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NILPOTENT, WEAKLY ABELIAN AND HAMILTONIAN LATTICE ORDERED GROUPS

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1. INTRODUCTION

In [2] Hollister studied the class of nilpotent lattice ordered groups and showed that nilpotent lattice ordered groups are representable. In [4], Martinez introduced two varieties of lattice ordered groups which to date have appeared somewhat isolated from the main developments in the theory of the lattice of varieties of lattice ordered groups. The first, denoted by $\mathscr{C}$, is the variety defined by the law $[x, x^y] = 1$ and the second, denoted by W, is defined by the law $(x \vee 1)^2 \vee x^y = (x \vee 1)^2$. Discussed implicitly in [4] is a fourth class $\mathscr{H}$ consisting of the Hamiltonian lattice ordered groups; that is, the lattice ordered groups for which every convex l-subgroup is normal. We show that these four classes are intimately connected. In particular, $\mathscr{C}$ is the variety of l-groups which are nilpotent of class at most 2. Furthermore, W contains the class of nilpotent l-groups and is the largest variety contained in $\mathscr{H}$.

For lattice ordered groups, henceforth *l-groups*, we adopt the notation and terminology of [1] and for groups we adopt the notation and terminology of [5]. In particular, for $a, b, b_i \in G$, G a group, $a^b = b^{-1}ab$, $[a, b] = a^{-1}b^{-1}ab$ and $[a, b_1, \ldots, b_n] = [[a, b_1, \ldots, b_{n-1}], b_n]$. Also, we will write $[a, b, \ldots, b]_n$ for $[a_1, b_1, \ldots, b_n]$ with $b_i = b$, $i = 1, \ldots, n$. If $[a, b_1, \ldots, b_n] = 1$ for all $a, b_i \in G$, then G is said to be *nilpotent* of class at most n.

Certain varieties (by „variety“ we will always mean „variety of l-groups“) and classes of l-groups figure prominently in our discussions. An l-group is *representable* if it is a subdirect product of totally ordered groups. The class of all representable l-groups is a variety defined by the law $(x \wedge y^{-1}x^{-1}y) \vee 1 = 1$. The following important result was established by Hollister [2].

Lemma 1.1. *Nilpotent l-groups are representable.*

In establishing this result, Hollister used the following observation which we will also require.

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Lemma 1.2. *Let G be a group and a and b be elements of G. Then any commutator of the form $[a, b, \ldots, b]$ can be written in the form ca^{b^n} where n is the length of the commutator and c is the product of $2^n - 1$ elements of the form a^{b^k} or $(a^{-1})^{b^k}$ with $0 \leqq k < n$.*

The variety $\mathscr{C}$ in which elements commute with their conjugates, the variety of *commuting conjugates*, is defined by the law $[x, x^y] = 1$. The *weakly abelian* variety W, is defined by the law

$$(x \vee 1)^2 \vee (x \vee 1)^y = (x \vee 1)^2\,, \quad \text{or equivalently}\,, \quad (x \vee 1)^y \leqq (x \vee 1)^2\,.$$

We say that a group or a variety is *weakly abelian* if it is contained in W. The varieties $\mathscr{C}$ and W sere introduced by Martinez [4].

Extending the terminology of group theory, we will say that an l-group is *Hamiltonian* if every convex l-subgroup is normal and we will say that a variety is *Hamiltonian* if every member is Hamiltonian.

2. NILPOTENT, WEAKLY ABELIAN AND HAMILTONIAN l-GROUPS

In the next theorem, we establish some equivalent formulations for weakly abelian l-groups and show that any weakly abelian l-group is Hamiltonian. The following lemma summarizes some relevant observations from [4].

Lemma 2.1. *Let G be an l-group and $n \geqq 2$ an integer. Of the following statements, (1), (2), (3) and (4) are equivalent and imply (5) and (6) which are also equivalent. Furthermore, (1)–(6) imply (7).*

(1) $(x \vee 1)^y \leqq (x \vee 1)^2$, *for all $x, y \in G$, that is, G is weakly abelian*;
(2) $x^y < x^2$, *for all $x, y \in G$, $x > 1$*;
(3) $(x \vee 1)^y \leqq (x \vee 1)^n$, *for all $x, y \in G$*;
(4) $x^y < x^n$, *for all $x, y \in G$, $x > 1$*;
(5) *All regular subgroups of G are normal*;
(6) *All convex l-subgroups of G are normal, that is, G is Hamiltonian*;
(7) *G is representable.*

The missing implication in Lemma 2.1, namely (6) implies (1), is not valid in general. An example is provided in [4].However, for varieties this implication does hold, as the following theorem demonstrates.

Theorem 2.2. *For a variety $\mathscr{V}$ of l-groups, the following statements are equivalent*:

(1) $(x \vee 1)^y \leqq (x \vee 1)^2$ *is a law in $\mathscr{V}$, that is $\mathscr{V}$ is weakly abelian*;
(2) *For some integer $n \geqq 2$, $(x \vee 1)^y \leqq (x \vee 1)^n$ is a law in $\mathscr{V}$*;
(3) *If $G \in \mathscr{V}$, then all regular subgroups of G are normal in G*;
(4) *If $G \in \mathscr{V}$, then all convex l-subgroups of G are normal in G; that is, $\mathscr{V}$ is Hamiltonian.*

Proof. From Lemma 2.1, it follows that it only remains to show that (4) implies (1). Let (4) hold and suppose that (1) does not hold. Then there exist $x, y \in G \in \mathscr{V}$ with $x > 1$ such that $x^y \nleqq x^2$. From the hypothesis that (4) holds and Lemma 2.1, it follows that G is representable, and can therefore be considered as a subdirect product of totally ordered groups T_i $(i \in I)$. For one of these groups, T say, the components a, b of x and y, respectively, must be such that $a > 1$ and $a^b > a^2$. Now, since T is a homomorphic image of G, $T \in \mathscr{V}$. Let N denote the natural numbers; for each $i \in N$ let $S_i = T$ and let $H = \Pi\{S_i : i \in N\}$. Then $H \in \mathscr{V}$. Let $f, g \in H$ be such that $f(i) = a$, $g(i) = b^i$ $(i \in N)$. Then f^g is not contained in the convex l-subgroup of H generated by f, contradicting the hypothesis that (4) holds. The result follows.

Corollary 2.3. *The variety W of all weakly abelian l-groups is the largest variety of Hamiltonian l-groups.*

It is important to note that although the class of all weakly abelian l-groups is a variety, namely, W, the class of all Hamiltonian l-groups is not. The last example in the paper by Kopitov and Medvedev [3] illustrates this point.

Theorem 2.4. *Every nilpotent l-groups is weakly abelian.*

Proof. Let G be a nilpotent l-group of class at most n. It follows from Lemma 1.1, that G is representable and so may be considered as a subdirect product ΠG_i of totally ordered groups G_i. We will show that for any $a, b \in G$, with $a > 1$, $a^b \leqq a^m$ where $m = 2^{n-1}$. To do this for G, it suffices to do it for each G_i or simply for G with the assumption that G is totally ordered. Since G is nilpotent of class at most n,

$$(1) \qquad [a, b, \ldots, b]_n = 1 .$$

By Lemma 1.2, $[a, b, \ldots, b]_n = c_n a^{b^n}$ where c_n is the product of $2^n - 1$ elements of the form a^{b^k} or $(a^{-1})^{b^k}$ with $0 \leqq k < n$. In fact, it is not difficult to verify that c_n involves 2^{n-1} terms of the form $(a^{-1})^{b^k}$ and $2^{n-1} - 1$ of the form a^{b^k}. Since G is totally ordered, either $a \leqq a^b$ or $a^b \leqq a$. If the latter holds then we already have the required result. So suppose that $a \leqq a^b$. Then

$$(2) \qquad a \leqq a^b \leqq a^{b^2} \leqq \ldots \leqq a^{b^{n-1}} .$$

By equation (1) and Lemma 1.2, we have

$$(3) \qquad a^{b^n} = c_n^{-1} .$$

Here c_n^{-1} contains $2^{n-1} - 1$ conjugates of a^{-1}. Since $a^{-1} < 1$, if we denote by d_n the element obtained from c_n^{-1} by deleting all conjugates of a^{-1} then $c_n^{-1} \leqq d_n$. If we now replace each conjugate of a in d_n systematically by $a^{b^{n-1}}$ and denote the resulting element by e_n, then by Lemma 1.2 and (2), $d_n \leqq e_n$ where e_n is simply a product of 2^{n-1} elements each equal to $a^{b^{n-1}}$. Thus

$$e_n = (a^{2^{n-1}})^{b^{n-1}}$$

and from (3), we have

$$a^{b^n} = c_n^{-1} \leqq d_n \leqq e_n = (a^{2^{n-1}})^{b^{n-1}} .$$

Thus

$$a^b \leqq a^{2^{n-1}}$$

as required. By Lemma 2.1, the result follows.

The situations where order theoretic assumptions on l-groups imply group theoretic conclusions are not numerous. However, we conclude this section by showing that, in relation to weakly abelian l-groups, one can derive some interesting conclusions.

Notation. For any regular subgroup M of an l-group G, we denote by M^* the convex l-subgroup of G covering M. See [1] for details.

The next result shows that weakly abelian l-groups have some properties resembling nilpotency (see [4]).

Lemma 2.5. *Let $G \in W$. Then for any regular subgroup M of G, M^*/M is contained in the centre of G/M.*

With regard to the next result we note that, since every convex l-subgroup is the intersection of regular subgroups, an l-group has finitely many regular subgroups if and only if it has finitely many convex l-subgroups.

Theorem 2.6. *Let G be a totally ordered group with n $(n < \infty)$ regular subgroups. Then G is solvable of length n. If, in addition, G is weakly abelian, then G is nilpotent of length n.*

Proof. Since G is totally ordered the regular subgroups must form a chain $\{V_i\}$,

$$V_1 = \{1\} \subset V_2 \subset \ldots \subset V_n \subset G .$$

Let $V_{n+1} = G$. Since a conjugate of a regular subgroup is again a regular subgroup and since we only have a finite chain, each V_i must be normal in G. Also each V_i is a maximal convex l-subgroup of V_{i+1}. Hence V_{i+1}/V_i is isomorphic to a subgroup of the reals and so is abelian. Hence G has a finite invariant series of length n with abelian factors and the first conclusion holds.

If, in addition, G is weakly abelian, then by Lemma 2.5 each factor V_{i+1}/V_i is in the centre of G/V_i and so G must be nilpotent of class at most n.

Since any representable l-group is a subdirect product of totally ordered groups, the following corollary is a simple consequence of Theorem 2.6.

Corollary 2.7. *Let G be a representable l-group such that any chain of regular subgroups contains at most n elements $(n < \infty)$. Then G is solvable of length at most n. If, in addition, G is weakly abelian, then G is nilpotent of class at most n.*

Relaxing the fixed upper bound to the length of chains of regular subgroups, but still requiring that each be finite, we obtain the following corollary.

Corollary 2.8. *Let G be a representable l group such that any chain of regular subgroups is finite. Then G is residually solvable. If, in addition, G is weakly abelian, then G is residually nilpotent.*

3. THE VARIETY OF *l*-GROUPS WITH COMMUTING CONJUGATES

In this section we establish precisely the connection between the variety of commuting conjugates and nilpotent *l*-groups and therefore, by Theorem 2.4, weakly abelian *l*-groups.

Theorem 3.1. *The variety $\mathscr{C}$ is the variety of nilpotent groups of class at most two.*

Proof. Let $G \in \mathscr{C}$. Then G satisfies the law

$$x(y^{-1}xy) = (y^{-1}xy)\,x\,. \tag{4}$$

Clearly, (4) implies that x commutes with any conjugate of any power of x. We shall apply this repeatedly below. To simplify the following calculation we place in parenthesis the elements ot be permuted in the next step by invoking (4) and underline elements that will cancel as a result of the permutation. Let $x, y, z \in G$. Then

$$[y, [z, x]]\,[[x, y], z] = y^{-1}[z, x]^{-1}\, y[z, x]\,[x, y]^{-1}\, z^{-1}[x, y]\, z =$$
$$= y^{-1}(x^{-1}z^{-1}xz)\,(yz^{-1}x^{-1}zxy^{-1})\, x^{-1}yxz^{-1}x^{-1}y^{-1}xyz =$$
$$= (z^{-1}x^{-1}zxy^{-1}x^{-1}z^{-1}xz)\,(x^{-1}yx)\, z^{-1}x^{-1}y^{-1}xyz =$$
$$= x^{-1}y(xz^{-1}x^{-1})\,(z)\, xy^{-1}x^{-1}z^{-1}y^{-1}xyz =$$
$$= x^{-1}(yz)\,(xz^{-1}y^{-1}x^{-1})\, z^{-1}y^{-1}xyz = x^{-1}xz^{-1}y^{-1}x^{-1}xyz = 1\,.$$

Thus

$$[x, y, z] = [y, [z, x]]^{-1} = [z, x, y]$$

and we may permute the order of the elements in any commutator of length 2. In the following computation, we denote by $\uparrow$ a point at which $y^{-1}y$ is inserted. We now have

$$[x, y, z]^3 = [x, y, z]\,[y, z, x]\,[z, x, y] =$$
$$= [x, y]^{-1}\, z^{-1}[x, y]\, z[y, z]^{-1}\, x^{-1}[y, z]\, x[z, x]^{-1}\, y^{-1}[z, x]\, y =$$
$$= y^{-1}x^{-1}yxz^{-1}x^{-1}y^{-1}xyzz^{-1}y^{-1}zyx^{-1}y^{-1}z^{-1}yzxx^{-1}z^{-1}xzy^{-1}z^{-1}x^{-1}zxy =$$
$$= y^{-1}(x^{-1}yx)\,(z^{-1}x^{-1}y^{-1}xz)\, yx^{-1}y^{-1}z^{-1}yxzy^{-1}z^{-1}x^{-1}zxy =$$
$$= y^{-1}z^{-1}x^{-1}y^{-1}xz(x^{-1}yx)\,(y)\, x^{-1}y^{-1}z^{-1}yxzy^{-1}z^{-1}x^{-1}zxy =$$
$$= (y^{-1})\,(z^{-1}x^{-1}y^{-1}xz)\, yx^{-1}z^{-1}yxzy^{-1}z^{-1}x^{-1}zxy =$$
$$= z^{-1}x^{-1}y^{-1}xzx^{-1}z^{-1}(y)\,(xzy^{-1}z^{-1}x^{-1})\, zxy =$$

$$= z^{-1}x^{-1}y^{-1} \uparrow xz(x^{-1})(z^{-1}xz) y^{-1}z^{-1}x^{-1}yzxy =$$
$$= z^{-1}x^{-1}y^{-1}y^{-1}(yxxzx^{-1}y^{-1})(z^{-1}x^{-1}) yzxy =$$
$$= z^{-1}x^{-1}y^{-1}y^{-1}z^{-1}x^{-1}yx(xzx^{-1})(z) xy =$$
$$= z^{-1}x^{-1}y^{-1}(y^{-1}z^{-1}x^{-1}y)(xz) xzy = (z^{-1}x^{-1}y^{-1}xz)(y^{-1}) z^{-1}x^{-1}yxzy =$$
$$= y^{-1}z^{-1}x^{-1}y^{-1}xzz^{-1}x^{-1}yxzy = 1\,.$$

But G is an l-group and therefore torsion free. Hence $[x, y, z] = 1$ and G is nilpotent of class at most 2.

Conversely, let G be nilpotent of class at most 2. Then $[x, y, z] = 1$, for all $x, y, z \in G$. In particular, $[y, x, x] = 1$. Therefore,

$$1 = x^{-1}y^{-1}xyx^{-1}y^{-1}x^{-1}yxx$$

and conjugating by $x^{-1}y^{-1}xy$ we obtain

$$1 = x^{-1}y^{-1}x^{-1}yxy^{-1}xy = [x, y^{-1}xy]\,.$$

Thus $G \in \mathscr{C}$.

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