2^{j-2} + 1, \ldots, 2^{j-1}, \quad j \geqq 2.$$

An analogous consideration holds for the spaces $B^s_{p,q,+}$. Thus, we have established the following

Theorem 4.2. *a) If* $1/2 < p \leqq \infty$, $0 < q \leqq \infty$, *and* $s > 0$, *then*

$$\|f\|_{L_p} + \{\sum_{n \geqq 1} n^{sq-1}\|f - V_n f\|^q_{L_p}\}^{1/q} \tag{4.11}$$

is an equivalent quasinorm in $B^s_{p,q,+}$.

b) If $1/2 < p < \infty$, $1/2 < q \leqq \infty$, $s > 0$, *then*

$$\|f\|_{L_p} + \|\{\sum_{n \geqq 1} n^{sq-1}|f(e^{it}) - V_n f(e^{it})|^q\}^{1/q}\|_{L_p} \tag{4.12}$$

is an equivalent quasinorm in $F^s_{p,q,+}$.

In the above proof of theorem 4.2 we only used the special properties (4.8) and $\eta^n_m = 1$, $\eta^n_m = 0$ for $m \leqq n$, $m \geqq 2n$, respectively, of the Vallee Poussin means. Thus, the following statement is obvious.

Theorem 4.3. *For a function* $\varphi(x) \in C^\infty(0, \infty)$ *with the properties* $\varphi(x) = 1$, $x \in [0, 1]$, *and* supp $\varphi(x) \subset [0, 2]$ *we introduce the means*

$$V^\varphi_n f(e^{it}) = \sum_{n \geqq 0} \varphi(m/n)\, c_m e^{imt}, \quad n \geqq 1, \quad f \in D'_+, \tag{4.13}$$

of Vallee Poussin type. If $s > 0$ then the quantities

$$\|f\|_{L_p} + \{\sum_{n\geqq 1} n^{sq-1}\|f - V_n^{\varphi}f\|_{L_p}^q\}^{1/q} \tag{4.14}$$

$$\|f\|_{L_p} + \|\{\sum_{n\geqq 1} n^{sq-1}|f(e^{it}) - V_n^{\varphi}f(e^{it})|^q\}^{1/q}\|_{L_p} \tag{4.15}$$

yield equivalent quasi norms in $B^s_{p,q,+}(0 < p,\ q \leqq \infty)$ and $F^s_{p,q,+}(0 < p < \infty,$ $0 < q \leqq \infty)$, respectively.

Remark 4.1. It should be mentioned that the means (4.13) have excellent approximation properties. E.g., if $f \in H_p = F^0_{p,2,+}$, $0 < p < \infty$, then

$$E_{2n-1}(f)_p \leqq \|f - V_n^{\varphi}f\|_{H_p} \leqq C\, E_n(f)_p\,, \quad n \geqq 1\,. \tag{4.16}$$

This immediately follows from the above considerations and (3.15). It would be of some interest to investigate in more detail approximation and summability properties in the spaces $B^s_{p,q,+}$, $F^s_{p,q,+}$, especially, for $0 < p \leqq 1$. In this connection we refer to [4], [16], [20].

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Author's address: Sektion Mathematik, Technische Universität Dresden 8027 Dresden, Mommsenstr. 13, DDR.