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SYMONS' d-CONGRUENCE ON SANDWICH SEMIGROUPS

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1. INTRODUCTION

Sandwich semigroups were essentially introduced in [2] and [3]. For more recent results, related work and additional references, one may also consult [9], [10], [13], [14] and [15]. In some of these papers the domains of the functions in the semigroup did not necessarily all coincide but in this paper our sandwich semigroup $S(X, Y, \alpha)$ will always consist of all continuous functions whose domains are all of X and which map X into Y. The multiplication is, of course, given by $fg = f \circ \alpha \circ g$ for all $f, g \in S(X, Y, \alpha)$. In the special case $X = Y$, we will use the simpler notation $S(X, \alpha)$.

J. S. V. Symons [14] defined three congruences l, r and d for sandwich semigroups and proved several results which show that they behave remarkably like Green's relations $\mathscr{L}$, $\mathscr{R}$ and $\mathscr{D}$. The definitions are flg iff $f \circ \alpha = g \circ \alpha$, frg iff $\alpha \circ f = \alpha \circ g$ and fdg iff $\alpha \circ f \circ \alpha = \alpha \circ g \circ \alpha$ where $f, g \in S(X, Y, \alpha)$. As the title suggests, the d-equivalence is the one we investigate here and our principal efforts are directed towards determining precisely when two d-quotients $S(X, Y, \alpha)/d$ and $S(V, W, \beta)/d$ are isomorphic. In Section 2, we pretty much settle this question when the two quotient semigroups both have identities. For in this case, $S(X, Y, \alpha)/d$ and $S(V, W, \beta)/d$ will be isomorphic if and only if Ran α and Ran β (the ranges of α and β respectively) are homeomorphic. We do put some conditions on the spaces Ran α and Ran β but they are quite mild. This result depends heavily on a previous result which lists several conditions all equivalent to the condition that $S(X, Y, \alpha)/d$ have an identity. One of these is that $S(X, Y, \alpha)/d$ and $S(\text{Ran}\ \alpha)$ (the full semigroup of continuous selfmaps on Ran α) are isomorphic. The result discussed above follows from this and theorems already in the literature.

In Section 3, we consider what we call *preapproximation systems*. The terminology is essentially due to Symons [14] who used the term *approximation system*. We wish to define these but some motivation is appropriate and we can't do better than to discuss the example discussed by Symons. Suppose f and g are any two functions mapping the reals into the reals (say polynomials) and suppose we wish to compute $f \circ g$ at a number x which may well be transcendental. We don't really compute

$f(g(x))$ but we do compute something close enough to it to serve our purposes. What we do is to choose a decimal approximation to x, compute g at that number, round off, compute f at the latter number and again round off. In other words, we really compute $\alpha \circ f \circ \alpha \circ g \circ \alpha$ at the point x where $\alpha(x)$ is x correct to the appropriate number of decimal places. Moreover, when we do compute, we do not distinguish between two functions f and g such that $\alpha \circ f \circ \alpha = \alpha \circ g \circ \alpha$. In fact, quite the contrary is often true. If, for example, we wish to compute $\sin \sqrt{2}$ and can't find tables to the degree of accuracy that we need, what we do is to select a polynomial P such that $\alpha \circ P \circ \alpha = \alpha \circ \sin \circ \alpha$ and we compute $\alpha \circ P \circ \alpha$ at $\sqrt{2}$. So what we have here is the sandwich semigroup $S(R_D, \alpha)$ (where R_D denotes the reals with the discrete topology) in which we identify two functions if they agree after being multiplied on both sides by α. In other words, we are really dealing with the quotient semigroup $S(R_D, \alpha)/d$ where d is Symons' congruence. Symons notes further that the sandwich function α is in this particular case, idempotent and he reserves the term approximation system for sandwich semigroups of the form $S(X, \alpha)$ whenever α is idempotent. We will take the liberty here of using the term not for the sandwich semigroup itself but for the quotient of the sandwich semigroup by d. All this leads us to

Definition (1.1). Let X be any topological space and let α be a continuous selfmap of X. Then the semigroup $S(X, \alpha)/d$ is referred to as a *preapproximation system* on, or supported by, X. If α is idempotent, then $S(X, \alpha)/d$ is referred to as an *approximation system.*

As we mentioned earlier, Section 3 is devoted primarily to the investigation of preapproximation systems and there again the main results are isomorphism theorems. In several of them, we replace the condition that $S(X, \alpha)/d$ contain an identity with another condition and we show that certain preapproximation systems, $S(X, \alpha)/d$ and $S(Y, \beta)/d$ are isomorphic if and only if there exist homeomorphisms h and k from X onto Y such that $h \circ \alpha = \beta \circ k$. When P and Q are two odd degree polynomials mapping the reals R onto R, this latter result, together with a result from [6] enables us to tell if $S(R, P)/d$ and $S(R, Q)/d$ are isomorphic merely by inspecting the local maxima and minima of the two polynomials.

Section 4 consists entirely of applications of the previous two sections. Among other things, we show in that section that the number of approximation systems a space supports serves to charaterize certain spaces among the Peano continua. Specifically, a Peano continuum supports exactly two (up to isomorphism) approximation systems if and only if it is an arc and it supports exactly three if and only if it is either a triod or a simple closed curve.

2. d-QUOTIENTS OF SANDWICH SEMIGROUPS

We recall that a retract of a topological space X is by definition, the range of an idempotent continuous selfmap of X.

Theorem (2.1). *Let $S(X, Y, \alpha)$ be any sandwich semigroup whatsoever. Then the following statements are all equivalent.*

(2.1.1) *$S(X, Y, \alpha)/d$ has an identity.*

(2.1.2) *$S(X, Y, \alpha)/d$ has a left identity.*

(2.1.3) Ran α *is retract of X and α maps some subspace of Y homeomorphically onto* Ran α.

(2.1.4) *$S(X, Y, \alpha)/d$ and $S(\text{Ran } \alpha)$ are isomorphic.*

Proof. We first show that (2.1.2) implies (2.1.3). Let l denote a left identity of $S(X, Y, \alpha)/d$. Then for any $f \in S(X, Y, \alpha)$, we have

(2.1.5) $\alpha \circ l \circ \alpha \circ f \circ \alpha = \alpha \circ f \circ \alpha$.

We use this to verify

(2.1.6) $\alpha \circ l(x) = x$ for all $x \in \text{Ran } \alpha$.

Suppose $x \in \text{Ran } \alpha$. Then $x = \alpha(y)$ for some $y \in Y$ and we let $\langle y \rangle$ denote the constant function which maps all of X into the point y. We then have

$$\alpha \circ l(x) = \alpha \circ l \circ \alpha(y) = \alpha \circ l \circ \alpha \circ \langle y \rangle \circ \alpha(y) = \alpha \circ \langle y \rangle \circ \alpha(y) = \alpha(y) = x$$

and (2.1.6) has been verified. This immediately implies two things. Since we already have $\text{Ran } \alpha \circ l \subset \text{Ran } \alpha$, it implies that, in fact,

(2.1.7) $\text{Ran } \alpha \circ l = \text{Ran } \alpha$.

Moreover, it implies that $\alpha \circ l$ is the identity on its range. In other words, we have

(2.1.8) $\alpha \circ l$ is idempotent.

We want to show next that α maps $\text{Ran } l \circ \alpha$ homeomorphically onto $\text{Ran } \alpha$. Evidently, α maps $\text{Ran } l \circ \alpha$ into $\text{Ran } \alpha$ and l maps $\text{Ran } \alpha$ into $\text{Ran } l \circ \alpha$. Take any $y \in \text{Ran } l \circ \alpha$. Then $y = l \circ \alpha(z)$ for some $z \in Y$ and we use (2.1.6) to get

$$l \circ \alpha(y) = l \circ \alpha \circ l \circ \alpha(z) = l \circ \alpha(z) = y .$$

That is,

(2.1.9) $l \circ \alpha(y) = y$ for all $y \in \text{Ran } l \circ \alpha$.

Statements (2.1.6) and (2.1.9) together imply that α maps $\text{Ran } l \circ \alpha$ homeomorphically onto $\text{Ran } \alpha$ and we have now verified that (2.1.2) implies (2.1.3).

Next we show that (2.1.3) implies (2.1.4). Define a mapping ϕ from the sandwich semigroup $S(X. Y, \alpha)$ into $S(\text{Ran } \alpha)$ by

(2.1.10) $\phi(f) = \alpha \circ (f/\text{Ran } \alpha)$.

For any $f, g \in S(X, Y, \alpha)$ we have

$$\phi(fg) = \alpha \circ (fg/\text{Ran } \alpha) = \alpha \circ (f \circ \alpha \circ g/\text{Ran } \alpha) =$$
$$= \alpha \circ (f/\text{Ran } \alpha) \circ \alpha \circ (g/\text{Ran } \alpha) = \phi(f) \circ \phi(g) .$$

That is, ϕ is a homomorphism from $S(X, Y, \alpha)$ into $S(\text{Ran } \alpha)$. We wish to show that, ϕ is onto so let any $f \in S(\text{Ran } \alpha)$ be given. Since Ran α is a retract of X there exists an idempotent continuous selfmap v of X such that Ran v = Ran α. Moreover, α maps some subspace Z of Y homeomorphically onto Ran α nd we define $\delta = (\alpha/Z)^{-1}$. Then $\delta \circ f \circ v$ belongs to $S(X, Y, \alpha)$ and one easily checks that $\phi(\delta \circ f \circ v) = f$. Finally, we have $\phi(f) = \phi(g)$ if and only if $\alpha \circ f \circ \alpha = \alpha \circ g \circ \alpha$ and we conclude that $S(X, Y, \alpha)/d$ is isomorphic to $S(\text{Ran } \alpha)$.

It is immediate that (2.1.1) implies (2.1.2) and that (2.1.4) implies (2.1.1) so we have proved the theorem.

This generalizes Theorem (1.7) of Symons [14]. He showed that $S(X, Y, \alpha)/d$ is isomorphic to $S(\text{Ran } \alpha)$ whenever both X and Y are discrete. To get his result from our theorem, simply note that in this case, every subspace of X is a retract of X and any function α from Y to X maps a subspace of Y homeomorphically onto Ran X. Then use the fact that (2.1.3) and (2.1.4) are equivalent.

Recall from [16] that a topological space X is a *generated space* if it is T_1 and $\{f^{-1}(y) : f \in S(X),\ y \in \text{Ran} f\}$ is a subbasis for the closed subsets of X. According to results in [5] pages 198–200, generated spaces include all completely regular Hausdorff spaces which contain an arc *as* well as all 0-dimensional Hausdorff spaces. Moreover, two generated spaces X and Y are homeomorphic if and only if their full semigroups of continuous selfmaps $S(X)$ and $S(Y)$ are isomorphic. These observations, together with Theorem (2.1) immediately result in

Theorem (2.2). *Suppose that both quotient semigroups $S(X, Y, \alpha)/d$ and $S(V, W, \beta)/d$ have identiteis. Suppose further that both* Ran α *and* Ran β *are generated spaces. Then the following statements are equivalent*

(2.2.1) $S(X, Y, \alpha)/d$ *and* $S(V, W, \beta)/d$ *are isomorphic.*

(2.2.2) $S(\text{Ran } \alpha)$ *and* $S(\text{Ran } \beta)$ *are isomorphic.*

(2.2.3) Ran α *and* Ran β *are homeomorphic.*

When $S(X, Y, \alpha)/d$ does not have an identity, it cannot possibly be isomorphic to $S(\text{Ran } \alpha)$. Nevertheless, it is isomorphic to a subsemigroup of $S(\text{Ran } \alpha)$. Before we prove this, we introduce some notation and terminology.

Definition (2.3). Let $S(X, Y, \alpha)$ be any sandwich semigroup. The *associative* of α denoted by $A(\alpha)$ is the subsemigroup of $S(\text{Ran } \alpha)$ consisting of all those functions $g \in S(\text{Ran } \alpha)$ such that $g = \alpha \circ (f/\text{Ran } \alpha)$ for some $f \in S(X, Y, \alpha)$.

Theorem (2.4). *Let $S(X, Y, \alpha)$ be any sandwich semigroup whatsoever. Then the quotient semigroup $S(X, Y, \alpha)/d$ is isomorphic to $A(\alpha)$ the associative of α.*

Proof. Define a function ϕ from the sandwich semigroup $S(X, Y, \alpha)$ into $A(\alpha)$ by $\phi(f) = \alpha \circ (f/\text{Ran } \alpha)$. One checks that ϕ is a homeomorphism just as in the proof of Theorem (2.1). The map ϕ is certainly surjective and moreover, $\phi(f) = \phi(g)$ if and only if $\alpha \circ f \circ \alpha = \alpha \circ g \circ \alpha$. Consequently, $S(X, Y, \alpha)/d$ is isomorphic to $A(\alpha)$.

This result will be used in the next section when we get an isomorphism theorem for preapproximation systems without the requirement that they have identities.

3. PREAPPROXIMATION SYSTEMS

We recall that if, in the sandwich semigroup $S(X, Y, \alpha)$, we have $X = Y$, then the quotient semigroup $S(X, Y, \alpha)/d$ is denoted more simply by $S(X, \alpha)/d$ and is referred to as a preapproximation system. For these semigroups, we can extend somewhat the first two theorems of the previous section. Specifically, we have

Theorem (3.1). *The following statements about a preapproximation system $S(X, \alpha)/d$ are equivalent*

(3.1.1) *$S(X, \alpha)/d$ has an identity.*

(3.1.2) *$S(X, \alpha)/d$ has a left identity.*

(3.1.3) Ran α *is a retract of X and α maps some subspace of X homeomorphically onto* Ran α.

(3.1.4) *$S(X, \alpha)/d$ and $S(\text{Ran}\ \alpha)$ are isomorphic.*

(3.1.5) *α is a regular element of $S(X)$.*

Theorem (3.2). *Let $S(X, \alpha)/d$ and $S(X, \beta)/d$ be two preapproximation systems on X and suppose that both have identities. Moreover, suppose that both* Ran α *and* Ran β *are generated spaces. Then the following statements are equivalent*

(3.2.1) *$S(X, \alpha)/d$ and $S(X, \beta)/d$ are isomorphic.*

(3.2.2) *$S(\text{Ran}\ \alpha)$ and $S(\text{Ran}\ \beta)$ are isomorphic.*

(3.2.3) Ran α *and* Ran β are homeomorphic.

(3.2.4) *α and β are $\mathscr{D}$-equivalent in $S(X)$.*

The mutual equivalence of (3.1.1) to (3.1.4) inclusive follows immediately from Theorem (2.1). Theorem (3.1) of [7] assures us that conditions (3.1.3) and (3.1.5) are equivalent. Similarly, it follows from Theorem (2.2) that (3.2.1) to (3.2.3) inclusive are equivalent and Theorem (3.2) of [7] allows us to conclude that (3.2.3) and (3.2.4) are equivalent.

Some remarks are in order at this point. First of all, it seems to be almost poetic that $S(X, \alpha)/d$ and $S(X, \beta)/d$ are isomorphic if and only if α and β are $\mathscr{D}$-equivalent. After all the d-congruence (together with Symons' l and r congruences) bears a striking resemblence to Green's $\mathscr{D}$-equivalence as Symons [14] shows in his paper. Secondly, it is immediate that an approximation system on a space may well be isomorphic to a preapproximation system on that same space which is not an approximation system. In fact, we have the following

Theorem (3.3). *A preapproximation system on X is isomorphic to an approximation system on X if and only if the preapproximation system has an identity.*

Proof. Necessity is evident. On the other hand, if the preapproximation system $S(X, \alpha)/d$ has an identity, it follows from Theorem (3.1) that α is a regular element of $S(X)$. Now it is well known that any regular element is $\mathscr{R}$-equivalent to an idempotent so let w be an idempotent in $S(X)$ which is $\mathscr{R}$-equivalent to α. Then it follows that $\operatorname{Ran} w = \operatorname{Ran} \alpha$ and therefore that $S(\operatorname{Ran} w)$ is isomorphic to $S(\operatorname{Ran} \alpha)$. Condition (3.1.4) now implies that $S(X, \alpha)/d$ and $S(X, w)/d$ are isomorphic.

We note that this theorem was proven without any conditions whatsoever on the space X.

As we have seen, the condition that $S(X, \alpha)/d$ have an identity is equivalent to requiring that $\operatorname{Ran} \alpha$ is a retract of X and that α map some subspace of X homeomorphically onto $\operatorname{Ran} \alpha$. Of course, many functions satisfy these conditions but at the same time, many do not, including some of the functions we wish to consider most such as odd degree polynomials from the reals onto the reals which are not homeomorphisms. We do get an isomorphism theorem for preapproximation systems without requiring them to have identities. Actually, we get two such results and they apply to odd degree polynomials from R onto R. Some definitions are in order first.

Definition (3.4). Let α be a continuous selfmap of a topological space X. We say that α is *X-compatible* if for each continuous selfmap f of X, there exists a continuous selfmap g of X which satisfies the following two conditions

(3.4.1) $g[\operatorname{Ran} \alpha] \subset \operatorname{Ran} \alpha$,

(3.4.2) $\alpha \circ g = \alpha \circ f$.

Note that any continuous function mapping X *onto* X is X-compatible. One need only take $g = f$. Consequently, all odd degree polynomials from R onto R are R-compatible. If α is an idempotent continuous self map of X, then it is X-compatible. Just take $g = \alpha \circ f$. The next result gives us a recipe for finding still further, examples.

Lemma (3.5). *Let α be a continuous selfmap of a topological space X. For every continuous selfmap f of X, there exists a continuous selfmap g of X satisfying*

(3.5.1) $\operatorname{Ran} g \subset \operatorname{Ran} \alpha$,

(3.5.2) $\alpha \circ g = \alpha \circ f$,

if and only if there exists a continuous selfmap β of X such that $\operatorname{Ran} \beta \subset \operatorname{Ran} \alpha$ *and* $\alpha \circ \beta = \alpha$.

Proof. Suppose such a map β exists and let $f \in S(X)$ be given. Define $g = \beta \circ f$. It is immediate that both (3.5.1) and (3.5.2) are satisfied. Conversely, suppose that for each $f \in S(X)$ there exists a $g \in S(X)$ satisfying both (3.5.1) and (3.5.2). Take β to be the function one gets when f is the identity map. Evidently $\operatorname{Ran} \beta \subset \operatorname{Ran} \alpha$ and $\alpha \circ \beta = \alpha$.

From the preceeding lemma we immediately get

Corollary (3.6). *Let α be a continuous selfmap of X and suppose there exists a continuous selfmap β of X such that* $\operatorname{Ran} \beta \subset \operatorname{Ran} \alpha$ *and* $\alpha \circ \beta = \alpha$. *Then α is X-compatible.*

Example (3.7). Let t be any continuous map of the closed interval $[0, 1/2]$ *onto* itself and define a continuous selfmap α of the closed unit interval I by $\alpha(x) = t(x)$ for $0 \leqq x \leqq 1/2$ and $\alpha(x) = t(1 - x)$ for $1/2 \leqq x \leqq 1$. The map α is I-compatible. To see this, define $\beta(x) = x$ for $0 \leqq x \leqq 1/2$ and $\beta(x) = 1 - x$ for $1/2 \leqq x \leqq 1$. Then $\operatorname{Ran} \beta = \operatorname{Ran} \alpha = [0, 1/2]$ and $\alpha \circ \beta = \alpha$. The conclusion now follows from Corollary (3.6).

Definition (3.8). A subspace Y of a topological space X is said to be *properly embedded* in X if each continuous map from Y into X can be extended to a continuous selfmap of X.

Any closed subspace of an absolute retract X is properly embedded in X and the same is true for closed subspaces of 0-dimensional metric spaces [1, p. 169]. For an entirely different type of example, note that every completely regular Hausdorff space is properly embedded in its Stone-Čech compactification.

Definition (3.9). A preapproximation system $S(X, \alpha)/d$ is said to be *admissible* if the following conditions are satisfied:

(3.9.1) α is X-compatible.
(3.9.2) $\operatorname{Ran} \alpha$ is a properly embedded, generated subspace of X.
(3.9.3) $\{\alpha \circ f : f \in S(X)\}$ separates points of $\operatorname{Ran} \alpha$.

And now we are in a position to state and prove our next isomorphism theorem.

Theorem (3.10). *Two admissible preapproximation systems $S(X, \alpha)/d$ and $S(Y, \beta)/d$ are isomorphic if and only if there exists a homeomorphism h from* $\operatorname{Ran} \alpha$ *onto* $\operatorname{Ran} \beta$ *and continuous selfmaps f and g of* $\operatorname{Ran} \alpha$ *and* $\operatorname{Ran} \beta$ *respectively such that the following two conditions are satisfied.*

(3.10.1) $h \circ \alpha = \beta \circ g \circ h$ *on* $\operatorname{Ran} \alpha$.
(3.10.2) $\beta \circ h = h \circ \alpha \circ f$ *on* $\operatorname{Ran} \alpha$.

Proof. Suppose first that functions exist satisfying both (3.10.1) and (3.10.2). Define a mapping ϕ on the associative $A(\alpha)$ of α by $\phi(k) = h \circ k \circ h^{-1}$. Now any $k \in A(\alpha)$ is of the form $k = \alpha \circ (t/\operatorname{Ran} \alpha)$ for some $t \in S(X)$. By X-compatibility of α we may assume $t[\operatorname{Ran} \alpha] \subset \operatorname{Ran} \alpha$. We use this and (3.10.1) to get

$$\phi(k) = \phi(\alpha \circ (t/\operatorname{Ran} \alpha)) = h \circ \alpha \circ (t/\operatorname{Ran} \alpha) \circ h^{-1} = \beta \circ g \circ h \circ (t/\operatorname{Ran} \alpha) \circ h^{-1}.$$

Now $g \circ h \circ (t/\mathrm{Ran}\,\alpha) \circ h^{-1}$ is a continuous selfmap of $\mathrm{Ran}\,\beta$ and since the latter is properly embedded in Y, there exists a continuous selfmap v of Y such that

$$v/\mathrm{Ran}\,\beta = g \circ h \circ (t/\mathrm{Ran}\,\alpha) \circ h^{-1} .$$

Thus, $\phi(k) = \beta \circ (v/\mathrm{Ran}\,\beta)$. That is to say, ϕ maps $A(\alpha)$ into $A(\beta)$.

Now let k be any element in $A(\beta)$. Then $k = \beta \circ (t/\mathrm{Ran}\,\beta)$ for some $t \in S(Y)$. Again we may assume that $t[\mathrm{Ran}\,\beta] \subset \mathrm{Ran}\,\beta$. The function $f \circ h^{-1} \circ (t/\mathrm{Ran}\,\beta) \circ h$ is a continuous selfmap of $\mathrm{Ran}\,\alpha$ and since the latter is properly embedded in X, there exists a continuous selfmap v of X such that

$$v/\mathrm{Ran}\,\alpha = f \circ h^{-1} \circ (t/\mathrm{Ran}\,\beta) \circ h .$$

Thus, $\alpha \circ (v/\mathrm{Ran}\,\alpha) \in A(\alpha)$. We use (3.10.2) and we get

$$\begin{aligned}\phi(\alpha \circ (v/\mathrm{Ran}\,\alpha)) &= h \circ \alpha \circ (v/\mathrm{Ran}\,\alpha) \circ h^{-1} = \\ &= (h \circ \alpha \circ f) \circ h^{-1} \circ (t/\mathrm{Ran}\,\beta) \circ h \circ h^{-1} = \\ &= (\beta \circ h) \circ h^{-1} \circ (t/\mathrm{Ran}\,\beta) = \beta \circ (t/\mathrm{Ran}\,\beta) = k .\end{aligned}$$

Consequently, ϕ maps $A(\alpha)$ onto $A(\beta)$. Since ϕ is given by $\phi(k) = h \circ k \circ h^{-1}$ for each $k \in A(\alpha)$ where h is a homeomorphism from $\mathrm{Ran}\,\alpha$ onto $\mathrm{Ran}\,\beta$, it readily follows that ϕ is a bijection from $A(\alpha)$ onto $A(\beta)$ and, moreover, is an isomorphism. It now follows from Theorem (2.4) that $S(X, \alpha)/d$ and $S(Y, \beta)/d$ are isomorphic.

Conversely assume that $S(X, \alpha)/d$ and $S(Y, \beta)/d$ are isomorphic. Then $A(\alpha)$ and $A(\beta)$ are isomorphic by Theorem (2.4). Let ϕ be any isomorphism from $A(\alpha)$ onto $A(\beta)$. For a point $x \in \mathrm{Ran}\,\alpha$, let $\langle x \rangle$ denote the constant function whose domain is $\mathrm{Ran}\,\alpha$ and which maps everything into the point x. Let $\langle\langle x \rangle\rangle$ denote the constant function which maps all of X into the point x. Similarly, for $y \in \mathrm{Ran}\,\beta$, $\langle y \rangle$ maps all of $\mathrm{Ran}\,\beta$ into y and $\langle\langle y \rangle\rangle$ maps all of Y into y. We assert that

(3.10.3) $\langle x \rangle \in A(\alpha)$ for all $x \in \mathrm{Ran}\,\alpha$

and

(3.10.4) $\langle y \rangle \in A(\beta)$ for all $y \in \mathrm{Ran}\,\beta$.

It is sufficient to verify (3.10.3). Let $x \in \mathrm{Ran}\,\alpha$ be given. Then $x = \alpha(y)$ for some $y \in X$. Now α is X-compatible and we take the f in Definition (3.4) to be $\langle\langle y \rangle\rangle$. Then there exists a continuous selfmap k of X such that $k[\mathrm{Ran}\,\alpha] \subset \mathrm{Ran}\,\alpha$ and $\alpha \circ \langle\langle y \rangle\rangle = \alpha \circ k$. But $\alpha \circ \langle\langle y \rangle\rangle = \langle\langle \alpha(y) \rangle\rangle = \langle\langle x \rangle\rangle$ and it readily follows that $\langle x \rangle = \alpha \circ (k/\mathrm{Ran}\,\alpha)$. This proves (3.10.3) and, of course, (3.10.4) follows in the same manner. Next, we assert that

(3.10.5) An element $t \in A(\alpha)$ $(A(\beta))$ is a left zero of $A(\alpha)$ $(A(\beta))$ if and only if $t = \langle x \rangle$ for some $x \in \mathrm{Ran}\,\alpha$ $(\mathrm{Ran}\,\beta)$.

Sufficiency follows immediately from (3.10.3) and (3.10.4). As for necessity, suppose t is a left zero of $A(\alpha)$ and take any $x, y \in \mathrm{Ran}\,\alpha$. Then $\langle x \rangle$ and $\langle t(x) \rangle$ both belong to $A(\alpha)$ and we get

$$t(x) = \langle t(x)\rangle\,(y) = (t \circ \langle x\rangle)\,(y) = t(y)$$

which implies $t = \langle t(y)\rangle$.

Now the isomorphism ϕ must map the left zeros of $A(\alpha)$ bijectively onto the left zeros of $A(\beta)$ so that for each $x \in \operatorname{Ran} \alpha$, there exists a unique $y \in \operatorname{Ran} \beta$ such that $\phi\langle x\rangle = \langle y\rangle$. Define a bijection h from $\operatorname{Ran} \alpha$ onto $\operatorname{Ran} \beta$ by $h(x) = y$ and note that

(3.10.6) $\phi\langle x\rangle = \langle h(x)\rangle$ for each $x \in \operatorname{Ran} \alpha$.

Now let any $k \in A(\alpha)$ and $y \in \operatorname{Ran} \beta$ be given. We are assured by (3.10.3) and (3.10.4) that all constant functions associated with points in $\operatorname{Ran} \alpha$ and $\operatorname{Ran} \beta$ belong to $A(\alpha)$ and $A(\beta)$ respectively. We use this fact and (3.10.6) several times when we write the following string of equalities

$$\begin{aligned} h \circ k \circ h^{-1}(y) &= \langle h(k(h^{-1}(y)))\rangle\,(y) = \\ &= \phi\langle k(h^{-1}(y))\rangle\,(y) = \phi(k \circ \langle h^{-1}(y)\rangle)\,(y) = \\ &= (\phi(k) \circ \phi\langle h^{-1}(y)\rangle)\,(y) = (\phi(k) \circ \langle y\rangle)\,(y) = \phi(k)\,(y)\,. \end{aligned}$$

Consequently, we have

(3.10.7) $\phi(k) = h \circ k \circ h^{-1}$ for each $k \in A(\alpha)$.

Next, we assert that

(3.10.8) $h[k^{-1}(x)] = (\phi(k))^{-1}(h(x))$

for each $k \in A(\alpha)$ and $x \in \operatorname{Ran} \alpha$. Suppose $y \in h[k^{-1}(x)]$. Then $y = h(z)$ for some $z \in k^{-1}(x)$. Thus,

$$x = k(z) = k \circ h^{-1}(y)$$

and we use (3.10.7) to get

$$h(x) = h \circ k \circ h^{-1}(y) = \phi(k)\,(y)\,.$$

Consequently, $y \in (\phi(k))^{-1}\,(h(x))$. One then reverses the argument and concludes that (3.10.8) is valid. In a similar manner, one verifies

(3.10.9) $h^{-1}[t^{-1}(y)] = (\phi^{-1}(t))^{-1}\,(h^{-1}(y))$

for each $t \in A(\beta)$ and $y \in \operatorname{Ran} \beta$.

Our next task is to show that $\{k^{-1}(x) : k \in A(\alpha),\ x \in \operatorname{Ran} \alpha\}$ is a subbasis for the closed subsets of $\operatorname{Ran} \alpha$. Of course, the analogous family will also form a subbasis for the closed subsets of $\operatorname{Ran} \beta$ and this together with (3.10.8) and (3.10.9) will imply that h is a homeomorphism. So let H be a closed subset of $\operatorname{Ran} \alpha$ and let p be a point of $\operatorname{Ran} \alpha$ such that $p \notin H$. Since $\operatorname{Ran} \alpha$ is a generated space, there exists a finite number of continuous selfmaps $\{f_i\}_{i=1}^N$ of $\operatorname{Ran} \alpha$ and points $\{z_i\}_{i=1}^N$ of $\operatorname{Ran} \alpha$ such that

(3.10.10) $H \subset \bigcup\limits_{i=1}^{N} \{f_i^{-1}(z_i)\}$,

(3.10.11) $p \in \bigcap\limits_{i=1}^{N} \mathscr{C}\{f_i^{-1}(z_i)\}$.

Since Ran α is properly embedded in X, each f_i can be extended to a continuous selfmap $\hat{f}_i$ of X. Now (3.10.11) implies that $\hat{f}_i(p) \neq z_i$ for every $i = 1, 2, \ldots, N$. By (3.9.3), for each $i = 1, 2, \ldots, N$, there exists a continuous selfmap g_i of X such that $\alpha \circ g_i(\hat{f}_i(p)) \neq \alpha \circ g_i(z_i)$. By X-compatibility of α we may assume that $g_i(\text{Ran}\ \alpha) \subset$ $\subset$ Ran α. Let $k_i = \alpha \circ (g_i \circ \hat{f}_i \,/\, \text{Ran}\ \alpha)$ and $y_i = \alpha \circ g_i(z_i)$. Then $k_i \in A(\alpha)$ and

(3.10.12) $H \subset \bigcup_{i=1}^{N} k_i^{-1}(y_i)$,

while

(3.10.13) $p \in \bigcap_{i=1}^{N} \mathscr{C} k_i^{-1}(y_i)$.

This proves that $\{k^{-1}(x) : k \in A(\alpha),\ x \in \text{Ran}\ \alpha\}$ does indeed form a subbasis for the closed subsets of Ran α and it follows that h is a homeomorphism from Ran α onto Ran β.

Now let $\hat{\alpha} = \alpha/\text{Ran}\ \alpha$. Then $\hat{\alpha} = \alpha \circ (i/\text{Ran}\ \alpha)$ where i is the identity map on X. This means that $\hat{\alpha} \in A(\alpha)$ and with (3.10.7) we get

(3.10.14) $h \circ \hat{\alpha} \circ h^{-1} = \phi(\hat{\alpha}) = \beta \circ (k/\text{Ran}\ \beta)$

for some $k \in S(Y)$. Since β is Y-compatible, there exists a function $t \in S(Y)$ such that $t[\text{Ran}\ \beta] \subset \text{Ran}\ \beta$ and $\beta \circ t = \beta \circ k$. It follows that

(3.10.15) $h \circ \hat{\alpha} \circ h^{-1} = \beta \circ (t/\text{Ran}\ \beta)$.

We now take $g = t/\text{Ran}\ \beta$ and this is a continuous selfmap of Ran β ($k/\text{Ran}\ \beta$ may possibly map points in Ran β outside of Ran β). This results in

(3.10.16) $h \circ \hat{\alpha} \circ h^{-1} = \beta \circ g$

and (3.10.1) readily follows.

To get the function f and to verify that (3.10.2) holds, one notes that $\hat{\beta} = \beta/\text{Ran}\ \beta$ belongs to $A(\beta)$ and gets

(3.10.17) $h^{-1} \circ \beta \circ h = \phi^{-1}(\hat{\beta}) = \alpha \circ (k/\text{Ran}\ \alpha)$

for some $k \in S(X)$ and then proceeds exactly as above. Therefore conditions (3.10.1) and (3.10.2) are not only sufficient but also necessary and the proof is complete.

For certain spaces and certain functions, we can simplify somewhat the conditions (3.10.1) and (3.10.2). A definition is in order.

Definition (3.11). A topological space is *reversible* if each continuous bijection of the space onto itself is a homeomorphism.

According to results in [12], reversible spaces include not only all compact Hausdorff spaces but all locally Euclidean spaces as well.

We need a lemma before we can prove our next isomorphism theorem. It concerns finite-to-one functions. By that, we mean simply that preimages of points are finite. In particular, we do not require that there be an upper bound for the cardinalities of these sets. The lemma we are about to prove although not formally stated in [8] was verified in the course of proving Theorem (2.7) of that paper. The proof we give here seems to be shorter and simpler.

Lemma (3.12). *Let X be any set and let α be any finite-to-one function mapping X into X. Let t be any function mapping X onto X and suppose that $\alpha \circ t = \alpha$. Then t is a permutation of X.*

Proof. We must show that t is injective. Suppose, to the contrary, that $t(a) = t(b) = x_1$ where $a \neq b$. Now $\alpha^{-1}(\alpha(x_1)) = \{x_1, x_2, \ldots, x_N\}$ for some positive integer N. Then $\alpha \circ t = \alpha$ implies

$$\{x_1, x_2, \ldots, x_N\} = \alpha^{-1}\ (\alpha(x_1)) = t^{-1}(\alpha^{-1}(\alpha(x_1))) = t^{-1}\{x_1, x_2, \ldots, x_N\}\,.$$

However, card $\{x_1, x_2, \ldots, x_N\} = N$ while card $t^{-1}\{x_1, x_2, \ldots, x_N\} = \sum_{i=1}^{N} \text{card}\,.\, t^{-1}(x_i) \geqq N + 1$ since card $t^{-1}(x_1) \geqq 2$ and card $t^{-1}(x_i) \geqq 1$ for $i > 1$. This contradiction implies that t is injective.

Theorem (3.13). *Let $S(X, \alpha)/d$ and $S(Y, \beta)/d$ be two admissible preapproximation systems and let X and Y be reversible. Suppose that α is finite-to-one and maps X onto X but does not map any proper subspace onto X which is the range of a continuous selfmap of X. Suppose that the analogous condition holds for β Then $S(X, \alpha)/d$ and $S(Y, \beta)/d$ are isomorphic if and only if there exist two homeomorphisms h and k from X onto Y such that $h \circ \alpha = \beta \circ k$.*

Proof. Suppose that $S(X, \alpha)/d$ and $S(Y, \beta)/d$ are isomorphic. Since the hypothesis of Theorem (3.10) is satisfied, there exists a homeomorphism h from X onto Y and continuous selfmaps f and g of X and Y respectively such that

(3.13.1) $h \circ \alpha = \beta \circ g \circ h$
and
(3.13.2) $\beta \circ h = h \circ \alpha \circ f$.

From (3.13.1) we get $\alpha = h^{-1} \circ \beta \circ g \circ h$ and from (3.13.2) we get $\beta = h \circ \alpha \circ f \circ h^{-1}$. These equalities, in turn, lead to

(3.13.3) $\alpha = \alpha \circ (f \circ h^{-1} \circ g \circ h)$
and
(3.13.4) $\beta = \beta \circ (g \circ h \circ f \circ h^{-1})$.

Since α does not map any proper subspace of X onto X which is the range of a continuous selfmap of X, it follows that $f \circ h^{-1} \circ g \circ h$ maps X onto X. Lemma (3.12) now applies and we conclude that

(3.13.5) $f \circ h^{-1} \circ g \circ h$ is a bijection from X onto X, and similarly,
(3.13.6) $g \circ h \circ f \circ h^{-1}$ is a bijection from Y onto Y.

Let us consider (3.13.5). Since h maps X bijectively onto Y, it follows immediately that g is injective. Suppose, however, that g does *not* map Y *onto* Y. Then $h^{-1} \circ g \circ$

$\circ\, h[x]$ is a proper subset of X which, in view of (3.13.5), f maps *onto* X. But this means f is not injective which, together with (3.13.6) results in a contradiction. Consequently, g does map Y onto Y. Since g is a continuous bijection of Y and Y is reversible, it follows that g is a homeomorphism from Y onto Y. Thus $k = g \circ h$ is a homeomorphism from X onto Y which, by (3.13.1) satisfies $h \circ \alpha = \beta \circ k$.

Conversely, suppose there exist homeomorphisms h and k from X onto Y such that $h \circ \alpha = \beta \circ k$. Define $f = k^{-1} \circ h$ and $g = k \circ h^{-1}$. One then verifies that $h \circ \alpha = \beta \circ g \circ h$ and $\beta \circ h = h \circ \alpha \circ f$ and it follows from Theorem (3.10) that $S(X, \alpha)/d$ and $S(Y, \beta)/d$ are isomosphic.

From the previous theorem, we immediately get the following

Corollary (3.14). *Let R be the space of real numbers and let P and Q be two odd degree polynomials on R. Then the two preapproximation systems $S(R, P)/d$ and $S(R, Q)/d$ are isomorphic if and only if there exist two homeomorphisms h and k from R onto R such that $h \circ P = Q \circ k$.*

4. APPLICATION OF PREVIOUS RESULTS

Corollary (3.14) combines with a theorem from [6] to produce a result which permits us to determine whether or not $S(R, P)/d$ and $S(R, Q)/d$ are isomorphic simply by inspecting the local maxima and minima of P and Q. We need some definitions before we can state the result.

Definition (4.1). Let P be any odd degree polynomial with $\{a_i\}_{i=0}^N$ denoting the x-coordinates, in increasing order, of the local maxima and minima. Denote the corresponding y-coordinates by $\{y_i\}_{i=0}^N$. Let $\{y_{j_i}\}_{i=0}^N$ be the rearrangement of the y-coordinates of the local maxima and minima so that $y_{j_i} \leqq y_{j_{i+1}}$ and if $y_{j_i} = y_{j_{i+1}}$, then $j_i < j_{i+1}$. The sequence of subscripts $\{j_i\}_{i=0}^N$ is referred to as the *direct signature* of the polynomial P.

Definition (4.2). Now let $\{y_{k_i}\}_{i=0}^N$ be the rearrangement of the y-coordinates of the local maxima and minima of P so that $y_{k_i} \leqq y_{k_{i+1}}$ but now require that $k_{i+1} < k_i$ if $y_{k_i} = y_{k_{i+1}}$. This sequence of subscripts $\{k_i\}_{i=0}^N$ is referred to as the *reverse signature* of the polynomial P.

Let P and Q be odd degree polynomials. It follows immediately from corollary (3.14) that if one of them is a homeomorphism (which is equivalent to saying that it has no local maxima or minima) then the preapproximation systems $S(R, P)/d$ and $S(R, Q)/d$ are isomorphic if and only if the other is a homeomorphism also. Consequently we may give our attention to the case where both P and Q do have local maxima and minima. In what follows, $\{a_i\}_{i=0}^M$ will denote the x-coordinates in increasing order of the local maxima and minima of P and $\{y_i\}_{i=0}^M$ will denote the corresponding y-coordinates. The sequences $\{b_i\}_{i=0}^N$ and $\{z_i\}_{i=0}^N$ will have the analogous relationship to the polynomial Q. The sequences $\{j_i\}_{i=0}^{M}$ and $\{k_i\}_{i=0}^{M}$ will denote respectively the direct and reverse signatures of P and $\{t_i\}_{i=0}^{N}$ will denote the direct signature of Q.

We can now state

Theorem (4.3). *The preapproximation systems $S(R, P)/d$ and $S(R, Q)/d$ are isomorphic if and only if $M = N$ and at least one of the four following conditions holds*:

(4.3.1) $j_i = t_i$ *for each i and* $y_{j_i} = y_{j_{i+1}}$ *implies* $z_{j_i} = z_{j_{i+1}}$,

(4.3.2) $k_i = t_{N-i}$ *for each i and* $y_{k_i} = y_{k_{i+1}}$ *implies* $z_{k_i} = z_{k_{i+1}}$,

(4.3.3) $k_i + t_i = N$ *for each i and* $y_{k_i} = y_{k_{i+1}}$ *implies* $z_{N-k_i} = z_{N-k_{i+1}}$,

(4.3.4) $j_i + t_{N-i} = N$ *for each i and* $y_{j_i} = y_{j_{i+1}}$ *implies* $z_{N-j_i} = z_{N-j_{i+1}}$.

Proof. According to Corollary (3.14), $S(R, P)/d$ and $S(R, Q)/d$ are isomorphic if and only if there exist homeomorphisms h and k from R onto R such that $h \circ P = Q \circ k$ and according to Theorem (3.8) of [6] the latter condition holds if and only if $M = N$ and at least one of the four conditions listed above holds.

In certain instances the latter result can be stated somewhat more simply. In particular, the direct and reverse signatures of a polynomial coincide whenever the functional values at its local maxima and minima are all distinct. As a consequence, the next result follows immediately from the previous theorem.

Corollary (4.4). *Suppose the functional values of the local maxima and minima of P are all distinct and that the same is true of Q. Then the preapproximation systems $S(R, P)/d$ and $S(R, Q)/d$ are isomorphic if and only if $M = N$ and at least one of the four following conditions is satisfied*:

(4.4.1) $j_i = t_i$ *for each i.*

(4.4.2) $j_i = t_{N-i}$ *for each i.*

(4.4.3) $j_i + t_i = N$ *for each i.*

(4.4.4) $j_i + t_{N-i} = N$ *for each i.*

We discuss several examples and we continue to assume that neither P nor Q is a homeomorphism. If both P and Q are cubics then the latter corollary applies. The direct signature of P will be either $\{0, 1\}$ or $\{1, 0\}$ and the same is true of Q. Consequently, either (4.4.1) is satisfied or (4.4.3) is satisfied and $S(R, P)/d$ and $S(R, Q)/d$ will be isomorphic. We borrow some examples of fifth degree polynomials from [6]. Let

$$P(x) = 6x^5 - 75x^4 + 350x^3 - 750x^2 + 720x - 210\,,$$

$$Q(x) = 2x^5 - 35x^4 + 210x^3 - 530x^2 + 560x + 100\,,$$

$$T(x) = 2x^5 + 35x^4 + 210x^3 + 530x^2 + 560x + 100\,.$$

The x-coordinates and corresponding y-coordinates of the local maxima and minima of these three polynomials are as follows:

$$
\begin{aligned}
P\colon\ & x_0 = 1, & x_1 &= 2, & x_2 &= 3, & x_3 &= 4,\\
& y_0 = 41, & y_1 &= 22, & y_2 &= 33, & y_3 &= 14,\\
Q\colon\ & x_0 = 1, & x_1 &= 2, & x_2 &= 4, & x_3 &= 7,\\
& y_0 = 307, & y_1 &= 284, & y_2 &= 388, & y_3 &= -341,\\
T\colon\ & x_0 = -7, & x_1 &= -4, & x_2 &= -2, & x_3 &= -1,\\
& y_0 = 541, & y_1 &= -188, & y_2 &= -84, & y_3 &= -107\,.
\end{aligned}
$$

The direct signatures $S_D(P)$, $S_D(Q)$ and $S_D(T)$ then turn out to be

$$
\begin{aligned}
S_D(P) &= \{3, 1, 2, 0\}\,,\\
S_D(Q) &= \{3, 1, 0, 2\}\,,\\
S_D(T) &= \{1, 3, 2, 0\}\,.
\end{aligned}
$$

Since the functional values of the local maxima and minima are all distinct, we can again apply Corollary (4.4). One easily checks that the pairs $S_D(P)$, $S_D(Q)$ and $S_D(P)$, $S_D(T)$ satisfy none of the conditions of Corollary (4.4) so that $S(R, P)/d$ is not isomorphic to either of $S(R, Q)/d$ or $S(R, T)/d$. However, $S_D(Q)$ and $S_D(T)$ do satisfy (4.4.4) so that the preapproximation systems $S(R, Q)/d$ and $S(R, T)/d$ are isomorphic.

Before we turn our attention to approximation systems, we prove one more result about preapproximation systems. It relates the automorphism group of such a semigroup to the maximal subgroup containing the sandwich function.

Theorem (4.5). *Suppose that α belongs to some subgroup of $S(X)$ and suppose further that* Ran α *is a generated space. Then* Aut $S(X, \alpha)/d$ *is isomorphic to the maximal subgroup in $S(X)$ which contains α.*

Proof. According to [4], Ran α is a retract of X and α maps Ran α homeomorphically onto itself. Moreover, the maximal subgroup in $S(X)$ to which α belongs is isomorphic to $G(\text{Ran}\ \alpha)$, the group of all homeomorphisms on Ran α. According to Theorem (2.1), $S(X, \alpha)/d$ is isomorphic to $S(\text{Ran}\ \alpha)$ and hence Aut $S(X, \alpha)/d$ is isomorphic to Aut $S(\text{Ran}\ \alpha)$. It follows from Theorem (2.3) p. 198 of [5] that Aut $S(\text{Ran}\ \alpha)$ is isomorphic to $G(\text{Ran}\ \alpha)$ and the proof is complete.

And now we focus our attention on approximation systems, i.e., preapproximation systems $S(X, \alpha)/d$ where the sandwich function α is idempotent. Our next result will give us information about the number of approximation systems a particular space supports. Some additional notation is convenient. For any topological space X, let $\mathscr{R}(X)$ denote the collection of all retracts of X, let $\mathscr{A}(X)$ denote the collection of approximation systems on X and let $\Delta(S(X))$ denote the family of all regular $\mathscr{D}$-classes of $S(X)$. Declare two retracts in $\hat{\mathscr{R}}(X)$ to be equivalent if they are homeomorphic and denote the resulting collection of equivalence classes by $\hat{\mathscr{R}}(X)$. Similarly, declare two isomorphic approximation systems on X to be equivalent and denote that collection of equivalence classes by $\hat{\mathscr{A}}(X)$. We then have

Theorem (4.6). *Suppose that X is a generated space. Then* card $(\hat{\mathscr{A}}X) =$ $=$ card $\hat{\mathscr{R}}(X) =$ card $\Delta(S(X))$.

Proof. Let $S(X, \alpha)/d$ be a approximation system and let $[S(X, \alpha)/d]$ denote the equivalence class in $\hat{\mathscr{A}}(X)$ to which $S(X, \alpha)/d$ belongs. Define $\Gamma[S(X, \alpha)/d] =$ $= [\text{Ran } \alpha]$ where $[\text{Ran } \alpha]$ is the equivalence class in $\hat{\mathscr{R}}(X)$ to which Ran α belongs. It follows from Theorem (3.2) that Γ is well defined and injective. To see that it is onto let W be any retract of X. Then $W = \text{Ran } w$ for some idempotent $w \in S(X)$. Hence $S(X, w)/d$ is an approximation system on X and $\Gamma[S(X, w)/d] = [W]$.

Now for any retract H of X, let $D_H = \{f \in S(X) : f$ is regular and Ran f is homeomorphic to $H\}$. By Theorem (3.3) of [7], $\{D_H : H$ is a retract of $X\} = \mathscr{E}$ is precisely the collection of regular $\mathscr{D}$-classes of $S(X)$. Define a mapping ϕ from $\hat{\mathscr{A}}(X)$ into $\mathscr{E}$ by $\phi[S(x, \alpha)/d] = D_{\text{Ran}\,\alpha}$. In this case also Theorem (3.2) assures us that ϕ is a bijection from $\hat{\mathscr{A}}(X)$ onto $\mathscr{E}$.

In the statements of the following results, we will not distinguish between isomorphic approximation systems. The first one follows quite quickly from Theorem (4.6).

Theorem (4.7). *Let X be any Peano continuum. Then X supports exactly two approximation systems if and only if X is an arc.*

Proof. According to Theorem (4.6) it is sufficient to show that X contains (up to homeomorphism) exactly two retracts if and only if X is an arc. If X is an arc it is immediate that it has only two retracts, namely a point and an arc. On the other hand, suppose X has exactly two retracts. Then it must be nondegenerate and since it is a Peano continuum, it contains an arc A. Certainly A is a retract of X and any point of X is also a retract of X. Since X is also a retract of itself, it must be an arc, otherwise X would contain at least three retracts.

Remark. It is easy to see that the previous theorem is valid if one requires only that X contains an arc.

We recall that a triod is the union of three arcs all of which have precisely one endpoint in common.

Theorem (4.8). *Let X be a Peano continuum. Then X supports exactly three approximation systems if and only if X is either a triod or a simple closed curve.*

Proof. By Corollary 2 of [11], X has exactly three regular $\mathscr{D}$-classes if and only if it is either a triod or a simple closed curve. The result now follows from Theorem (4.6).

We conclude the paper with two more corollaries which follow immediately from the previous theorem.

Corollary (4.9). *Let X be a Peano continuum with at least one cut point. Then X supports exactly three approximation systems if and only if X is a triod.*

Corollary (4.10). *Let X be a Peano continuum with no cut points. Then X supports exactly three approximation systems if and only if it is a simple closed curve.*

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