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ON CERTAIN ASYMPTOTIC PROPERTIES OF THE SOLUTIONS OF THE EQUATION $\dot{z} = f(t, z)$ WITH A COMPLEX-VALUED FUNCTION f

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1. INTRODUCTION

The purpose of this paper is to present certain results concerning the asymptotic properties of the solutions of an equation

$$(1.1) \qquad \dot{z} = f(t, z)\,, \quad \dot{} = \frac{\mathrm{d}}{\mathrm{d}t}\,,$$

where f is a continuous complex-valued function of a real variable t and a complex variable z. Some results dealing with the asymptotic behaviour of the solutions of (1.1) are established in [1], [2]. The principial tool used in these papers is the technique of Liapunov-like functions.

In the present paper, we give conditions under which a solution $z(t)$ of (1.1) satisfies

$$\int_{t_1}^{\infty} D(t)\,|z(t)|^{\alpha}\,\mathrm{d}t < \infty \quad \left(\text{in particular} \quad \int_{t_1}^{\infty} |z(t)|^{\alpha}\,\mathrm{d}t < \infty\right),$$

where $D(t)$ is a continuous nonnegative function. It is convenient to write the equation (1.1) in the form

$$(1.2) \qquad \dot{z} = G(t, z)\,[h(z) + g(t, z)]\,,$$

where G is a real-valued function and g, h are complex-valued functions. We shall assume that the function h is holomorphic and that the right-hand side of (1.2) is in a suitable sense "close" to this function.

The paper consists of four sections. In Section 2 we recall the definition of the Liapunov-like function $W(z)$ and of the sets $\hat{K}(\lambda)$, $K(\lambda)$, $K(\lambda_1, \lambda_2)$ which were useful in [1], [2]. For our further purposes, we also quote some theorems from [1] concerning the asymptotic behaviour of the solutions of (1.2). The fundamental results are stated in Section 3. The fourth section is devoted to the equation

$$\dot{z} = q(t, z) - p(t)\,z^2\,.$$

Applying the results of Section 3 to this equation we generalize some results of [3] and [4].

2. NOTATION AND PRELIMINARIES

Throughout the paper we use the following notation:

$\mathbb{C}$	Set of all complex numbers
$\mathbb{N}$	Set of all positive integers
$\operatorname{Re} b$	Real part of a complex number b
$\operatorname{Im} b$	Imaginary part of a complex number b
$\bar{b}$	Conjugate of b
$\lvert b\rvert$	Absolute value of b
$\operatorname{Bd}\Gamma$	Boundary of a set $\Gamma \subset \mathbb{C}$
$\operatorname{Cl}\Gamma$	Closure of a set $\Gamma \subset \mathbb{C}$
$\operatorname{Int}\Gamma$	Interior of a Jordan curve $z = z(t)$, $t \in [\alpha, \beta]$ whose points z form a set Γ; Γ will be called the *geometric image* of the Jordan curve $z = z(t)$, $t \in \in [\alpha, \beta]$
I	Interval $[t_0, \infty)$
Ω	Simply connected region in $\mathbb{C}$ such that $0 \in \Omega$
$C[\alpha, \infty)$	Class of all continuous real-valued functions defined on the interval $[\alpha, \infty)$
$C(\Gamma)$	Class of all continuous real-valued functions defined on the set Γ
$\tilde{C}(\Gamma)$	Class of all continuous complex-valued functions defined on the set Γ
$\mathscr{H}(\Gamma)$	Class of all complex-valued functions defined and holomorphic in the region Γ.

Suppose that $h(z) \in \mathscr{H}(\Omega)$ is a function such that $h'(0) \neq 0$ and $h(z) = 0 \Leftrightarrow z = 0$. Following [1] we define

$$r(z) = \begin{cases} \dfrac{z\,h'(0) - h(z)}{z\,h(z)} & \text{for} \quad z \in \Omega\,, \quad z \neq 0\,, \\[2ex] -\dfrac{h''(0)}{2\,h'(0)} & \text{for} \quad z = 0\,, \end{cases}$$

$$w(z) = z \exp\left[\int_0^z r(z^*)\,\mathrm{d}z^*\right]$$

and

$$W(z) = |w(z)|\,.$$

All of these functions are well-defined on Ω. Let Ξ be the system of all simply connected regions $\Gamma \subset \Omega$ with the property $0 \in \Gamma$. For any $\Gamma \in \Xi$ put

$$\lambda_0^\Gamma = \lim_{M\to\infty}\ \inf_{z\in\Gamma_M} W(z)\,,$$

where

$$\Gamma_M = \{z \in \Gamma : \inf_{z^* \in \mathrm{Bd}\,\Gamma} |z - z^*| < M^{-1}\} \cup \{z \in \Gamma : |z| > M\} .$$

Denote

$$\lambda_0 = \sup_{\Gamma \in \Xi} \lambda_0^\Gamma .$$

Obviously $0 < \lambda_0 \leqq \infty$.

For $0 < \lambda < \lambda_0$ define sets $\hat{K}(\lambda) \subset \Omega$ in the following way: choose $\Gamma \in \Xi$ so that $\lambda_0^\Gamma > \lambda$ and put

$$\hat{K}(\lambda) = \{z \in \Gamma : W(z) = \lambda\} .$$

According to [1], this definition is correct, and, denoting

$$\hat{K}(0) = \{0\} ,$$

$$K(\lambda) = \bigcup_{0 \leqq \mu < \lambda} \hat{K}(\mu) \quad \text{for} \quad 0 < \lambda \leqq \lambda_0 ,$$

$$K(\lambda_1, \lambda_2) = \bigcup_{\lambda_1 < \mu < \lambda_2} \hat{K}(\mu) \quad \text{for} \quad 0 \leqq \lambda_1 < \lambda_2 \leqq \lambda_0 ,$$

we have the following statement:

Theorem 2.1. *$K = K(\lambda_0)$ is a simply connected region and $\lambda_0^K = \lambda_0$. Every set $\hat{K}(\lambda)$, where $0 < \lambda < \lambda_0$, is the geometric image of a certain Jordan curve, and,*

$$\hat{K}(\lambda) = \{z \in K(\lambda_0) : W(z) = \lambda\} ,$$

$$\operatorname{Int} \hat{K}(\lambda) = \{z \in K(\lambda_0) : W(z) < \lambda\} .$$

Moreover,

$$K(\lambda) = \operatorname{Int} \hat{K}(\lambda) \quad \textit{for} \quad 0 < \lambda < \lambda_0 ,$$

$$K(\lambda_1, \lambda_2) = K(\lambda_2) - \operatorname{Cl} K(\lambda_1) \quad \text{for} \quad 0 < \lambda_1 < \lambda_2 \leqq \lambda_0 ,$$

and

$$K(0, \lambda) = K(\lambda) - \{0\} \quad \textit{for} \quad 0 < \lambda \leqq \lambda_0 .$$

Now, for our further purposes, we recall Theorems 2.2, 2.3 and 2.5 of [1]. Assume that $G \in C(I \times (\Omega - \{0\}))$, $g \in \tilde{C}(I \times (\Omega - \{0\}))$, $G(t, z)[h(z) + g(t, z)] \in \tilde{C}(I \times \Omega)$ and consider the equation

$$\dot{z} = G(t, z)[h(z) + g(t, z)] . \tag{2.1}$$

Theorem 2.2. *Let $\delta \geqq 0$, $\vartheta \leqq \lambda_0$. Suppose there is an $E(t) \in C[t_0, \infty)$ such that the conditions*

$$\sup_{t_0 \leqq s \leqq t < \infty} \int_s^t E(\xi)\,\mathrm{d}\xi = \varkappa < \infty ,$$

$$\delta \mathrm{e}^\varkappa < \vartheta$$

are fulfilled and

$$-G(t, z)\ \mathrm{Re}\left\{h'(0)\left[1 + \frac{g(t, z)}{h(z)}\right]\right\} \leqq E(t)$$

holds for $t \geqq t_0$, $z \in K(\delta, \vartheta)$.

If a solution $z(t)$ *of* (2.1) *satisfies*

$$z(t_1) \in \hat{K}(\gamma),$$

where $t_1 \geqq t_0$ *and* $\delta e^{\varkappa} < \gamma < \vartheta$, *then*

$$z(t) \notin K(\gamma e^{-\varkappa})$$

for all $t \geqq t_1$ *for which* $z(t)$ *is defined.*

Theorem 2.3. *Suppose* $\delta_n \geqq 0$, $\vartheta \leqq \lambda_0$, $s_n \in I$ *for* $n \in \mathbb{N}$ *and* $\vartheta < \infty$. *Assume that there are functions* $E_n(t) \in C[t_0, \infty)$ *such that:*

(i) *for* $n \in \mathbb{N}$ *the following conditions are fulfilled:*

$$\int_{t_0}^{\infty} E_n(s)\, \mathrm{d}s = -\infty,$$

$$\sup_{s_n \leqq s \leqq t < \infty} \int_s^t E_n(\xi)\, \mathrm{d}\xi = \varkappa_n < \infty,$$

$$\delta_n \mathrm{e}^{\varkappa_n} < \vartheta;$$

(ii) *for* $t \geqq s_n$, $z \in K(\delta_n, \vartheta)$, $n \in \mathbb{N}$ *the following inequality holds*

$$G(t, z)\ \mathrm{Re}\left\{h'(0)\left[1 + \frac{g(t, z)}{h(z)}\right]\right\} \leqq E_n(t).$$

Denote

$$\delta = \inf_{n \in \mathbb{N}} [\delta_n \mathrm{e}^{\varkappa_n}].$$

If a solution $z(t)$ *of* (2.1) *satisfies*

$$z(t_1) \in K(\vartheta \mathrm{e}^{-\varkappa_1}),$$

where $t_1 \geqq s_1$, *then for any* ε, $\delta < \varepsilon < \lambda_0$, *there is a* $T = T(\varepsilon, t_1) > 0$ *independent of* $z(t)$ *such that*

$$z(t) \in K(\varepsilon)$$

for $t \geqq t_1 + T$.

Theorem 2.4. *Let* $\delta > 0$, $\vartheta_n \leqq \lambda_0$, $s_n \in I$ *for* $n \in \mathbb{N}$. *Suppose there are functions* $E_n(t) \in C[t_0, \infty)$ *such that:*

(i) *for* $n \in \mathbb{N}$ *the following conditions are fulfilled*:

$$\int_{t_0}^{\infty} E_n(s)\, \mathrm{d}s = -\infty ,$$

$$\sup_{s_n \leqq s \leqq t < \infty} \int_s^t E_n(\xi)\, \mathrm{d}\xi = \varkappa_n < \infty ,$$

$$\delta \mathrm{e}^{\varkappa_n} < \vartheta_n ;$$

(ii) *for* $t \geqq s_n$, $z \in K(\delta, \vartheta_n)$, $n \in \mathbb{N}$ *the following inequality holds*

$$-G(t, z) \operatorname{Re} \left\{ h'(0) \left[1 + \frac{g(t, z)}{h(z)} \right] \right\} \leqq E_n(t) .$$

Denote

$$\vartheta = \sup_{n \in \mathbb{N}} \left[\vartheta_n \mathrm{e}^{-\varkappa_n} \right] .$$

If a solution $z(t)$ *of* (2.1) *satisfies*

$$z(t_1) \in K(\delta \mathrm{e}^{\varkappa_1}, \lambda_0) ,$$

where $t_1 \geqq s_1$, *then for any* ε, $0 < \varepsilon < \vartheta$, *there exists a* $T = T(\varepsilon, t_1) > 0$ *independent of* $z(t)$ *such that*

$$z(t) \notin \operatorname{Cl} K(\varepsilon)$$

for all $t \geqq t_1 + T$ *for which* $z(t)$ *is defined.*

3. MAIN RESULTS

Consider the equation

$$\dot{z} = G(t, z) \left[h(z) + g(t, z) \right] , \tag{3.1}$$

where $G \in C(I \times \Omega)$, $g \in \tilde{C}(I \times \Omega)$, $h \in \mathscr{H}(\Omega)$. Assume that $h'(0) \neq 0$ and $h(z) = 0 \Leftrightarrow z = 0$. Let $W(z)$, λ_0, $\hat{K}(\lambda)$, $K(\lambda)$, $K(\lambda_1 . \lambda_2)$ be defined as before.

Note. *Suppose* $E(t) \in C[t_0, \infty)$, $0 < \gamma_n < \lambda_0$,

$$\inf_{n \in \mathbb{N}} \gamma_n = 0 .$$

If

$$G(t, z) \operatorname{Re} \left\{ h'(0) \left[1 + \frac{g(t, z)}{h(z)} \right] \right\} \leqq E(t) \tag{3.2}$$

or

$$-G(t, z) \operatorname{Re} \left\{ h'(0) \left[1 + \frac{g(t, z)}{h(z)} \right] \right\} \leqq E(t) \tag{3.3}$$

holds for $t \geqq t_0$, $z \in \hat{K}(\gamma_n)$, $n \in \mathbb{N}$, *then* $G(t, 0)\, g(t, 0) = 0$ *for* $t \geqq t_0$.

Proof. Notice that $h(z) = h'(0)\,[z + q(z)]$, where $q(z) = o(|z|)$ as $z \to 0$. Now,

$$G(t, z) \operatorname{Re}\left\{h'(0)\left[1 + \frac{g(t, z)}{h(z)}\right]\right\} =$$

$$= G(t, z) \operatorname{Re} h'(0) + G(t, z) \operatorname{Re}\left\{g(t, z)\, \frac{\bar{z} + \overline{q(z)}}{|z|^2 + 2 \operatorname{Re}[\bar{z}\, q(z)] + |q(z)|^2}\right\} =$$

$$= G(t, z) \operatorname{Re} h'(0) + G(t, z)\, \frac{X\varphi + Y\psi + \varphi \operatorname{Re} q(z) + \psi \operatorname{Im} q(z)}{|z|^2 + 2 \operatorname{Re}[\bar{z}\, q(z)] + |q(z)|^2},$$

where $X = \operatorname{Re} z$, $Y = \operatorname{Im} z$, $\varphi = \varphi(t, X, Y) = \operatorname{Re} g(t, z)$, $\psi = \psi(t, X, Y) = \operatorname{Im} g(t, z)$. Using (3.2) and (3.3), we get

$$\varepsilon G(t, X + \mathrm{i}Y)\,[X\varphi + Y\psi + \varphi \operatorname{Re} q(z) + \psi \operatorname{Im} q(z)] \leqq$$
$$\leqq [E(t) - \varepsilon G(t, z) \operatorname{Re} h'(0)]\,\{|z|^2 + 2 \operatorname{Re}[\bar{z}\, q(z)] + |q(z)|^2\}$$

for $t \geqq t_0$, $z = X + \mathrm{i}Y \in \hat{K}(\gamma_n)$, $n \in \mathbb{N}$, where $\varepsilon = 1$ or $\varepsilon = -1$. Hence

$$\varepsilon G(t, X + \mathrm{i}Y)\left[X(X^2 + Y^2)^{-1/2}\, \varphi + Y(X^2 + Y^2)^{-1/2}\, \psi + \varphi\, \frac{\operatorname{Re} q(z)}{|z|} + \right.$$

$$\left. + \psi\, \frac{\operatorname{Im} q(z)}{|z|}\right] \leqq [E(t) - \varepsilon G(t, z) \operatorname{Re} h'(0)]\left\{|z| + \frac{2 \operatorname{Re}[\bar{z}\, q(z)]}{|z|} + \frac{|q(z)|^2}{|z|}\right\}.$$

Putting $Y = 0$ and letting $X \to 0\pm$, we observe that $G(t, 0)\, \varphi(t, 0, 0) = 0$. Similarly $G(t, 0)\, \psi(t, 0, 0) = 0$. Therefore $G(t, 0)\, g(t, 0) = 0$.

Theorem 3.1. *Assume that* $0 < \vartheta < \lambda_0$, $\alpha > 0$. *Suppose there is a function* $E(t) \in C[t_0, \infty)$ *such that*

$$\int_{t_0}^{\infty} \exp\left[\alpha \int_{t_0}^{s} E(\xi)\, \mathrm{d}\xi\right] \mathrm{d}s < \infty\,, \tag{3.4}$$

and that

$$G(t, z) \operatorname{Re}\left\{h'(0)\left[1 + \frac{g(t, z)}{h(z)}\right]\right\} \leqq E(t) \tag{3.5}$$

holds for $t \geqq t_0$, $z \in K(0, \vartheta)$. *For* $\alpha \in (0, 1)$ *suppose in addition that any initial value problem* (3.1), $z(\tau) = 0$, *where* $\tau \geqq t_0$, *possesses the unique solution* ($z(t) \equiv 0$).

If a solution $z(t)$ *of* (3.1) *satisfies*

$$z(t) \in K(\vartheta) \quad \textit{for} \quad t \geqq t_1\,, \tag{3.6}$$

where $t_1 \geqq t_0$, *then*

$$\int_{t_1}^{\infty} |z(t)|^\alpha \, dt < \infty .$$

Proof. Let $z(t)$ be any solution of (3.1) satisfying (3.6). If $\alpha \in (0, 1)$ we may assume that $z(t) \neq 0$ for $t \geqq t_1$. For $t \geqq t_1$ we have

$$\frac{d}{dt} W^2(z) = \frac{d}{dt} [w(z) \overline{w(z)}] = 2 \operatorname{Re} [w'(z) \overline{w(z)} \dot{z}] =$$

$$= 2 \operatorname{Re} \{w(z) \overline{w(z)} [z^{-1} + r(z)] \dot{z}\} = 2 W^2(z) \operatorname{Re} [h'(0) h^{-1}(z) \dot{z}] ,$$

where $z = z(t)$. Therefore

$$\dot{W}(z) = W(z) \operatorname{Re} [h'(0) h^{-1}(z) \dot{z}] =$$

$$= G(t, z) W(z) \operatorname{Re} \{h'(0) h^{-1}(z) [h(z) + g(t, z)]\} =$$

$$= G(t, z) W(z) \operatorname{Re} \left\{h'(0) \left[1 + \frac{g(t, z)}{h(z)}\right]\right\}$$

for $t \geqq t_1$. This together with (3.5) yields

$$\frac{d}{dt} W^\alpha(z(t)) = \alpha W^{\alpha-1}(z(t)) \dot{W}(z(t)) \leqq \alpha E(t) W^\alpha(z(t))$$

for $t \geqq t_1$. Hence

$$\frac{d}{dt} \left\{W^\alpha(z(t)) \exp\left[-\alpha \int_{t_1}^{t} E(\xi) \, d\xi\right]\right\} \leqq 0 , \quad t \geqq t_1 .$$

Integrating this inequality from t_1 to t, we obtain

$$W^\alpha(z(t)) \exp\left[-\alpha \int_{t_1}^{t} E(\xi) \, d\xi\right] - W^\alpha(z(t_1)) \leqq 0 .$$

Thus

$$W^\alpha(z(t)) \leqq W^\alpha(z(t_1)) \exp\left[\alpha \int_{t_1}^{t} E(s) \, ds\right], \quad t \geqq t_1 .$$

Integration over $[t_1, t]$ gives

$$\int_{t_1}^{t} W^\alpha(z(s)) \, ds \leqq W^\alpha(z(t_1)) \int_{t_1}^{t} \exp\left[\alpha \int_{t_1}^{s} E(\xi) \, d\xi\right] ds , \quad t \geqq t_1 .$$

Consequently,

$$\int_{t_1}^{\infty} W^\alpha(z(t)) \, dt \leqq W^\alpha(z(t_1)) \exp\left[-\alpha \int_{t_0}^{t_1} E(\xi) \, d\xi\right] \int_{t_0}^{\infty} \exp\left[\alpha \int_{t_0}^{s} E(\xi) \, d\xi\right] ds .$$

This inequality together with (3.4) implies

$$\int_{t_1}^{\infty} W^{\alpha}(z(t))\, \mathrm{d}t < \infty .$$

Since

$$W(z) = \left| z \exp \left[\int_0^z r(z^*)\, \mathrm{d}z^* \right] \right| ,$$

and $\mathrm{Cl}\, K(\vartheta) \subset K(\lambda_0)$ is a compact set, there exists a constant $L > 0$ such that

$$W(z) \geqq L|z| \quad \text{for} \quad z \in \mathrm{Cl}\, K(\vartheta) .$$

Accordingly

$$\int_{t_1}^{\infty} |z(t)|^{\alpha}\, \mathrm{d}t \leqq L^{-\alpha} \int_{t_1}^{\infty} W^{\alpha}(z(t))\, \mathrm{d}t < \infty .$$

Theorem 3.2. *Assume that* $0 < \vartheta < \lambda_0$, $\alpha \geqq 1$. *Suppose there are functions* $D(t)$, $E(t) \in C[t_0, \infty)$, $E(t) \geqq 0$, *such that*

$$\int_{t_0}^{\infty} \exp \left[\alpha \int_{t_0}^{s} D(\xi)\, \mathrm{d}\xi \right] \mathrm{d}s < \infty ,$$

$$\int_{t_0}^{\infty} \left\{ \int_{t_0}^{s} E(\xi) \exp \left[\alpha \int_{\xi}^{s} D(\eta)\, \mathrm{d}\eta \right] \mathrm{d}\xi \right\} \mathrm{d}s < \infty ,$$

and that

(3.7) $$G(t, z) \operatorname{Re} h'(0) \leqq D(t) ,$$

(3.8) $$W(z)\, G(t, z) \operatorname{Re} \left[g(t, z) \frac{h'(0)}{h(z)} \right] \leqq E(t)$$

hold for $t \geqq t_0$, $z \in K(0, \vartheta)$.

If a solution $z(t)$ *of* (3.1) *satisfies*

(3.6) $$z(t) \in K(\vartheta) \quad \textit{for} \quad t \geqq t_1 ,$$

where $t_1 \geqq t_0$, *then*

$$\int_{t_1}^{\infty} |z(t)|^{\alpha}\, \mathrm{d}t < \infty .$$

Proof. Let $z(t)$ be any solution of (3.1) satisfying (3.6). Put $\mathcal{M} = \{t \geqq t_1 : z(t) \in K(0, \vartheta)\}$, $\mathcal{M}_0 = \{t \geqq t_1 : z(t) \in K(\vartheta)\} = [t_1, \infty)$. We have

$$\dot{W}(z) = G(t, z)\, W(z) \operatorname{Re} \left\{ h'(0) \left[1 + \frac{g(t, z)}{h(z)} \right] \right\}$$

for $t \in \mathcal{M}$. Let $\tau \geqq t_1$ be such a number that $z(\tau) = 0$. Then

$$\dot{W}_+(z(\tau)) = \lim_{t\to\tau+} \frac{W(z(t))}{t-\tau} = \lim_{t\to\tau+} \frac{|z(t)| \left|\exp\left[\int_0^{z(t)} r(z^*)\,\mathrm{d}z^*\right]\right|}{t-\tau} =$$

$$= \lim_{t\to\tau+} \left\{\left|\frac{z(t)}{t-\tau}\right| \left|\exp\left[\int_0^{z(t)} r(z^*)\,\mathrm{d}z^*\right]\right|\right\} = |\dot{z}(\tau)| =$$

$$= |G(\tau, 0)\, g(\tau, 0)|\,.$$

Similarly

$$\dot{W}_-(z(\tau)) = -|G(\tau, 0)\, g(\tau, 0)|\,.$$

Hence $\dot{W}(z(\tau))$ exists if and only if $G(\tau, 0)\, g(\tau, 0) = 0$. In this case $\dot{W}(z(\tau)) = 0$.

Let $\mathcal{M}_1 = \{t \geqq t_1 : z(t) = 0,\ G(t, 0)\, g(t, 0) = 0\}$. The set $\mathcal{M}_0 - (\mathcal{M} \cup \mathcal{M}_1)$ is at most countable. For $t \in \mathcal{M}$

$$\frac{\mathrm{d}}{\mathrm{d}t} W^\alpha(z) = \alpha\, W^{\alpha-1}(z)\, \dot{W}(z) = \alpha\, G(t, z)\, W^\alpha(z) \operatorname{Re}\left\{h'(0)\left[1 + \frac{g(t, z)}{h(z)}\right]\right\}$$

holds. Notice that $h(z) = z\, q(z)$, where $q \in \mathscr{H}(\Omega)$ and $q(z) \neq 0$ for $z \in \Omega$. Using (3.7) and (3.8), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} W^\alpha(z) \leqq \alpha\, D(t)\, W^\alpha(z) + \alpha\, W^{\alpha-1}(z)\, E(t) \leqq$$

$$\leqq \alpha\, D(t)\, W^\alpha(z) + \alpha\vartheta^{\alpha-1}\, E(t) \quad \text{for} \quad t \in \mathcal{M} \cup \mathcal{M}_1$$

and

$$\left|\frac{\mathrm{d}}{\mathrm{d}t} W^\alpha(z) - \alpha\, D(t)\, W^\alpha(z)\right| \leqq \alpha|G(t, z) \operatorname{Re} h'(0) - D(t)|\, W^\alpha(z) +$$

$$+ \alpha\vartheta^{\alpha-1}|G(t, z)\, g(t, z)\, h'(0)| \left|\exp\left[\int_0^z r(z^*)\,\mathrm{d}z^*\right]\right| |q(z)|^{-1}$$

for $t \in \mathcal{M} \cup \mathcal{M}_1$.

Define

$$B(t) = \begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}\left\{W^\alpha(z(t)) \exp\left[-\alpha \displaystyle\int_{t_1}^t D(s)\,\mathrm{d}s\right]\right\} & \text{whenever} \quad t \in \mathcal{M} \cup \mathcal{M}_1\,, \\ 0 \quad \text{whenever} \quad t \in \mathcal{M}_0 - (\mathcal{M} \cup \mathcal{M}_1)\,. \end{cases}$$

$B(t)$ satisfies the estimates

$$B(t) \leqq \alpha\vartheta^{\alpha-1}\, E(t) \exp\left[-\alpha \int_{t_1}^t D(s)\,\mathrm{d}s\right], \tag{3.9}$$

$$|B(t)| \leqq \alpha\{|G(t, z) \operatorname{Re} h'(0) - D(t)|\, W^\alpha(z) +$$

$$+\ \vartheta^{\alpha-1}|G(t,z)\,g(t,z)\,h'(0)|\,\frac{\left|\exp\left[\int_0^z r(z^*)\,\mathrm{d}z^*\right]\right|}{|q(z)|}\Bigg\}\left|\exp\left[-\alpha\int_{t_1}^t D(s)\,\mathrm{d}s\right]\right|$$

for $t \in \mathcal{M}_0$. Thus $B(t)$ is continuous for $t \in \mathcal{M} \cup \mathcal{M}_1$. Let $\mathcal{M}_2$ be the set of all $t \geqq t_1$ for which $B(t)$ is discontinuous. Since $\mathcal{M}_2 \subset \mathcal{M}_0 - (\mathcal{M} \cup \mathcal{M}_1)$, the set $\mathcal{M}_2$ is at most countable. Moreover, $B(t)$ is bounded on any compact subinterval of $[t_1, \infty)$. Therefore

$$\int_{t_1}^t B(s)\,\mathrm{d}s = W^\alpha(z(t))\exp\left[-\alpha\int_{t_1}^t D(s)\,\mathrm{d}s\right] - W^\alpha(z(t_1))$$

for $t \geqq t_1$.

Integration of (3.9) yields

$$W^\alpha(z(t))\exp\left[-\alpha\int_{t_1}^t D(s)\,\mathrm{d}s\right] - W^\alpha(z(t_1)) \leqq$$

$$\leqq \alpha\vartheta^{\alpha-1}\int_{t_1}^t E(s)\exp\left[-\alpha\int_{t_1}^s D(\xi)\,\mathrm{d}\xi\right]\mathrm{d}s$$

for $t \geqq t_1$. Hence

$$\int_{t_1}^\infty W^\alpha(z(s))\,\mathrm{d}s \leqq W^\alpha(z(t_1))\int_{t_1}^\infty \exp\left[\alpha\int_{t_1}^s D(\xi)\,\mathrm{d}\xi\right]\mathrm{d}s +$$

$$+\ \alpha\vartheta^{\alpha-1}\int_{t_1}^\infty\left\{\int_{t_1}^s E(\xi)\exp\left[\alpha\int_\xi^s D(\eta)\,\mathrm{d}\eta\right]\mathrm{d}\xi\right\}\mathrm{d}s \leqq$$

$$\leqq W^\alpha(z(t_1))\exp\left[-\alpha\int_{t_0}^{t_1} D(\xi)\,\mathrm{d}\xi\right]\int_{t_0}^\infty \exp\left[\alpha\int_{t_0}^s D(\xi)\,\mathrm{d}\xi\right]\mathrm{d}s +$$

$$+\ \alpha\vartheta^{\alpha-1}\int_{t_0}^\infty\left\{\int_{t_0}^s E(\xi)\exp\left[\alpha\int_\xi^s D(\eta)\,\mathrm{d}\eta\right]\mathrm{d}\xi\right\}\mathrm{d}s < \infty\,.$$

The rest of the proof is the same as that of Theorem 3.1.

Theorem 3.3. *Assume that* $0 < \vartheta \leqq \lambda_0$, $\vartheta < \infty$, $\alpha \geqq 1$, $\operatorname{Re} h'(0) \neq 0$. *Suppose there are nonnegative functions* $D(t)$, $E(t) \in C[t_0, \infty)$ *such that*

$$\int_{t_0}^\infty D(t)\,\mathrm{d}t = \infty\,,$$

$$\int_{t_0}^\infty E(t)\,\mathrm{d}t < \infty\,, \tag{3.10}$$

and that

$$G(t, z) \geqq D(t),$$

$$-\operatorname{sgn}[\operatorname{Re} h'(0)]\, W(z)\, G(t, z) \operatorname{Re}\left[g(t, z)\frac{h'(0)}{h(z)}\right] \leqq E(t)$$

hold for $t \geqq t_0$, $z \in K(0, \vartheta)$.

If a solution $z(t)$ *of* (3.1) *satisfies*

$$z(t) \in K(\vartheta) \quad \textit{for} \quad t \geqq t_1, \tag{3.6}$$

where $t_1 \geqq t_0$, *then*

$$\int_{t_1}^{\infty} D(t)\, |z(t)|^{\alpha}\, \mathrm{d}t < \infty$$

and

$$\lim_{t \to \infty} z(t) = 0.$$

Proof. Without loss of generality we may assume that $\alpha = 1$. Proceeding similarly as in the proof of Theorem 3.2 and defining

$$B(t) = \begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t} W(z(t)) & \text{whenever} \quad t \in \mathcal{M} \cup \mathcal{M}_1, \\ 0 & \text{whenever} \quad t \in \mathcal{M}_0 - (\mathcal{M} \cup \mathcal{M}_1), \end{cases}$$

we observe that

$$\int_{t_1}^{t} B(s)\, \mathrm{d}s = W(z(t)) - W(z(t_1)), \quad t \geqq t_1$$

and

$$-\operatorname{sgn}[\operatorname{Re} h'(0)]\, B(t) \leqq -D(t)\, |\operatorname{Re} h'(0)|\, W((z(t)) + E(t)$$

for $t \geqq t_1$. Integrating this inequality over $[t_1, t]$ and letting $t \to \infty$, we infer, in view of (3.10) and $0 \leqq W(z) \leqq \vartheta$, that

$$\int_{t_1}^{\infty} D(t)\, W(z(t))\, \mathrm{d}t < \infty.$$

Therefore

$$\liminf_{t \to \infty} W(z(t)) = \liminf_{t \to \infty} |z(t)| = 0. \tag{3.11}$$

Let $\operatorname{Re} h'(0) < 0$. For $n \in \mathbb{N}$ choose $s_n \geqq t_0$ such that

$$\int_{s_n}^{\infty} E(t)\, \mathrm{d}t < \frac{\vartheta}{2(n+1)} \ln(n+1), \quad n \in \mathbb{N}.$$

By using Theorem 2.3 with $\delta_n = \vartheta/(n+1)$, $E_n(t) = D(t) \operatorname{Re} h'(0) + (n+1)\, E(t)/\vartheta$, we obtain

$$\lim_{t \to \infty} z(t) = 0 . \tag{3.12}$$

We shall prove that (3.12) holds also if Re $h'(0) > 0$. Suppose this is not the case. Then

$$\limsup_{t \to \infty} W(z(t)) = \beta > 0 .$$

For $n \in \mathbb{N}$ define $s_n \geqq t_0$ such that

$$\int_{s_n}^{\infty} E(t)\, \mathrm{d}t < \frac{\beta}{2n} \mathrm{e}^{-1} .$$

Using Theorem 2.4 with $\delta = \beta \mathrm{e}^{-1}/2$, $\vartheta_n = \vartheta$, $E_n(t) = -D(t) \operatorname{Re} h'(0) + 2e\, E(t)/\beta$ we get

$$\liminf_{t \to \infty} W(z(t)) > 0 ,$$

which contradicts (3.11). This proves (3.12).

Now, there exists a positive constant L such that

$$W(z(t)) = |z(t)| \left| \exp\left[\int_0^{z(t)} r(z^*)\, \mathrm{d}z^* \right] \right| \geqq L|z(t)|$$

for $t \geqq t_1$. Therefore

$$\int_{t_1}^{\infty} D(t)\, |z(t)|\, \mathrm{d}t \leqq L^{-1} \int_{t_1}^{\infty} D(t)\, W(z(t))\, \mathrm{d}t < \infty .$$

4. APPLICATION TO THE EQUATION $\dot{z} = q(t, z) - p(t)\, z^2$

In this section we propose establishing certain results concerning the asymptotic behaviour of the equation

$$\dot{z} = q(t, z) - p(t)\, z^2 , \tag{4.1}$$

where $p \in \tilde{C}(I)$, $q \in \tilde{C}(I \times \mathbb{C})$. Some results of this type are given in [1], [2]. The special case of (4.1) is studied in [3], [4], where M. Ráb has obtained results describing the asymptotic properties of the Riccati differential equation

$$\dot{z} = q(t) - p(t)\, z^2$$

with complex-valued coefficients p, q.

If $a, b \in \mathbb{C}$, $\psi(t) \in C[t_0, \infty)$, $\psi(t) > 0$, then (4.1) can be written in the form

$$\dot{z} = \psi(t) \left[(\bar{b} - \bar{a})(z - a)(z - b) + \frac{q(t, z)}{\psi(t)} - \frac{p(t)}{\psi(t)} z^2 + \right. \\ \left. + (\bar{a} - \bar{b})(z - a)(z - b) \right] . \tag{4.2}$$

Suppose $a \neq b$ and denote $c = a - b$. Substituting $z_1 = z - a$ or $z_2 = z - b$, we get

$$(4.2_1) \qquad \dot{z}_1 = G_1(t, z_1)\left[h_1(z_1) + g_1(t, z_1)\right]$$

or

$$(4.2_2) \qquad \dot{z}_2 = G_2(t, z_2)\left[h_2(z_2) + g_2(t, z_2)\right]$$

respectively, where

$$G_1(t, z_1) = \psi(t)\,, \quad h_1(z_1) = -\bar{c}z_1(z_1 + c)\,,$$

$$g_1(t, z_1) = \frac{q(t, z_1 + a)}{\psi(t)} - \frac{p(t)}{\psi(t)}(z_1 + a)^2 + \bar{c}z_1(z_1 + c)\,,$$

$$G_2(t, z_2) = \psi(t)\,, \quad h_2(z_2) = -\bar{c}z_2(z_2 - c)\,,$$

$$g_2(t, z_2) = \frac{q(t, z_2 + b)}{\psi(t)} - \frac{p(t)}{\psi(t)}(z_2 + b)^2 + \bar{c}z_2(z_2 - c)\,.$$

Put

$$\Omega_1 = \{z_1 \in \mathbb{C} : 2\,\mathrm{Re}\,[\bar{c}z_1] > -|c|^2\}\,,$$

$$\Omega_2 = \{z_2 \in \mathbb{C} : 2\,\mathrm{Re}\,[\bar{c}z_2] < |c|^2\}\,.$$

I. First we shall consider the equation (4.2_1) on the set $I \times \Omega_1$. We find out that $W(z_1) = |c|\,|z_1|\,|z_1 + c|^{-1}$, $\lambda_0 = |c|$ and $K(\lambda_0) = \Omega_1$. Moreover, we have

$$\hat{K}(\lambda) = \{z_1 \in \Omega_1 : |c|\,|z_1| = \lambda|z_1 + c|\}$$

for $0 \leqq \lambda < \lambda_0$. Notice that

$$|z_1 + c| > \frac{|c|^2}{|c| + \lambda}$$

for $z_1 \in K(\lambda)$, where $0 < \lambda \leqq \lambda_0$, and

$$|z_1| > \frac{|c|\lambda}{|c| + \lambda}$$

for $z_1 \in K(\lambda, \lambda_0)$, where $0 \leqq \lambda < \lambda_0$.

Suppose that there is an $H(t) \in C[t_0, \infty)$ such that

$$|q(t, z_1 + a) + ab\,p(t) - (a + b)\,p(t)\,(z_1 + a)| \leqq H(t)$$

for $t \geqq t_0$, $z_1 \in \Omega_1$.

1° Assume that

$$(4.3) \qquad \mathrm{Re}\,[c\,p(t)] > 0 \quad \text{for} \quad t \geqq t_0\,,$$

$$(4.4) \qquad \int_{t_0}^{\infty} \mathrm{Re}\,[c\,p(t)]\,\mathrm{d}t = \infty$$

and

$$(4.5) \qquad \int_{t_0}^{\infty} H(t)\,\mathrm{d}t < \infty .$$

Let $s_n \geqq t_0$ be such that

$$\int_{s_n}^{\infty} H(t)\,\mathrm{d}t < \frac{|c|}{4n}\,\mathrm{e}^{-1}, \quad n \in \mathbb{N} .$$

Put $\psi(t) \equiv 1$ and

$$\delta_n = \frac{|c|}{n}\,\mathrm{e}^{-1} \quad \text{for} \quad n \in \mathbb{N} .$$

We have

$$\operatorname{Re}\left\{h_1'(0)\left[1 + \frac{g_1(t, z_1)}{h_1(z_1)}\right]\right\} =$$

$$= \operatorname{Re}\left\{[q(t, z_1 + a) - a^2\,p(t) - (a + b)\,p(t)\,z_1]\,\frac{c}{z_1(z_1 + c)}\right\} +$$

$$+ \operatorname{Re}\left\{[-c\,p(t)\,z_1 - p(t)\,z_1^2]\,\frac{c}{z_1(z_1 + c)}\right\} =$$

$$= \operatorname{Re}\left\{[q(t, z_1 + a) + ab\,p(t) - (a + b)\,p(t)\,(z_1 + a)]\,\frac{c}{z_1(z_1 + c)}\right\} -$$

$$- \operatorname{Re}\,[c\,p(t)] \leqq H(t)\,\frac{|c|}{|z_1|\,|z_1 + c|} - \operatorname{Re}\,[c\,p(t)] \leqq$$

$$\leqq H(t)\,|c|\left[\frac{|c|\,\delta_n}{|c| + \delta_n}\,\tfrac{1}{2}|c|\right]^{-1} - \operatorname{Re}\,[c\,p(t)] \leqq \frac{4}{\delta_n}\,H(t) - \operatorname{Re}\,[c\,p(t)]$$

for $t \geqq s_n$, $z_1 \in K(\delta_n, \lambda_0)$, $n \in \mathbb{N}$.

Using Theorem 2.3 (with $\vartheta = \lambda_0 = |c|$, $G(t, z) \equiv 1$, $E_n(t) = 4\,H(t)/\delta_n - \operatorname{Re}\,[c\,p(t)]$), we get the following assertion:

If a solution $z_1(t)$ of (4.2_1) satisfies the condition

$$|z_1(t_1)| < \exp\left[-\frac{4\mathrm{e}}{|c|}\int_{s_1}^{\infty} H(t)\,\mathrm{d}t\right]|z_1(t_1) + c| ,$$

where $t_1 \geqq s_1$, then

$$\lim_{t\to\infty} z_1(t) = 0 .$$

2° Suppose that (4.3), (4.4) and (4.5) hold. Put

$$\psi(t) = \frac{\operatorname{Re}\,[c\,p(t)]}{|c|^2} .$$

Then

$$W(z_1)\,\psi(t)\,\mathrm{Re}\left[g_1(t, z_1)\,\frac{h_1'(0)}{h_1(z_1)}\right] =$$

$$= W(z_1)\,\mathrm{Re}\left\{[q(t, z_1 + a) + ab\,p(t) - (a + b)\,p(t)\,(z_1 + a)]\,\frac{c}{z_1(z_1 + c)}\right\} \leqq$$

$$\leqq \frac{|c|\,|z_1|}{|z_1 + c|}\,H(t)\,\frac{|c|}{|z_1|\,|z_1 + c|} \leqq \frac{|c|^2}{|z_1 + c|^2}\,H(t) \leqq 4\,H(t)$$

for $t \geqq t_0$, $z_1 \in K(0, \lambda_0)$.

Applying Theorem 3.3 (with $\vartheta = \lambda_0 = |c|$, $D(t) = G(t, z) = \psi(t)$, $E(t) = 4\,H(t)$), we obtain the following statement:

If a solution $z_1(t)$ of (4.2_1) satisfies

$$2\,\mathrm{Re}\,[\bar{c}\,z_1(t)] > -|c|^2 \quad \textit{for} \quad t \geqq t_1\,,$$

where $t_1 \geqq t_0$, then

$$\int_{t_1}^{\infty} \mathrm{Re}\,[c\,p(t)]\,|z_1(t)|\,\mathrm{d}t < \infty$$

and

$$\lim_{t \to \infty} z_1(t) = 0\,.$$

II. Consider the equation (4.2_2) on the set $I \times \Omega_2$. In this case we have $W(z_2) = |c|\,|z_2|\,|z_2 - c|^{-1}$, $\lambda_0 = |c|$ and $K(\lambda_0) = \Omega_2$. Further,

$$\hat{K}(\lambda) = \{z_2 \in \Omega_2 : |c|\,|z_2| = \lambda|z_2 - c|\}$$

for $0 \leqq \lambda < \lambda_0$. Notice that

$$|z_2 - c| > \frac{|c|^2}{|c| + \lambda}$$

for $z_2 \in K(\lambda)$, where $0 < \lambda \leqq \lambda_0$, and,

$$|z_2| > \frac{|c|\,\lambda}{|c| + \lambda}$$

for $z_2 \in K(\lambda, \lambda_0)$, where $0 \leqq \lambda < \lambda_0$.

Suppose there is an $H(t) \in C[t_0, \infty)$ such that

$$|q(t, z_2 + b) + ab\,p(t) - (a + b)\,p(t)\,(z_2 + b)| \leqq H(t)$$

for $t \geqq t_0$, $z_2 \in \Omega_2$.

3° Assume that (4.3), (4.4) and (4.5) hold. Put $\psi(t) \equiv 1$ and choose $\delta \in (0, |c|\,\mathrm{e}^{-1})$. Define $S \geqq t_0$ so that

$$\int_S^{\infty} H(t)\,\mathrm{d}t < \frac{\delta}{4}\,.$$

Then

$$- \operatorname{Re} \left\{ h_2'(0) \left[1 + \frac{g_2(t, z_2)}{h_2(z_2)} \right] \right\} \leqq H(t) \frac{|c|}{|z_2| |z_2 - c|} - \operatorname{Re} [c\, p(t)] \leqq$$

$$\leqq H(t) |c| \left[\frac{|c| \delta}{|c| + \delta} \tfrac{1}{2} |c| \right]^{-1} - \operatorname{Re} [c\, p(t)] \leqq$$

$$\leqq \frac{4}{\delta} H(t) - \operatorname{Re} [c\, p(t)]$$

holds for $t \geqq S$ ad $z_2 \in K(\delta, \lambda_0)$.

Making use of Theorem 2.2 (with $\vartheta = \lambda_0 = |c|$, $E(t) = 4\, H(t)/\delta - \operatorname{Re} [c\, p(t)]$, $G(t, z) \equiv 1$), we get:

If a solution $z_2(t)$ of (4.2_2) satisfies

$$|c| |z_2(t_1)| > \delta e |z_2(t_1) - c| ,$$

where $t_1 \geqq S$, then

$$|c| |z_2(t)| > \delta |z_2(t) - c|$$

for all $t \geqq t_1$ for which $z_2(t)$ is defined.

4° Suppose that (4.3), (4.4) and (4.5) hold. Putting

$$\psi(t) = \frac{\operatorname{Re} [c\, p(t)]}{|c|^2} ,$$

we obtain

$$- W(z_2)\, \psi(t) \operatorname{Re} \left[g_2(t, z_2) \frac{h_2'(0)}{h_2(z_2)} \right] \leqq \frac{|c| |z_2|}{|z_2 - c|} H(t) \frac{|c|}{|z_2| |z_2 - c|} \leqq$$

$$\leqq \frac{|c|^2}{|z_2 - c|^2} H(t) \leqq 4\, H(t)$$

for $t \geqq t_0$, $z_2 \in K(0, \lambda_0)$.

Applying Theorem 3.3 (with $\vartheta = \lambda_0 = |c|$, $D(t) = G(t, z) = \psi(t)$, $E(t) = 4\, H(t)$) we get the following assertion:

If a solution $z_2(t)$ of (4.2_2) satisfies

$$2 \operatorname{Re} [\bar{c}\, z_2(t)] < |c|^2 \quad \textit{for} \quad t \geqq t_1 ,$$

where $t_1 \geqq t_0$, then

$$\int_{t_1}^{\infty} \operatorname{Re} [c\, p(t)] |z_2(t)| \, dt < \infty$$

and

$$\lim_{t \to \infty} z_2(t) = 0 .$$

By virtue of 1°, 2°, 3°, 4° we can prove the following generalization of Theorem 5 of [3] and Theorem 6 of [4]:

Theorem 4.1. *Suppose there exist* $a, b \in \mathbb{C}$ *and* $H(t) \in C[t_0, \infty)$ *such that*

$$|q(t, z) + ab\, p(t) - (a + b)\, p(t)\, z| \leqq H(t) \quad \text{for} \quad t \geqq t_0, \quad z \in \mathbb{C},$$

$$\operatorname{Re}\,[(a - b)\, p(t)] > 0 \quad \text{for} \quad t \geqq t_0,$$

$$\int_{t_0}^{\infty} \operatorname{Re}\,[(a - b)\, p(t)]\, \mathrm{d}t = \infty$$

and

$$\int_{t_0}^{\infty} H(t)\, \mathrm{d}t < \infty. \tag{4.5}$$

Then each solution $z(t)$ *of* (4.1) *defined for* $t \to \infty$ *satisfies either*

$$\lim_{t \to \infty} z(t) = a, \quad \int^{\infty} \operatorname{Re}\,[(a - b)\, p(t)]\, |z(t) - a|\, \mathrm{d}t < \infty \tag{4.6}$$

or

$$\lim_{t \to \infty} z(t) = b, \quad \int^{\infty} \operatorname{Re}\,[(a - b)\, p(t)]\, |z(t) - b|\, \mathrm{d}t < \infty. \tag{4.7}$$

Let $S \geqq t_0$ *be such that*

$$\int_S^{\infty} H(t)\, \mathrm{d}t < (4e)^{-1}\, |a - b|.$$

Then each solution $z(t)$ *of* (4.1) *satisfying*

$$|z(t_1) - a| < \exp\left[-\frac{4e}{|a - b|} \int_S^{\infty} H(t)\, \mathrm{d}t\right] |z(t_1) - b|,$$

where $t_1 \geqq S$, *is defined for all* $t \geqq t_1$, *and*

$$\lim_{t \to \infty} z(t) = a.$$

Proof. Denote $c = a - b$. Suppose there is a solution $z(t)$ of (4.1) such that

$$\operatorname{Re}\,\{\bar{c}[2\, z(\tilde{t}_n) - a - b]\} = 0, \quad n \in \mathbb{N},$$

where

$$\lim_{n \to \infty} \tilde{t}_n = \infty.$$

Using 1°, 3°, it can be easily verified that there exists an $L > 0$ with the following property:

$$|z(t) - a|\, |z(t) - b| \geqq L$$

for sufficiently large $t \in I$. For these t's we get

$$\frac{\mathrm{d}}{\mathrm{d}t} \frac{|z(t) - a|}{|z(t) - b|} = \frac{|z(t) - a|}{|z(t) - b|} \operatorname{Re} \left\{ \frac{c}{(z - a)(z - b)} [q(t, z) - p(t) z^2] \right\} \leqq$$

$$\leqq \frac{|z(t) - a|}{|z(t) - b|} \left\{ \frac{|c| \, |q(t, z) + ab\, p(t) - (a + b) p(t) z|}{|z - a| \, |z - b|} - \right.$$

$$\left. - \operatorname{Re} [c\, p(t)] \right\} \leqq$$

$$\leqq \frac{|z(t) - a|}{|z(t) - b|} \left\{ \frac{|c|}{L} H(t) - \operatorname{Re} [c\, p(t)] \right\}.$$

Hence

$$\frac{\mathrm{d}}{\mathrm{d}t} \left\{ \exp \left[- \int_{t_1}^{t} \left[\frac{|c|}{L} H(s) - \operatorname{Re} [c\, p(s)] \right] \mathrm{d}s \right] \frac{|z(t) - a|}{|z(t) - b|} \leqq 0 \, .\right.$$

Integration and the limiting process $t \to \infty$ yield

$$\lim_{t \to \infty} \frac{|z(t) - a|}{|z(t) - b|} = 0 \, ,$$

which contradicts our initial supposition. Consequently, there is a $\tau \geqq t_0$ such that either

$$\operatorname{Re} \{ \bar{c}[2\, z(t) - a - b] \} > 0 \quad \text{for} \quad t \geqq \tau$$

or

$$\operatorname{Re} \{ \bar{c}[2\, z(t) - a - b] \} < 0 \quad \text{for} \quad t \geqq \tau \, .$$

In view of 2° and 4° the solution $z(t)$ satisfies either (4.6) or (4.7). The rest of the proof results from 1°.

References

[1] *J. Kalas:* Asymptotic Behaviour of the Solutions of the Equation $\mathrm{d}z/\mathrm{d}t = f(t, z)$ with a Complex-Valued Function f, Proceedings of the Colloquium on Qualitative Theory of Differential Equations, August 1979, Szeged - Hungary, Seria Colloquia Mathematica Societatis János Bolyai & North-Holland Publishing Company, pp. 431—462.

[2] *J. Kalas:* On the Asymptotic Behaviour of the Equation $\mathrm{d}z/\mathrm{d}t = f(t, z)$ with a Complex-Valued Function f, Archivum Mathematicum (Brno) *17* (1981), 11—22.

[3] *M. Ráb:* Geometrical Approach to the Study of the Riccati Differential Equation with Complex-Valued Coefficients, Journal of Differential Equations *25* (1977), 108—114.

[4] *M. Ráb:* The Riccati Differential Equation with Complex-Válued Coefficients, Czechoslovak Mathematical Journal *20* (1970), 491—503.

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