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DOUBLE COVERS AND LOGICS OF GRAPHS

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In this paper we shall study logics of graphs [2] with help of double covers of graphs [3]. We consider finite undirected graphs without loops and multiple edges.

First we shall prove auxiliary results on double covers of graphs. For our purposes we shall use the definition given in [1].

Given a map $f : E(G) \to Z_2$, the graph $D = dc(G, f)$ is a double cover of G when $V(D) = V(G) \times Z_2$ and $[(u, x), (v, y)] \in E(D)$ if and only if $[u, v] \in E(G)$ and $f([u, v]) = xy$.

Here Z_2 denotes a group of the order 2.

The elements of Z_2 will be denoted by e and a so that $e^2 = a^2 = e$, $ea = ae = a$. For the mapping f mentioned in the definition there are two extreme cases. If f is a constant mapping which maps each element of $E(G)$ onto e, the double cover of G consists of two disjoint copies of G. If f maps each element of $E(G)$ onto a, the double cover of G is a bipartite graph.

We shall (according to [3]) denote $[v, e]$ as v and $[v, a]$ as v' for each $v \in V(G)$. We describe the double cover $dc(G, f)$ of G corresponding to the mapping f which maps each edge of G onto a. Denote $V = V(G)$, $V' = \{v' \mid v \in V(G)\}$. The vertex set of $dc(G, f)$ is $V \cup V'$. If $u \in V$, $v \in V$, $u \neq v$, then u is adjacent to v' and v is adjacent to u' in $dc(G, f)$ if and only if u is adjacent to v in G. There are no pairs of adjacent vertices in $dc(G, f)$ except those just described.

This graph $dc(G, f)$ is evidently a bipartite graph on the sets V, V'. We shall denote it by $B(G)$ and call it a bipartite double cover of G. Note that a double cover $dc(G, f)$ may be a bipartite graph even if f is not the described mapping. But in this paper a bipartite double cover of G will always mean the above described graph.

Now we describe some notation from [2].

If G is a graph and A a non-empty subset of the vertex set $V(G)$ of G, then by $A^\perp$ we denote the set of all vertices of G which are adjacent to all vertices of A. If $A = \emptyset$, we put $A^\perp = V(G)$. Further $A^{\perp\perp} = (A^\perp)^\perp$. For a one-element subset $\{x\}$ of $V(G)$ we write $x^\perp$ and $x^{\perp\perp}$ instead of $\{x\}^\perp$ and $\{x\}^{\perp\perp}$.

We shall consider the following properties of a graph G.

Property P 1. For each vertex x of a graph G and for each subset Y of the vertex set of G the equality $x^\perp = Y^\perp$ implies $x \in Y$.

Property $P\,2$. For any two vertices x, y of a graph G there is $x^\perp = y^\perp$ if and only if $x = y$.

Evidently $P\,1 \Rightarrow P\,2$, but not conversely.

Proposition 1. *Let G be a graph with the property $P\,2$. Let the vertex set $V(G)$ of G and the family of all subsets of $V(G)$ which are equal to $x^\perp$ for some $x \in V(G)$ be given. Then the bipartite double cover $B(G)$ of G is uniquely determined.*

Proof. Let $\{A_1, \ldots, A_n\}$ be the mentioned family of subsets of $V(G)$. Take the set $V(G)$ and a set $\{a_1, \ldots, a_n\}$ disjoint with $V(G)$. For each $i = 1, \ldots, n$ join a_i by edges with all vertices of A_i; denote the resulting graph by H. We shall prove that $H \cong B(G)$. Each A_i is equal to $u^\perp$ for some $u \in V(G)$; the property $P\,2$ implies that this u is unique for each A_i. Thus we may put $a_i = u'$. If u and v are two adjacent vertices in G, let $u^\perp = A_i$, $v^\perp = A_j$ for some i and j. We have $u \in A_j$, $v \in A_i$, therefore $a_i = u'$ is adjacent to v and $a_j = v'$ is adjacent to u in H. If u and v are not adjacent in G, then obviously u is not adjacent to v' and v is not adjacent to u'. Not wo vertices of $V(G)$ and no two vertices of $\{a_1, \ldots, a_n\}$ are adjacent in H. Hence $H \cong B(G)$.

Proposition 2. *There exist non-isomorphic graphs G_1, G_2 such that $B(G_1) \cong B(G_2)$.*

Proof. Let G_1 be a circuit of the length 6, let G_2 be a graph with two connected components, each of which is a circuit of the length 3. Then both $B(G_1)$ and $B(G_2)$ are graphs with two connected components, each of which is a circuit of the length 6.

A characterization of graphs which are isomorphic to double covers of graphs was given in [5]. Here we shall prove some results which concern bipartite double covers.

Proposition 3. *Let G be a finite bipartite graph on vertex sets U, V. Then the following two assertions are equivalent:*

(i) *There exists a graph G_0 such that $G \cong B(G_0)$.*

(ii) *There exists an automorphism α of G such that $\alpha(V) = V$, $\alpha(V) = U$, $\alpha(\alpha(x)) = x$ for each vertex x of G and x is adjacent to $\alpha(x)$ for no vertex x of G.*

This proposition follows immediately from Theorem 1 in [5].

Theorem 1. *Let G be a finite bipartite graph on vertex sets U, V satisfying the conditions of Proposition 3. Then the following two assertions are equivalent:*

(i) *If G_1, G_2 are two graphs such that $G \cong B(G_1) \cong B(G_2)$, then $G_1 \cong G_2$.*

(ii) *Any two automorphisms α, β of G satisfying the conditions of Proposition 3 are conjugated in the group of all automorphisms of G.*

Proof. (i) $\Rightarrow$ (ii). Let (i) hold. Let α, β be two automorphisms of G satisfying the conditions of Proposition 3. Let x, y be two vertices of U. If x is adjacent to $\alpha(y)$, then $y = \alpha(\alpha(y))$ is adjacent to $\alpha(x)$ and conversely. Hence if we identify x with $\alpha(x)$

for each $x \in U$, we obtain a graph G_1 such that $G \cong B(G_1)$. Analogously if we identify x with $\beta(x)$ for each $x \in U$, then we obtain a graph G_2 such that $G \cong B(G_2)$. According to (i), we have $G_1 \cong G_2$. Let φ be an isomorphic mapping of G_1 onto G_2. Let φ_0 be a mapping defined so that for each $x \in U$ the vertex $\varphi_0(x)$ is the vertex y such that the vertex obtained by identifying y with $\beta(y)$ is the image in φ of the vertex obtained by identifying x with $\alpha(x)$ and the vertex $\varphi_0\, \alpha(x)$ is the vertex $\beta(y)$. If x_1, x_2 are two vertices of G such that x_1 is adjacent to $\alpha(x_2)$, then also x_2 is adjacent to $\alpha(x_1)$ and in G_1 the vertex obtained by identifying x_1 with $\alpha(x_1)$ is adjacent to the vertex obtained by identifying x_2 with $\alpha(x_2)$. Let $y_1 = \varphi_0(x_1)$, $y_2 = \varphi_0(x_2)$. As φ is an isomorphism, in G_2 the vertex obtained by identifying y_1 with $\beta(y_1)$ and the vertex obtained by identifying y_2 with $\beta(y_2)$ are adjacent. This means that $\varphi_0(x_1) = y_1$ is adjacent to $\varphi_0\, \alpha(x_2) = \beta(y_2)$ and $\varphi_0(x_2) = y_2$ is adjacent to $\varphi_0\, \alpha(x_1) = \beta(y_1)$ in G. If x_1, x_2 are not adjacent in G, then evidently neither y_1, $\beta(y_2)$, nor y_2, $\beta(y_1)$ are adjacent. Hence φ_0 is an autonorphism of G. If $y = \varphi_0(x)$, then $\varphi_0\, \alpha(x) = \beta(y) = \beta\, \varphi_0(x)$ for each $x \in U$. Each $z \in V$ equals to $\alpha(x)$ for some $x \in U$. If again $y = \varphi_0(x)$, we have $\varphi_0\, \alpha(z) = \varphi_0(x) = y = \beta\, \beta(y) = \beta\, \varphi_0(x) = \beta\, \varphi_0(z)$. Therefore $\varphi_0\alpha = \beta\varphi_0$ and $\beta = \varphi_0\alpha\varphi_0^{-1}$ and β is conjugated with α in the group of all automorphisms of G. As α, β were chosen arbitrarily, any two such automorphisms are conjugated.

(ii) $\Rightarrow$ (i). Let (ii) hold. Let G_1, G_2 be two graphs with the property $B(G_1) \cong B(G_2) \cong G$. If $x \in U$, then let x' (or x'') be the corresponding vertex of V in $B(G_1)$ (or $B(G_2)$ respectively). Define α, β so that $\alpha(x) = x'$, $\alpha(x') = x$, $\beta(x) = x''$, $\beta(x'') = x$, for each $x \in U$. The mappings α, β are automorphisms of G satisfying the conditions of Proposition 3. According to (ii) there exists an automorphism ψ_0 of G such that $\beta = \psi_0\alpha\psi_0^{-1}$. If x, y are two vertices of U such that x is adjacent to y', then x' is adajcent to y, $\psi_0(x)$ is adjacent to $\psi_0(y') = \psi_0\, \alpha(y) = \beta\, \psi_0(y)$ and $\psi_0(y)$ is adjacent to $\psi_0(x') = \psi_0\, \alpha(x) = \beta\, \psi_0(x)$. If we identify each vertex $x \in U$ with $\alpha(x)$ (or $\beta(x)$), we obtain a graph isomorphic to G_1 (or G_2); we may consider it to be G_1 (or G_2 respectively) itself. Let ψ be the mapping which maps the vertex of G_1 obtained by identifying x with $\alpha(x)$ onto the vertex of G_2 obtained by identifying $\psi_0(x)$ with $\beta\, \psi_0(x)$ for each $x \in U$. This ψ is an isomorphism of G_1 onto G_2 and $G_1 \cong G_2$.

In [4] the isotopy of directed graphs was defined. Let $\vec{G_1}$, $\vec{G_2}$ be two directed graphs, let $V(\vec{G_1})$, $V(\vec{G_2})$ be their vertex sets respectively. An isotopy of $\vec{G_1}$ onto $\vec{G_2}$ is an ordered pair $\langle \varphi_1, \varphi_2 \rangle$ of bijections of $V(\vec{G_1})$ onto $V(\vec{G_2})$ with the property that for any two vertices x, y of $\vec{G_1}$ a directed edge goes from $\varphi_1(x)$ into $\varphi_2(y)$ in $\vec{G_2}$ if and only if a directed edge from x into y goes in $\vec{G_1}$. Two graphs $\vec{G_1}$, $\vec{G_2}$ are called *isotopic*, if there exists an isotopy of $\vec{G_1}$ onto $\vec{G_2}$. If two graphs are isomorphic, they are also isotopic, but not conversely.

Theorem 2. *Let G_1, G_2 be two undirected graphs, let $\vec{G_1}$, $\vec{G_2}$ be the directed graphs obtained from G_1, G_2 respectively by substituting each undirected edge* *by a pair of oppositely directed edges joining the same pair of vertices. Then the following two assertions are equivalent:*

(i) $G_1^{\rightarrow}$ *and* $G_2^{\rightarrow}$ *are isotopic.*

(ii) *The bipartite double covers of* G_1 *and* G_2 *are isomorphic.*

Proof. (i) $\Rightarrow$ (ii). Let (i) hold. Then there exists an isotopy $\langle \varphi_1, \varphi_2 \rangle$ of $G_1^{\rightarrow}$ onto $G_2^{\rightarrow}$. Consider bipartite double covers $B(G_1)$, $B(G_2)$. To each vertex x of G_1 the vertices x, x' of $B(G_1)$ correspond and analogously for G_2 and $B(G_2)$. Define the mapping ψ of $V(B(G_1))$ onto $V(B(G_2))$ so that if $x \in V(G_1)$, then $\psi(x) = \varphi_1(x)$ and $\psi(x') = (\varphi_2(x))'$. Let x, y be two vertices of G_1. If x, y' are adjacent in $B(G_1)$, then also x', y are adjacent in $B(G_1)$ and x, y are adjacent in G_1. Further there are edges $(xy)^{\rightarrow}$, $(yx)^{\rightarrow}$ in $G_1^{\rightarrow}$ and edges $(\varphi_1(x)\, \varphi_2(y))^{\rightarrow}$, $(\varphi_1(y)\, \varphi_2(x))^{\rightarrow}$ in $G_2^{\rightarrow}$. In G_2 then there exist the edge $\varphi_1(x)\, \varphi_2(y)$ and in $B(G_2)$ the edge $\varphi_1(x)\, (\varphi_2(y))'$ and this edge is equal to $\psi(x)\, \psi(y')$. If x and y' are not adjacent in $B(G_1)$, evidently neither $\psi(x)$, $\psi(y')$ are adjacent in $B(G_2)$. Hence ψ is an isomorphism of $B(G_1)$ onto $B(G_2)$ and $B(G_1) \cong B(G_2)$.

(ii) $\Rightarrow$ (i). Let ψ be an isomorphism of $B(G_1)$ onto $B(G_2)$; without loss of generality suppose that the dashed elements of $B(G_1)$ are mapped by ψ onto the dashed elements of $B(G_2)$. For each x from G_1 let $\varphi_1(x) = \psi(x)$ and let $\varphi_2(x)$ be the vertex y of G_2 such that $\psi(x') = y'$. By the considerations inverse to those of the first part of the proof we prove that $\langle \varphi_1, \varphi_2 \rangle$ is an isotopy of $G_1^{\rightarrow}$ onto $G_2^{\rightarrow}$.

Now we turn our attention to the logics of graphs.

For each subset A of the vertex set $V(G)$ of a graph G we have defined the sets $A^{\perp}$ and $A^{\perp\perp}$. In [2] some properties of these sets are described. For each A we have $A \cap A^{\perp} = \emptyset$; this follows from the fact that G has no loops. Further $A \subseteq A^{\perp\perp}$, $(A^{\perp\perp})^{\perp} = A^{\perp}$, $(A^{\perp\perp})^{\perp\perp} = A^{\perp\perp}$. Therefore the set $A^{\perp\perp}$ is the closure of A in a certain sense. The sets A for which $A = A^{\perp\perp}$ holds will be called $\perp\perp$-*closed subsets of* $V(G)$ or shortly $\perp\perp$-*closed sets*. These sets form a lattice with respect to the set inclusion. The meet in this lattice is the set intersection, because the intersection of two $\perp\perp$-closed sets is a $\perp\perp$-closed set. The union of two $\perp\perp$-closed sets need not be $\perp\perp$-closed; the join of two $\perp\perp$-closed sets is the closure of their union.

The mapping $A \mapsto A^{\perp}$ is a unary operation on the set of all $\perp\perp$-closed sets in G. The mentioned lattice with this operation (which is an operation of complementation on it) is called the logic of G and denoted by $\mathscr{L}(G)$.

We shall investigate what information about G can be obtained from $\mathscr{L}(G)$.

An element a of a lattice L is called *join-irreducible*, if it is not the least element of L and for any two elements b, c of L the equality $b \vee c = a$ implies $b = a$ or $c = a$.

Theorem 3. *Let G be a finite graph with the property P 1, let $\mathscr{L}(G)$ be its logic. Let $A \in \mathscr{L}(G)$. The set $A = u^{\perp\perp}$ for some $u \in V(G)$ if and only if A is a join-irreducible element of $\mathscr{L}(G)$.*

Proof. Let $A = u^{\perp\perp}$ for some $u \in V(G)$, let B, C be elements of $\mathscr{L}(G)$ such that $B \vee C = A$. Evidently $B \vee C = (B \cup C)^{\perp\perp}$. This implies $A^{\perp} = (B \cup C)^{\perp} = u^{\perp}$. The property $P\,1$ then implies that $u \in B \cup C$. If $u \in B$, then, as $B^{\perp\perp} = B$, we have $A = u^{\perp\perp} \subseteq B$, therefore $A = B$. If $u \in C$, then $A = C$. We have proved that A is a join-irreducible element of $\mathscr{L}(G)$.

Now let D be a join-irreducible element of $\mathscr{L}(G)$. As $\mathscr{L}(G)$ is a finite lattice, there exists exactly one element E of $\mathscr{L}(G)$ which is covered by D, i.e. a $\perp\perp$-closed set E which is a proper subset of D and contains each proper subset of D as its subset. Let $x \in D - E$. As D is $\perp\perp$-closed, we have $x^{\perp\perp} \subseteq D$. If $x^{\perp\perp}$ is a proper subset of D, then $x^{\perp\perp} \subseteq E$ and $x \in E$, which is a contradiction. Therefore $x^{\perp\perp} = D$ and the assertion is proved.

Theorem 4. *Let G be a finite graph with the property $P\,1$. Then an element A of $\mathscr{L}(G)$ is meet-irreducible if and only if $A = u^{\perp}$ for some vertex u of G.*

Proof. Let $A = u^{\perp}$ for some $u \in V(G)$. Let B, C be elements of $\mathscr{L}(G)$ such that $A = B \wedge C$. Then $A = B \cap C$. As $A \subseteq B$, $A \subseteq C$, we have $B^{\perp} \subseteq A^{\perp} = u^{\perp\perp}$, $C^{\perp} \subseteq A^{\perp} = u^{\perp\perp}$. By Theorem 3 the element $u^{\perp\perp}$ is join-irreducible, therefore there exists an element D of $\mathscr{L}(G)$ such that D is a proper subset of $u^{\perp\perp}$ and each proper $\perp\perp$-closed subset of $u^{\perp\perp}$ is a subset of D. If $B^{\perp} \neq u^{\perp\perp}$, $C^{\perp} \neq u^{\perp\perp}$, then $B^{\perp} \subseteq D$, $C^{\perp} \subseteq D$. Hence $D^{\perp} \subseteq B^{\perp\perp} \cap C^{\perp\perp} = B \cap C = A$ and $D = D^{\perp\perp} \supseteq A^{\perp} = u^{\perp\perp}$, which is a contradiction with the assumption that D is a proper subset of $u^{\perp\perp}$. Therefore either $B^{\perp} = u^{\perp\perp}$ and $B = B^{\perp\perp} = u^{\perp} = A$, or $C = A$ and A is meet-irreducible.

Now let E be a meet-irreducible element of $\mathscr{L}(G)$. As $\mathscr{L}(G)$ is finite, there exists exactly one element F of $\mathscr{L}(G)$ which covers E, i.e. a $\perp\perp$-closed set F such that E is a proper subset of F and each $\perp\perp$-closed set which contains E as a proper subset contains F as a subset. Then $F^{\perp}$ is a proper subset H of $E^{\perp}$. For each proper subset H of $E^{\perp}$ the set $H^{\perp}$ contains E as a proper subset, hence it contains F as a subset and $H \subseteq F^{\perp}$. The element $E^{\perp}$ covers exactly one element $F^{\perp}$ of $\mathscr{L}(G)$ and therefore $E^{\perp}$ is join-irreducible. By Theorem 3 we have $E^{\perp} = v^{\perp\perp}$ for some vertex v of G and $E = E^{\perp\perp} = v^{\perp}$.

Theorem 5. *Let G be a finite graph with the property $P\,1$ and let its logic $\mathscr{L}(G)$ be given as an abstract lattice with a complementation. Then G can be reconstructed uniquely up to an isomorphism.*

Proof. In $\mathscr{L}(G)$ we find all join-irreducible elements. According to Theorem 3 there is a one-to-one correspondence between them and the vertices of G such that to each join-irreducible element A of $\mathscr{L}(G)$ the vertex a such that $a^{\perp\perp} = A$ is assigned. Thus the vertex set of G is reconstructed. Now let a, b be two vertices of G. Take the elements A, B of $\mathscr{L}(G)$ such that $a^{\perp\perp} = A$, $b^{\perp\perp} = B$. If a is adjacent to b in G, then $b \in a^{\perp} = A^{\perp}$; as $A^{\perp}$ is $\perp\perp$-closed, also $B = b^{\perp\perp} \subseteq A^{\perp}$ and analogously also $A \subseteq B^{\perp}$. On the other hand, $B \subseteq A^{\perp}$ implies that $b \in A^{\perp} = a^{\perp}$ and a is adjacent to b in G. In this way we reconstruct the edges of G.

Theorem 6. *Let G be a finite graph with the property P 1 and let the lattice of all $\perp\perp$-closed subsets of $V(G)$ be given as an abstract lattice (without the operation of complementation). Then the bipartite double cover $B(G)$ of the graph G can be reconstructed uniquely up to an isomorphism.*

Proof. Like in the proof of Theorem 5 we can reconstruct the vertex set of G and thus also the vertex set of $B(G)$. Analogously according to Theorem 4 we may find all meet-irreducible elements of $\mathscr{L}(G)$; they correspond to sets $x^{\perp}$ for $x \in V(G)$. If A is a join-irreducible element of $\mathscr{L}(G)$ and B a meet-irreducible element of $\mathscr{L}(G)$, then the vertex a such that $a^{\perp\perp} = A$ is contained in the set B if and only if $A \subseteq B$. Thus we have reconstructed the vertex set $V(G)$ of G and the family of all subsets of $V(G)$ which are equal to $x^{\perp}$ for some $x \in V(G)$. According to Proposition 1 we can reconstruct the bipartite double cover $B(G)$ of G.

Remark. The assertions of Theorems 5 and 6 are to be understood so that we do the reconstruction knowing *a priori* that G is a graph with the property P 1.

Theorem 7. *Let G be a graph, let $B(G)$ be its bipartite double cover. If $B(G)$ is given as an abstract graph (without the dash notation of vertices), then the lattice of all $\perp\perp$-closed sets of G is determined uniquely.*

Proof. Let $B(G)$ be a bipartite graph on the sets V, W. We may consider V to be $V(G)$. If A is a subset of V, we may find $A^{\perp\perp}$ in $B(G)$; this is a subset of V which is also $A^{\perp\perp}$ in G. In this way we find all subsets of $V(G)$ which are $\perp\perp$-closed and thus also the lattice of all such sets.

Corollary 1. *Let G_1, G_2 be two finite graphs with the property P 1. Then the lattices of $\perp\perp$-closed sets in G_1 and G_2 respectively are isomorphic if and only if the bipartite double covers of G_1 and G_2 respectively are isomorphic.*

Now consider the graphs in general, without supposing the property P 1.

Theorem 8. *Let G be a graph, let X be a $\perp\perp$-closed subset of $V(G)$. Let G' be the graph obtained from G by adding a new vertex w and joining it by edges with all vertices of X. Then $\mathscr{L}(G') \cong \mathscr{L}(G)$.*

Remark. Note that G' has not the property P 1.

Proof. The symbols $A^{\perp}$, $A^{\perp\perp}$ will have the usual meaning with respect to G. With respect to G' we shall use $(A^{\perp})'$, $(A^{\perp\perp})'$. If $A \subseteq V(G)$ and is not a subset of X, then evidently $(A^{\perp})' = A^{\perp}$; if $A \subseteq X$, then $(A^{\perp})' = A^{\perp} \cup \{w\}$. Further $((A \cup \{w\})^{\perp})' = (A^{\perp})' \cap (w^{\perp})' = A^{\perp} \cap X^{\perp}$ for each $A \subseteq V(G)$. Each $\perp\perp$-closed set in G (or in G') is of the form $A^{\perp}$ (or $(A^{\perp})'$) for some $A \subseteq V(G)$ (or $A \subseteq V(G')$ respectively). Thus each $\perp\perp$-closed set of G' is either of the form $A^{\perp}$ for $A \nsubseteq X$, or of the form $A^{\perp} \cup \{w\}$ for $A \subseteq X$; in other words, it is a set B, where $B \in \mathscr{L}(G)$, $B^{\perp} \nsubseteq X$, or a set $B \cup \{w\}$, where $B \in \mathscr{L}(G)$, $B^{\perp} \subseteq X$. For each $B \in \mathscr{L}(G)$ define $\varphi(B)$ so that $\varphi(B) = B$

for such B that $B^\perp \subseteq X$ and $\varphi(B) = B \cup \{w\}$ for such B that $B^\perp \subseteq X$. The mapping φ is a bijection of $\mathscr{L}(G)$ onto $\mathscr{L}(G')$. Let B, C be two elements of $\mathscr{L}(G)$. Suppose $B \subseteq C$. If $\varphi(B) = B$, then $\varphi(B) \subseteq \varphi(C)$, because $\varphi(C) = C$ or $\varphi(C) = C \cup \{w\}$. If $\varphi(B) = B \cup \{w\}$, then $B^\perp \subseteq X$. As $B \subseteq C$, we have $C^\perp \subseteq B^\perp \subseteq X$, which implies $\varphi(C) = C \cup \{w\}$ and we have $\varphi(B) = B \cup \{w\} \subseteq C \cup \{w\} = \varphi(C)$. We have proved that $B \subseteq C$ implies $\varphi(B) \subseteq \varphi(C)$ and analogously we can prove the inverse implication. Therefore φ preserves the ordering of $\mathscr{L}(G)$ and hence the lattice operations. It is easy to prove that φ preserves also the unary operation $A \mapsto A^\perp$ and that it is an isomorphism of $\mathscr{L}(G)$ onto $\mathscr{L}(G')$.

Corollary 2. *To each graph G there exist infinitely many graphs G' without the property P 1 such that $\mathscr{L}(G') \cong \mathscr{L}(G)$.*

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