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ON K-RADICAL CLASSES OF LATTICE ORDERED GROUPS

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The notion of radical class of lattice ordered groups was introduced in the author's paper [4]; cf. also [5]. A radical class X which can be defined by means of the properties of the lattice of closed convex l-subgroups of l-groups belonging to X is said to be a *K-radical class* (P. Conrad [1]).

Examples of K-radical classes are: the class X_A of all archimedean l-groups; the class X_B of all l-groups with a basis; the class X_C of all completely distributive l-groups; the class S_t of all l-groups G such that the lattice of all closed convex l-subgroups of G is a Stone lattice. These and some other examples of K-radical classes were thoroughly studied by Conrad [1].

Let $\mathscr{R}$ and $\mathscr{R}_K$ be the collections of all radical classes and K-radical classes, respectively. Both $\mathscr{R}$ and $\mathscr{R}_K$ are partially ordered by inclusion. Then $\mathscr{R}$ and $\mathscr{R}_K$ are complete lattices. The lattice $\mathscr{R}$ was investigated in [4]. In the present paper the lattice $\mathscr{R}_K$ will be dealt with.

For $X \in \mathscr{R}_K$ we denote by $a(X)$ the class of all elements of $\mathscr{R}_K$ covering X; further, let $a_1(X)$ be the class of all $Y \in \mathscr{R}_K$ such that $X < Y$ and no element of the interval $[X, Y]$ covers X. Let R_0 and $\mathscr{G}$ be the least element and the greatest element of $\mathscr{R}_K$, respectively. If $X \in \mathscr{R}_K$ is generated by a one-element class, then X is said to be *principal.*

Sample results: It will be shown that $\mathscr{R}_K$ fails to be a closed sublattice of $\mathscr{R}$. It will be proved that $\mathscr{R}_K$ is a Brouwer lattice and for each subclass X of $\mathscr{G}$ the K-radical class generated by X is equal to Join Lat Sub X (for denotations, cf. Sec. 1 below). There exist K-radical classes X_1, X_2 distinct from $\mathscr{G}$ such that $a(X_1) = \emptyset$ and $a_1(X_2) = \emptyset$. If $X \neq R_0$ is a principal K-radical class, then both $a(X)$ and $a_1(X)$ are nonempty (in fact, they are proper classes). For each $X \in \{X_A, X_B, X_C, S_t\}$ the class $a(X)$ is nonempty. K-radical classes generated by linearly ordered groups will be examined.

1. PRELIMINARIES

Let $\mathscr{G}$ be the class of all lattice ordered groups. When considering a subclass Y of $\mathscr{G}$ we always assume that Y is closed with respect to isomorphisms and that the zero group $\{0\}$ belongs to Y.

A subclass X of $\mathscr{G}$ is said to be a *radical class* if it is closed with respect to

a) convex l-subgroups, and

b) joins of convex l-subgroups.

Let $G \in \mathscr{G}$. We denote by $c(G)$ the system of all convex l-subgroups of G; the set $c(G)$ is partially ordered by inclusion. Then $c(G)$ is a complete lattice. The operation $\wedge$ in $c(G)$ coincides with the set-theoretical intersection. The join in $c(G)$ will be denoted by $\bigvee_{i\in I}^{c} G_i$.

For $G \in \mathscr{G}$ we denote by $K(G)$ the system of all closed convex l-subgroups of G. The system $K(G)$ is partially ordered by inclusion; then $K(G)$ is a complete lattice. The lattice operations in $K(G)$ will be denoted by $\wedge$ and $\vee$. For $M \subseteq G$ we denote by M^- the closed convex l-subgroup of G that is generated by M. The meet in $K(G)$ coincides with the set-theoretical intersection and for $\{G_i\}_{i\in I} \subseteq K(G)$ we have

$$\bigvee_{i\in I} G_i = \left(\bigvee_{i\in I}^{c} G_i\right)^- .$$

A subclass X of $\mathscr{G}$ is said to be a *K-class* if there exists a class T of lattices such that the equivalence

$$G \in X \Leftrightarrow K(G) \in T$$

is valid for each lattice ordered group G.

A K-class X which is at the same time a radical class is called a *K-radical class* (cf. [1]).

Let $\mathscr{R}$ and $\mathscr{R}_K$ be the collections of all radical classes and all K-radical classes, respectively. Both $\mathscr{R}$ and $\mathscr{R}_K$ are partially ordered by inclusion. The one-element class $R_0 = \{\{0\}\}$ is the least element in both $\mathscr{R}$ and $\mathscr{R}_K$, and $\mathscr{G}$ is the largest element in both $\mathscr{R}$ and $\mathscr{R}_K$.

For $X \subseteq \mathscr{G}$ we denote by

$\operatorname{Sub} X$ – the class of all convex l-subgroups of l-groups belonging to X;

$\operatorname{Join} X$ – the class of all l-groups G having a system $\{G_i\}_{i\in I}$ of closed convex l-subgroups with $G_i \in X$ for each $i \in I$ such that $\bigvee_{i\in I} G_i = G$;

$\operatorname{Join}_c X$ – the class of all l-groups G having a system $\{G_i\}_{i\in I}$ of convex l-subgroups with $G_i \in X$ for each $i \in I$ such that $\bigvee_{i\in I}^{c} G_i = G$;

$\operatorname{Lat} X$ – the class of all lattice ordered groups G such that $K(G)$ is isomorphic to $K(G_1)$ for some $G_1 \in X$;

$(X)^-$ – the class of all lattice ordered groups G having the property that there exists $G_1 \in c(G) \cap X$ such that $G = G_1^-$.

$\mathscr{R}$ is a complete lattice in which the meet coincides with the intersection of classes. (In fact, $\mathscr{R}$ is a proper class.) The join in $\mathscr{R}$ will be denoted by $\bigvee^c$. For $X \subseteq \mathscr{G}$ we denote by $T(X)$ the intersection of all $Y \in \mathscr{R}$ with $X \leqq Y$. Then $T(X)$ belongs to $\mathscr{R}$ and is said to be the *radical class generated by* X. The following two propositions were proved in [4].

1.1. Proposition. *Let* $X \subseteq \mathscr{G}$. *Then* $T(X) = \operatorname{Join}_c \operatorname{Sub} X$.

1.2. Proposition. *Let I be a nonempty class and for each $i \in I$ let $X_i \in \mathscr{R}$. Then* $\bigvee^c_{i\in I} X_i = \mathrm{Join}_c(\bigcup_{i\in I} X_i)$.

2. THE LATTICE $\mathscr{R}_K$

2.1. Proposition. *$\mathscr{R}_K$ is a complete lattice. The operation of meet in $\mathscr{R}_K$ coincides with the intersection of classes.*

Proof. Let I be a nonempty class and for each $i \in I$ let X_i be a K-radical class· Hence for each $i \in I$ there exists a class T_i of lattices such that

$$G \in X_i \Leftrightarrow K(G) \in T_i$$

is valid for zach $G \in \mathscr{G}$.

Put $X = \bigcap_{i\in I} X_i$, $T = \bigcap_{i\in I} T_i$. Then X is a radical class and for each $H \in \mathscr{G}$ we hawe

$$H \in X \Leftrightarrow K(H) \in T;$$

hence $X \in \mathscr{R}_K$. Thus X is the meet of the class $\{X_i\}_{i\in I}$ in $\mathscr{R}_K$. Because $\mathscr{R}_K$ is bounded, it is a complete lattice.

2.2. Lemma. *Let $G \in \mathscr{G}$, $H \in c(G)$. For each $H_1 \in K(H)$ and each $H'_1 \in K(H^-)$ put $\varphi(H_1) = H_1^-$, $\psi(H'_1) = H \cap H'_1$. Then φ is an isomorphism of the lattice $K(H)$ onto $K(H^-)$ and $\psi = \varphi^{-1}$.*

Proof. Let $H_1, H_2 \in K(H)$ and assume that $\varphi(H_1) = \varphi(H_2)$. Let $0 < g \in H_2$. Thus $g \in H_2^- = H_1^-$ and hence there are elements $\{x_i\}_{i\in I}$ in H_1 such that $0 < x_i$ and $\sup \{x_i\} = g$ holds in G. This implies that $\sup \{x_i\} = g$ holds in H as well and thus, because of $H_1 \in K(H)$, we have $g \in H_1$. Therefore the mapping φ is a monomosphism.

Let $H^* \in K(H^-)$. Then clearly $H^* \cap H \in K(H)$ and $\varphi(H^* \cap H) \subseteq H$. Let $0 < g \in$ $\in H^*$. There is a subset $\{x_i\}_{i\in I} \subseteq H$ with $0 < x_i$ such that $\bigvee x_i = g$ holds in G. Then $\{x_i\}_{i\in I} \subseteq H^* \cap H$ and hence $g \in \varphi(H^* \cap H)$. Therefore $\varphi(H^* \cap H) = H^*$. Hence φ is onto $K(H^-)$. At the same time we have verified that $\psi = \varphi^{-1}$.

If $H_1, H_2 \in K(H)$ and $H_1^*, H_2^* \in K(H^-)$, then

$$H_1 \subseteq H_2 \Rightarrow \varphi(H_1) \subseteq \varphi(H_2)$$

and

$$H_1^* \subseteq H_2^* \Rightarrow \psi(H_1^*) \subseteq \psi(H_2^*);$$

hence φ is an isomorphism.

2.3. Lemma. *Let $X \subseteq \mathscr{G}$, $Y =$ Lat Sub X. Then* Sub $Y = Y$.

Proof. Let $G \in Y$. There exists $G' \in \mathrm{Sub}\, X$ such that the lattice $K(G)$ is isomorphic to $K(G')$; let φ be the corresponding isomorphism. Let $H \in c(G)$. Put $H^* = \varphi(H^-)$. The lattice $K(H^*)$ coincides with the interval $[\{0\}, H^*]$ of the lattice $K(G')$; similarly, the lattice $K(H^-)$ coincides with the interval $[\{0\}, H^-]$ of the lattice $K(G)$. Hence $K(H^-)$ is isomorphic to $K(H^*)$. Because of $H^* \in \text{Sub}\, X$ we infer that $H^- \in \text{Lat Sub}\, X$ and in view of 2.2, $H \in Y$. Therefore $\text{Sub}\, Y = Y$.

2.4. Lemma. *Let* $X \subseteq \mathscr{G}$, $Y = \text{Join Sub}\, X$. *Then* $\text{Sub}\, Y = Y$.

Proof. Let $H \in \text{Sub}\, Y$. There is $G \in Y$ with $H \in c(G)$. Further, there are $G_i \in \in \text{Sub}\, X \cap K(G)$ with $G = \bigvee_{i \in I} G_i$. Hence $G_i \cap H \in \text{Sub}\, X \cap K(G)$. It suffices to verify that $H = \bigvee_{i \in I} (G_i \cap H)$ is valid.

Let $0 < h \in H$. Since $h \in G = (\bigvee^c_{i \in I} G_i)^-$ there are elements h_j $(j \in J)$ such that $h = \bigvee_{j \in J} h_j$ and $0 < h_j \in \bigvee^c_{i \in I} G_i$ for each $j \in J$.

Let $j \in J$ be fixed. There exist elements $g_{j1}, \ldots, g_{n(j)} \in \bigcup_{i \in I} G_i$ such that $0 < g_{jk}$ is valid for $k = 1, 2, \ldots, n(j)$ and

$$h_j = g_{j1} + \ldots + g_{jn(j)}\, .$$

Then $g_{jk} \leqq h_j$ holds for $k = 1, 2, \ldots, n(j)$ and hence $g_{jk} \in \bigcup_{i \in I} (G_i \cap H)$. Therefore all h_j belong to $\bigvee^c_{i \in I} (G_i \cap H)$ and so $h \in (\bigvee^c_{i \in I} (G_i \cap H))^- = \bigvee_{i \in I} (G_i \cap H)$. Thus $H = \bigvee_{i \in I} (G_i \cap H)$.

2.5. Lemma. *Let* $X \subseteq \mathscr{G}$, $Z = \text{Join Lat Sub}\, X$. *Then* $\text{Sub}\, Z = Z$.

Proof. In view of 2.3 we have

$$Z = \text{Join Sub Lat Sub}\, X\, ;$$

hence according to 2.4 the relation $\text{Sub}\, Z = Z$ is valid.

2.6. Lemma. *Let G and G' be the lattice ordered groups and let φ be an isomorphism of $K(G)$ onto $K(G')$. Let $H \in K(G)$. Then $K(H)$ is isomorphic to $K(\varphi(H))$.*

Proof. The image of the interval $[\{0\}, H]$ of $K(G)$ under the isomorphism φ is the interval $[\{0\}, \varphi(H)]$ of the lattice $K(G')$. Since $K(H) = [\{0\}, H]$ and $K(\varphi(H)) = = [0, \varphi(H)]$, the lattice $K(H)$ is isomorphic to $K(\varphi(H))$.

2.7. Lemma. *Let* $X \subseteq \mathscr{G}$, $Y = \text{Join Lat}\, X$. *Then* $\text{Lat}\, Y = Y$.

Proof. Let $G \in \text{Lat}\, Y$. There is $G' \in Y$ such that there exists an isomorphism φ of $K(G)$ onto $K(G')$. Further, there are G'_i $(i \in I)$ in $K(G') \cap \text{Lat}\, X$ such that $G' = = \bigvee_{i \in I} G'_i$. Put $G_i = \varphi^{-1}(G'_i)$. Then in view of 2.6 we hawe $G_i \in K(G) \cap \text{Lat}\, X$; clearly $G = \bigvee_{i \in I} G_i$, hence $G \in Y$.

2.8. Lemma. *Let* $X \subseteq \mathscr{G}$, $Z = \text{Join Lat Sub}\, X$. *Then Z is closed with respect to joins of convex l-subgroups.*

Proof. Let $G \in \mathscr{G}$ and $G_i \in c(G) \cap Z$ $(i \in I)$. Put $H = \bigvee^c_{i \in I} G_i$. Then we have $H^- = \bigvee_{i \in I} G_i^-$. According to 2.2 and 2.7, all G_i^- belong to Z and hence $H^- \in \in \text{Join}\, Z = Z$. Since $H \in \text{Sub}\, \{H^-\}$, from 2.5 we infer that $H \in Z$ is valid.

2.9. Theorem. *Let $X \subseteq \mathscr{G}$. Then*
(i) Join Lat Sub $X \in \mathscr{R}_K$;
(ii) *for each $Z \in \mathscr{R}_K$ with $X \subseteq Z$ we have* Join Lat Sub $X \subseteq Z$.

Proof. Put Join Lat Sub $X = Z$. In view of 2.5, Z is closed with respect to convex l-subgroups; according to 2.8, Z is also closed with respect to joins of convex l-subgroups. Therefore Z is a radical class. Since Z is a K-class (cf. 2.7), we have $Z \in \mathscr{R}_K$. Hence (i) is valid.

Let $Z' \in \mathscr{R}_K$, $X \subseteq Z'$. Since Z' is a radical class, we have Sub $Z' = Z'$; because Z is a K-class, Lat $Z' = Z'$ is valid. Hence we have to verify that Join $Z' = Z'$ holds. Let $G \in \mathscr{G}$, $\{G_i\}_{i \in I} \subseteq X \cap K(G)$, $\bigvee_{i \in I} G_i = G$. Put $\bigvee^c_{i \in I} G_i = H$. In view of 2.8, $H \in Z'$. Since $H^- = G$, from 2.2 we infer that $G \in Z'$. Thus (ii) holds true.

For each $X \subseteq \mathscr{G}$ we denote $T_K(X) =$ Join Lat Sub X. In view of 2.9, $T_K(X)$ is the K-radical class generated by X. If $G \in \mathscr{G}$, then the K-radical class generated by $\{G\}$ will be denoted by $T_K(G)$.

The join in the lattice $\mathscr{R}_K$ will be denoted by $\bigvee$.

2.10. Theorem. *Let I be a nonempty class and for each $i \in I$ let X_i be a K-radical class. Then $\bigvee_{i \in I} X_i =$ Join $\bigcup_{i \in I} X_i$.*

Proof. From 2.9 we infer that

$$\bigvee_{i \in I} X_i = \text{Join Lat Sub} \bigcup_{i \in I} X_i$$

is valid. Further, we hawe

$$\text{Lat Sub} \bigcup_{i \in I} X_i = \bigcup_{i \in I} \text{Lat Sub}\, X_i = \bigcup_{i \in I} X_i\,,$$

completing the proof.

Let us remark that for $\{X_i\}_{i \in I} \subseteq \mathscr{R}_K$ the relation

$$\bigvee^c_{i \in I} X_i = \bigvee_{i \in I} X_i$$

need not be valid. This will be shown by Example 3.8 below.

2.11. Theorem. *The lattice $\mathscr{R}_K$ fulfils the infinite distributive law*

$$X \wedge (\bigvee_{i \in I} Y_i) = \bigvee_{i \in I} (X \wedge Y_i)\,.$$

Proof. We have $\bigvee_{i \in I} (X \wedge Y_i) \leqq X \wedge (\bigvee_{i \in I} Y_i)$. Let $G \in X \wedge (\bigvee_{i \in I} Y_i)$. Hence $G \in X$ and $G \in \bigvee_{i \in I} Y_i$. In view of 2.10 there are G_j $(j \in J)$ belonging to $K(G) \cap \cap (\bigcup_{i \in I} Y_i)$ such that

$$G = \bigvee_{j \in J} G_j\,.$$

Then we have $G_j \in \bigcup_{i \in I} (X \wedge Y_i)$ for each $j \in J$, hence $G \in \bigvee_{i \in I} (X \wedge Y_i)$.

3.1. Proposition. *Let $X \subseteq \mathscr{G}$. Assume that* $\operatorname{Sub} X = X$ *and* $\operatorname{Lat} X = X$. *Then* $T_K(X) = (T(X))^-$.

Proof. a) Let $G \in (T(X))^-$. Hence there is $G_1 \in c(G) \cap T(X)$ such that $G = G_1^-$. In view of 1.1 there are $G_i \in X \cap c(G_1)$ such that $G_1 = \bigvee_{i\in I}^c G_i$. Hence $G = G_1^- = \bigvee G_i^-$ and according to 2.2, $G_i^- \in X$. Therefore $G \in T_K(X)$ and thus $(T(X))^- \subseteq T_K(X)$.

b) Assume that $G \in T_K(X)$. Hence $G \in \operatorname{Join} \operatorname{Lat} \operatorname{Sub} X = \operatorname{Join} X$. Thus there are $G_i \in X \cap K(G)$ with $G = \bigvee_{i\in I} G_i$. Put $H = \bigvee_{i\in I}^c G_i$; we hawe $H \in T(X)$ and $H^- \in (T(X))^-$. From

$$H^- = \bigvee_{i\in I} G_i^- = \bigvee_{i\in I} G_i = G$$

we infer that $G \in (T(X))^-$ and therefore $T_K(X) \subseteq (T(X))^-$.

3.2. Lemma. *Let $G \in \mathscr{G}$. Then the following conditions are equivalent:*

(i) *G is linearly ordered.*

(ii) *$c(G)$ is a chain.*

(iii) *$K(G)$ is a chain.*

The proof is simple, it will be omitted.

Let us recall the notion of the completely subdirect product of lattice ordered groups which was introduced by F. Šik [7].

Let $G \in \mathscr{G}$ and let $S = \{G_i\}_{i\in I}$ be a system of convex l-subgroups of G. The lattice ordered group G is said to be a *completely subdirect product of the system S* if $\sum_{i\in I} G_i \subseteq G \subseteq \prod_{i\in I} G_i$. (Cf. also Conrad [2].) (Recall that the symbol $\sum_{i\in I} G_i$ above denotes the restricted direct product of the system S.)

It is easy to verify that for $G \in \mathscr{G}$ and $S = \{G_i\}_{i\in I} \subseteq c(G)$ the following conditions are equivalent:

(i) G is a completely subdirect product of the system S;

(ii) $G_i \cap G_j = \{0\}$ whenever i and j are distinct elements of I, and for each $0 < g \in G$ there are $g_i \in G_i$ $(i \in I)$ with $g = \bigvee_{i\in I} g_i$.

(iii) $G_i \cap G_j = \{0\}$ whenver i and j are distinct elements of I, each G_i is a direct factor of G and $\bigvee_{i\in I} G_i = G$.

3.3. Lemma. *Let $G \in \mathscr{G}$ and $H \in c(G)$. Assume that H is a completely subdirect product of a system $S = \{H_i\}_{i\in I}$, where each H_i is linearly ordered. Then H^- is a completely subdirect product of the system S as well.*

Proof. Without loss of generality we may suppose that $H_i \neq \{0\}$ for each $i \in I$. Let $0 < g \in H^-$. Then g is a join of some positive elements of H; since each positive element of H is a join of positive elements belonging to $\bigcup_{i\in I} H_i$ there are elements $0 < h_j \in \bigcup_{i\in I} H_i$ such that $g = \bigvee h_j$.

Let i_1 be a fixed element of I. Assume that $x \leqq g$ for each $x \in H_{i_1}$. Choose $0 < y \in$

$\in H_{i_1}$. If $i \in I$ and $i \neq i_1$, $0 < z \in H_i$, then $z + y = z \vee y$; hence

$$g < g + y = (\bigvee h_j) + y = \bigvee(h_j + y) \leqq g\,,$$

which is a contradiction. Therefore for each $i \in I$ there is $0 < h'_i \in H_i$ such that $h'_i \nleqq g$. Then $h'_i \wedge g$ is the greatest element of the set $[0, g] \cap H_i$. Hence we infer that $g = \bigvee_{i \in I} h'_i$. Consequently, H^- is a completely subdirect product of the system S.

3.4. Proposition. (Cf. [5], Thm. 3.4.) *Let $X \subseteq \mathscr{G}$. Assume that each lattice ordered group belonging to X is linearly ordered. Let $G \in \mathscr{G}$. Then the following conditions are equivalent*:

(i) $G \in T(X)$.

(ii) *There are systems $\{A_i\}_{i \in I} \subseteq c(G)$ and $\{A_{ij}\}_{j \in J(i)} \subseteq c(G) \cap X$ for each $i \in I$, such that $A_i = \bigcup_{j \in J(i)} A_{ij}$ is valid for each $i \in I$, and $G = \sum_{i \in I} A_i$.*

For $X \subseteq \mathscr{G}$ we denote by X^0 the class of all $G \in \mathscr{G}$ having the property that there exists a linearly ordered system $\{G_i\}_{i \in I} \subseteq X \cap c(G)$ such that $G = \bigcup_{i \in I} G_i$ is valid. (The system under consideration is ordered by inclusion.)

3.5. Theorem. *Let $X \subseteq \mathscr{G}$, $X_1 =$ Lat Sub X. Assume that all elements of X are linearly ordered groups. Then*

(i) *all elements of X_1 are linearly ordered groups*;

(ii) *$T_K(X)$ is the class of all lattice ordered groups which can be expressed as completely subdirect products of linearly ordered groups belonging to X_1^0.*

Proof. (i) is a consequence of 3.2. In view of 2.2 we have Sub $X_1 = X_1$, and clearly Lat $X_1 = X_1$. Thus 3.1, 3.3 and 3.4 imply that (ii) is valid.

3.6. Corollary. *Let Y be a K-radical class. Then the following conditions are equivalent*:

(i) *Each $G \in Y$ is a completely subdirect product of linearly ordered groups.*

(ii) *There exists a class X of linearly ordered groups such that $Y = T_K(X)$.*

3.7. Proposition. *Let G, $G' \in \mathscr{G}$. Assume that* (i) *G is a completely subdirect product of linearly ordered groups G_i $(i \in I)$, and* (ii) *$K(G)$ is isomorphic to $K(G')$. Then G' can be expressed as a completely subdirect product of linearly ordered groups G'_i $(i \in I)$ such that, for each $i \in I$, $K(G_i)$ is isomorphic to $K(G'_i)$.*

Proof. Let φ be an isomorphism of the lattice $K(G)$ onto $K(G')$. Put $G'_i = \varphi(G_i)$ for each $i \in I$. According to 2.6, the lattices $K(G_i)$ and $K(G'_i)$ are isomorphic. Thus in view of 3.2, all G'_i are linearly ordered.

G_i is a direct factor of G and hence G_i has a complement in the lattice $K(G)$. Thus G'_i has a complement H'_i in the lattice $K(G')$. Without loss of generality we may suppose that all the lattice ordered groups G_i are non-zero; hence the same is valid for G'_i. Assume that there exists $g' \in G'$ such that g' is an upper bound for some G'_{i_1} $(i_1 \in I)$. We have

$$G' = (G'_{i_1} \vee H'_{i_1})^-\,,$$

hence g' is a join of some positive elements of $G'_{i_1} \vee H'_{i_1}$. Each positive element of $G'_{i_1} \vee H'_{i_1}$ is a join of some positive elements of the set $G'_{i_1} \cup H'_{i_1}$; hence there are elements $0 < g_j \in G'_{i_1} \cup H'_{i_1}$ with $g' = \bigvee g_j$. Choose $0 < k \in G'_{i_1}$. Then

$$g' < g' + k = \bigvee_j (g_j + k)\,.$$

If $g_j \in G'_{i_1}$, then $g_j + k \in G'_{i_1}$ and hence $g_j + k \leqq g'$; if $g_j \in H'_{i_1}$, then $g_j + k = g_j \vee k \leqq g'$. Thus $\bigvee_j (g_j + k) \leqq g'$, which is a contradiction. Therefore G'_{i_1} fails to be bounded in G' and hence, in view of the result of [6], G'_{i_1} is a direct factor in G'.

We have $\bigvee_{i \in I} G_i = G$, hence $\bigvee_{i \in I} G'_i = G'$. Therefore G' is a completely subdirect product of the system $\{G'_i\}$ $(i \in I)$.

Remark. Proposition 3.7 can be applied to obtain an alternative proof of Theorem 3.5.

3.8. Example. Let R be the additive group of all reals with the natural linear order. We denote by $\circ$ the operation of lexicographic product. Put $R_n = R \circ R \circ \ldots \circ R$ (n-times), $X_n = T_K(R_n)$ $(n = 1, 2, \ldots)$,

$$G = \prod R_n \quad (n = 1, 2, \ldots)\,,$$

$$X = \bigvee X_n\,, \quad Y = \bigvee X_n \quad (n = 1, 2, \ldots)\,.$$

Then 3.4 and 1.2 imply that G does not belong to X; on the other hand, according to 3.5 we have $G \in Y$. Hence if S is an infinite chain in the lattice $\mathscr{R}_K$, the join of S with respect to $\mathscr{R}_K$ need not coincide with the join of S with respect to $\mathscr{R}$. In particular, $\mathscr{R}_K$ fails to be a closed sublattice of $\mathscr{R}$.

4. ON ATOMS OF THE LATTICE $\mathscr{R}_K$

For $X \in \mathscr{R}_K$ we denote by $a(X)$ the class of all $Y \in \mathscr{R}_K$ such that (i) $X < Y$, and (ii) there exists no $Z \in \mathscr{R}_K$ with $X < Z < Y$. The elements of $a(X)$ are called *atoms over* X. If $X = \{\{0\}\} = R_0$, then $a(X)$ is the class of all atoms of $\mathscr{R}_K$; this class will be denoted by A_0. Also, for each $X \in \mathscr{R}_K$, we denote by $a_1(X)$ the class of all $Y \in \mathscr{R}_K$ such that (i) $X < Y$, and (ii) the interval $[X, Y]$ of $\mathscr{R}_K$ has no atoms. The following proposition shows that A_0 is nonempty.

4.1. Proposition. *Let $X \in \mathscr{R}_K$. Then the following conditions are equivalent*:

(i) *$X \in A_0$ and X contains an archimedean non-zero linearly ordered group.*

(ii) *X is the class of all lattice ordered groups which can be expressed as completely subdirect products of archimedean linearly ordered groups.*

Proof. Let us denote by Y the class of all lattice ordered groups G such that G is a completely subdirect product of archimedean linearly ordered groups. In view of 3.5, Y is a K-radical class. Let $Z \in \mathscr{R}_K$, $R_0 < Z \leqq Y$. There is $G_1 \in Z$ with $G_1 \neq \{0\}$;

further, there is an archimedean linearly ordered group $G_2 \neq \{0\}$ such that G_2 is a direct factor of G_1. If G_3 is any non-zero archimedean linearly ordered group, then $K(G_3)$ is isomorphic to $K(G_2)$, hence $G_3 \in Z$ and therefore (because $Z = \text{Join } Z$) we infer that $Y \subseteq Z$. Thus (i) is a consequence of (ii). The implication (i) $\Rightarrow$ (ii) is obvious.

The word "archimedean" cannot be omitted in the condition (i) of 4.1. This will be shown by the following construction and by 4.2.

Let α be an infinite cardinal. We denote by $\omega(\alpha)$ the first ordinal having the property that the set of all ordinals less then $\omega(\alpha)$ has the cardinality α. Let I_α be a linearly ordered set dual to $\omega(\alpha)$. Let G be a linearly ordered group, $G \neq \{0\}$. We put

$$G_\alpha = \Gamma_{i \in I(\alpha)} G_i ,$$

where Γ is the symbol denoting the lexicographic product and G_i is isomorphic to G for each $i \in I$.

From the definition of G_α we immediately obtain:

4.2. Lemma. *Let H be a convex l-subgroup of G_α, $H \neq \{0\}$. Then there exists a convex l-subgroup H_1 of H such that H_1 is isomorphic to G_α.*

4.3. Lemma. *Let $\{0\} \neq G \in \mathscr{G}$. Assume that for each $H \in c(G)$ there exists $H_1 \in c(H)$ such that H_1 is isomorphic to G. Then $T_K(G)$ is an atom in $\mathscr{R}_K$.*

Proof. We have $R_0 < T_K(G)$. Let $X \in \mathscr{R}_K$, $R_0 < X \leqq T_K(G)$. There exists $\{0\} \neq G' \in X$. According to 2.9 we have $G' \in \text{Join Lat Sub}\{G\}$. Hence there are $\{G_i\}_{i \in I} \subseteq K(G')$, $\{H_i\}_{i \in I} \subseteq \text{Sub}\{G\}$ such that $G' = \bigvee_{i \in I} G_i$ and for each $i \in I$ there exists an isomorphism φ_i of $K(G_i)$ onto $K(H_i)$. Without loss of generality we can suppose that $G_i \neq \{0\}$ for each $i \in I$; hence $H_i \neq \{0\}$ for each $i \in I$. In view of 2.2, $K(H_i)$ is isomorphic to $K(H_i^-)$, hence there is an isomorphism φ'_i of $K(G_i)$ onto $K(H_i^-)$. According to the assumption there exists $H_{i1} \in c(H_i^-)$ such that H_{i1} is isomorphic to G. Hence $K(H_{i1})$ is isomorphic to $K(G)$ and in view of 2.2, $K(H_{i1})$ is isomorphic to $K(G)$ as well. Thus $(\varphi')^{-1}(H_{i1}) \in T_K(G') \subseteq X$ and $K((\varphi')^{-1}(H_{i1}))$ is isomorphic to $K(G)$, implying $G \in T_K(G')$. Therefore $X = T_K(G)$.

4.4. Proposition. *Let $G \in \mathscr{G}$, $G \neq \{0\}$ and let α be an infinite cardinal. Then $T_K(G_\alpha)$ is an atom of the lattice $\mathscr{R}_K$.*

Proof. This is a consequence of 4.2 and 4.3.

4.5. Corollary. *The class of all atoms of $\mathscr{R}_K$ is a proper class.*

Let us remark that $T_K(G_\alpha)$ does not contain any nonzero archimedean l-group.

4.6. Proposition. *There exists a nonzero linearly ordered group G such that the interval $[R_0, T_K(G)]$ of the lattice $\mathscr{R}_K$ contains no atom.*

Proof. Let β be an infinite cardinal and let $I(\beta)$ be a linearly ordered set dual to $\omega(\beta)$. For each $i \in I(\beta)$ let α_i be a cardinal such that

$$i_1 > i_2 \Rightarrow \alpha_{i_1} < \alpha_{i_2} ;$$

further, let J_i be a linearly ordered set isomorphic to $\omega(\alpha_i)$. We denote by M the set of all pairs $m = (j, i)$ with $i \in I(\beta)$ and $j \in J_i$; we put $(j_1, i_1) < (j_2, i_2)$ if either $i_1 < i_2$, or $i_1 = i_2$ and $j_1 < j_2$. Let $G_0 \neq \{0\}$ be an archimedean linearly ordered group and for each $m \in M$ let G_m be a linearly ordered group isomorphic to G_0. Put

$$G = \Gamma_{m \in M} G_m .$$

Let $R_0 < X \leqq T_K(G)$. There exists $H_1 \in X$ with $H_1 \neq \{0\}$. In view of 3.5, there exists a convex l-subgroup $H_2 \neq \{0\}$ of G such that H_2 is isomorphic to a convex subgroup of H_1; thus $T_K(H_2) \leqq X$. The construction of G yields that there exists a convex l-subgroup $H_3 \neq \{0\}$ of H_2 such that for each nonzero convex l-subgroup H_4 of H_3 the lattice $K(H_4)$ fails to be isomorphic to $K(H_2)$. By using 3.5 again we infer that H_2 does not belong to $T_K(H_3)$; therefore $R_0 < T_K(H_3) < T_K(H_2)$. Thus X fails to be an atom of $\mathscr{R}_K$.

We have proved that the class $a_1(R_0)$ is nonempty. This result can be sharpened as follows. Let us apply the same notation as in the proof of 4.6.

Now if we choose a cardinal β_1 such that $\beta_1 > \beta$ and if we construct a linearly ordered group G_1 in the same way as we did for G (with the distinction that β is replaced by β_1), then in view of 3.5, G_1 does not belong to $T_K(G)$; therefore $T_K(G_1) \neq T_K(G)$. From this consideration we infer

4.7. Proposition. *$a_1(R_0)$ is a proper class.*

Denote $A = \sup a(R_0)$, $A_1 = \sup a_1(R_0)$ (the symbol sup being taken with respect to the complete lattice $\mathscr{R}_K$).

The following proposition is a consequence of 4.5, 4.7 and of the fact that $\mathscr{R}_K$ is a Brouwer lattice (cf. 2.11).

4.8. Proposition. *$a(A_1)$ and $a_1(A)$ are proper classes. The class $a_1(A_1)$ is empty. The lattice $[R_0, A]$ is complemented. A_1 belongs to $a_1(R_0)$.*

5. THE CLASS $a(A)$

Let $G \in \mathscr{G}$, $H \in c(G)$. If for each $g \in G \setminus H$ with $g > 0$ we have $g > h$ whenever $h \in H$, then G is said to be a *lexico extension* of the lattice ordered group H and we denote this fact by writing $G = \langle H \rangle$. The lattice ordered group G is called a *proper lexico extension* if there exists $H_1 \in c(H)$ with $G \neq H_1 \neq \{0\}$ such that $G = \langle H_1 \rangle$.

Let $G \neq \{0\}$ be an archimedean linearly ordered group. Let α be an infinite cardinal. Put

$$G_{[\alpha]} = \left(\prod_{i \in I} G_i\right) \circ G ,$$

where each G_i is isomorphic to G. Then $G_{[\alpha]}$ is a proper lexico extension.

5.1. Lemma. *$T_K(G)$ is covered by $T_K(G_{[\alpha]})$.*

Proof. There exists a convex l-subgroup of $G_{[\alpha]}$ isomorphic to G; thus $G \in T_K(G_{[\alpha]})$ and hence $T_K(G) \leqq T_K(G_{[\alpha]})$. Now 3.4 implies that $G_{[\alpha]}$ does not belong to $T_K(G)$ (in fact, each element of $T_K(G)$ is archimedean and $G_{[\alpha]}$ fails to be archimedean). Therefore $T_K(G) < T_K(G_{[\alpha]})$.

Let $X \in \mathscr{R}_K$, $T_K(G) < X \leqq T_K(G_{[\alpha]})$. Hence there is $H \in X \setminus T_K(G)$. In view of 2.9 there are $H_i \in K(H)$ $(i \in I)$ such that $H = \bigvee_{i \in I} H_i$ and each H_i belongs to Lat Sub $\{G_{[\alpha]}\}$. Hence for each $i \in I$ there is $H'_i \in \text{Sub}\,\{G_{[\alpha]}\}$ such that $K(H_i)$ is isomorphic to $K(H'_i)$.

For each H'_i we have either (i) $H'_i \subseteq \prod_{i \in I} G_i$, or (ii) $H'_i = G_{[\alpha]}$. If (i) is valid for each $i \in I$, then $H'_i \in T_K(G)$ holds for each $i \in I$, hence $H \in T_K(G)$, which is a contradiction. Therefore there is $i \in I$ fulfilling (ii) and so $X = T_K(G_{[\alpha]})$, completing the proof.

5.2. Lemma. *Let $X \in A_0$, $X \neq T_K(G)$. Then $T_K(G_{[\alpha]}) \wedge X = R_0$.*

Proof. By way of contradiction, assume that there is $H \in T_K(G_{[\alpha]}) \wedge X$ with $H \neq \{0\}$. The construction of $G_{[\alpha]}$ and 2.9 yields that each nonzero lattice ordered group belonging to $T_K(G_{[\alpha]})$ has a nonzero archimedean subgroup. Thus $T_K(G) \leqq X$, which is a contradiction.

5.3. Lemma. *$T_K(G_{[\alpha]}) \vee A$ covers A.*

Proof. Let A'_0 be the class of all elements of A_0 distinct from $T_K(G)$ and let $A' = \sup A'_0$. In view of 5.2 and 2.11 we have

$$T_K(G_{[\alpha]}) \wedge A' = R_0\,,$$

hence according to 5.1 we obtain

$$T_K(G_{[\alpha]}) \wedge A = T_K(G_{[\alpha]}) \wedge (T_K(G) \vee A') = T_K(G)\,.$$

Since the interval $[A, T_K(G_{[\alpha]}) \vee A]$ is transposed to the interval $[T_K(G), T_K(G_{[\alpha]})]$, it follows from 5.1 that $T_K(G_{[\alpha]}) \vee A$ covers A.

5.4. Lemma. *Let β be a cardinal, $\beta > \alpha$. Then $T_K(G_{[\alpha]}) \neq T_K(G_{[\beta]})$.*

Proof. By way of contradiction, assume that $T_K(G_{[\alpha]}) = T_K(G_{[\beta]})$. Hence $G_\beta \in T_K(G_{[\alpha]})$. In view of 2.9 there are $H_j \in K(G_{[\beta]})$ $(j \in J)$ such that $G_{[\beta]} = \bigvee_{j \in J} H_j$ and all H_j belong to Lat Sub $\{G_{[\alpha]}\}$. Let $j \in J$. There exists $H'_j \in c(G_{[\alpha]})$ such that $K(H_j)$ is isomorphic to $K(H'_j)$. We distinguish two cases.

a) $H'_j \subseteq \prod_{i \in I} G_i$ for each $j \in J$. Because $\prod_{i \in I} G_i = \dot{\bigvee}_{i \in I} G_i$ and each G_i is isomorphic to G, we infer that $G_{[\beta]}$ belongs to $T_K(G)$ which is a contradiction (in fact, G is linearly ordered and archimedean, hence in view of 3.4 all elements of $T_K(G)$ are archimedean, but $G_{[\beta]}$ fails to be archimedean).

b) There exists $j \in J$ such that $H'_j \nsubseteq \prod_{i \in I} G_i$. Then $H'_j = G_{[\alpha]}$. Thus $K(H_j)$ cannot be isomorphic to $K(H'_j)$, and we arrived at a contradiction.

5.5. Corollary. *Let β be as in 5.4. Then*

$$(T_K(G_{[\alpha]}) \vee A) \wedge (T_K(G_{[\beta]} \vee A) = A .$$

5.6. Theorem. *$a(A)$ is a proper class.*

Proof. For each infinite cardinal α we have $T_K(G_{[\alpha]}) \vee A \in a(A)$ (cf. 5.3). From 5.5 it follows that if $\beta > \alpha$, then $T_K(G_{[\alpha]} \vee A \neq T_K(G_{\iota\beta]}) \vee A$. Hence $a(A)$ is a proper class.

6. PRINCIPAL ELEMENTS OF $\mathscr{R}_K$

An element $X \in \mathscr{R}_K$ is said to be *principal* if there is $G \in \mathscr{G}$ such that $X = T_K(G)$. Let $\mathscr{P}$ be the class of all principal elements of $\mathscr{R}_K$.

To each $G \in \mathscr{G}$ we assign a cardinal $k(G)$ as follows: $k(G)$ is the least cardinal α having the property that there are $G_i \in K(G)$ $(i \in I)$ such that $\bigvee_{i \in I} G_i = G$ and $\operatorname{card} K(G_i) \leqq \alpha$ for each $i \in I$.

A subclass X of G is said to be *k-bounded* if there is a cardinal α such that $k(G) \leqq \alpha$ for each $G \in X$.

6.1. Proposition. *Let $X \in \mathscr{R}_K$. Then the following conditions are equivalent:*

(i) *X is a principal element of $\mathscr{R}_K$.*

(ii) *X is k-bounded.*

Proof. Suppose that X is principal, $X = T_K(G)$. Put $\operatorname{card} K(G) = \alpha$. Then 2.9 implies that for each $H \in X$ we have $k(H) \leqq \alpha$; hence X is k-bounded.

Conversely, assume that X is k-bounded, i.e., $k(H) \leqq \alpha$ for each $H \in X$. There exists a set S of lattices which has the following properties:

(i) If $L \in S$, then $\operatorname{card} L \leqq \alpha$.

(ii) If $L \in S$, then there is $H_1 \in X$ such that L is isomorphic to $K(H_1)$.

(iii) If $H_1 \in X$ and $\operatorname{card} K(H_1) \leqq \alpha$, then there is $L \in S$ such that L is isomorphic to $K(H_1)$.

For each $L \in S$ we choose a fixed H_1 fulfilling (ii) and we put $H_1 = H_1(L)$. We denote

$$G = \prod_{L \in S} H_1(L) .$$

Because $H_1(L) \in X$ and since X is closed with respect to direct products (cf. [1]) we infer that G belongs to X and hence $T_K(G) \leqq X$. Let $H \in X$. In view of the definition of S there are H_i $(i \in I)$ in $K(G)$ such that $\bigvee_{i \in I} H_i = H$ and for each $i \in I$, $K(H_i)$ is isomorphic to a certain element of S. Hence $H_i \in T_K(G)$ for each $i \in I$ and thus $H \in T_K(G)$. Therefore $X = T_K(G)$.

6.2. Corollary. *Let $X \in \mathscr{P}$, $Y \in \mathscr{R}_K$, $Y \leqq X$. Then $Y \in \mathscr{P}$.*

From 6.1 and 2.10 we also obtain:

6.3. Corollary. *Let I be a nonempty set and for each $i \in I$ let X_i be a principal K-radical class. Then $\bigvee_{i \in I} X_i$ is a principal K-radical class as well.*

6.4. Proposition. *Let A and A_1 be as in Sec. 4. Then $A \notin \mathscr{P}$ and $A_1 \notin \mathscr{P}$.*

Proof. Le: $G \in \mathscr{G}$, $G \neq \{0\}$ and let α be an infinite cardinal. Then $k(G_\alpha) \geqq \alpha$. Hence $\{k(H)\}_{H \in A}$ is a proper class of cardinals. Thus the class A fails to be k-bounded and in view of 6.1, A is not principal. Also, from the construction of elements of A_1 applied in the proof of 4.6 it follows that A_1 is not k-bounded, hence A_1 fails to be principal.

6.5. Theorem. *Let X be a principal K-radical class. Then both $a(X)$ and $a_1(X)$ are proper classes.*

Proof. According to 6.1 there is a cardinal α such that $k(G) < \alpha$ for each $G \in X$. Let $G' \in \mathscr{G}$, $G' \neq \{0\}$. For each $\beta > \alpha$ we have $k(G'_\beta) > \alpha$, hence $G'_\beta \notin X$ and thus, because of 4.4, $X \wedge T_K(G'_\beta) = R_0$ and thus X is covered by $X \vee T_K(G'_\beta)$. If $\beta_1 > \beta$, then (since $\mathscr{R}_K$ is a Brouwer lattice) we have $X \vee T_K(G'_{\beta_1}) \neq X \vee T_K(G'_\beta)$. Therefore $a(X)$ is a proper class. The proof for $a_1(X)$ is analogous.

7. K-RADICAL CLASSES HAVING NO COVER

In view of the above results the natural question arises: does there exist a K-radical class $X \neq \mathscr{G}$ having the property that $a(X) = \emptyset$? The following consideration (the idea of which is analogous to that performed in [4] for the lattice $\mathscr{R}$ of all radical classes) shows that the answer is affirmative.

7.1. Lemma. *Let $X, Y \in \mathscr{R}_K$ such that X is covered by Y. Then there are $G_1 \in Y \setminus X$, $G_2 \in X$ such that $T_K(G_2)$ is covered by $T_K(G_1)$.*

Proof. From $X < Y$ it follows that there is $G_1 \in Y \setminus X$. Hence $T_K(G_1) \leqq Y$ and $T_K(G_1) \nleqq X$. Put $T_K(G_1) \wedge X = Z$. In view of 6.2, Z is a principal K-radical class, i.e., there is $G_2 \in Z$ with $Z = T_K(G_2)$. Then $G_2 \in X$. Since X is covered by Y we have $X \vee T_K(G_1) = Y$. Since the interval $[T_K(G_2), T_K(G_1)]$ is transposed to $[X, Y]$, $T_K(G_2)$ is covered by $T_K(G_1)$.

7.2. Corollary. *Let $X \in \mathscr{R}_K$. Assume that for each $G_2 \in X$ and each $G_1 \in \mathscr{G}$ such that $T_K(G_1) \in a(T_K(G_2))$ we have $G_1 \in X$. Then $a(X) = \emptyset$.*

Let A_1 be as in Sec. 4.

7.3. Lemma. *Let $X \in \mathscr{R}_K$, $X \wedge A_1 = R_0$ and $Y \in a(X)$. Then $Y \wedge A_1 = R_0$.*

Proof. By way of contradiction, assume that $Y \wedge A_1 = Z > R_0$. According to 4.8, the interval $[R_0, Z]$ contains no atom. Because X is covered by Y and $Z \nleqq X$, we obtain $X \vee Z = Y$. Since $X \wedge Z = R_0$, the interval $[R_0, Z]$ is transposed to $[X, Y]$, hence R_0 is covered by Z, which is a contradiction.

For each $C \subseteq \mathscr{R}_K$ we denote by $J(C)$ the class of all $Y \in \mathscr{P}$ such that there are $Z_1 \in C$ and $G_1 \in Z_1$ with $Y \in a(T_K(G_1))$. Next we put $X_J(C) = \sup C \vee \sup J(C)$.

For each ordinal α we assign a K-radical class X_α as follows. We put $X_1 = R_0$. Suppose that we have defined X_β for each $\beta < \alpha$. If α is a limit ordinal, then we put $X_\alpha = \bigvee_{\beta<\alpha} X_\beta$. If α is non-limit, $\alpha = \gamma + 1$, then we set $X_\alpha = X_J([R_0 . X_\gamma])$.

Put $X_0 = \bigvee_\alpha X_\alpha$, where α runs over the class of all ordinals.

7.4. Proposition. *$a(X_0) = \emptyset$ and $X_0 \neq \mathscr{G}$.*

Proof. The relation $a(X_0) = \emptyset$ follows from the construction of X_0 and from 7.2. According to 7.3 we have $X_0 \wedge A_1 = R_0$, hence $X_0 \neq \mathscr{G}$.

Let us remark that if $G \in X \in \mathscr{R}_K$, $G' \in \mathscr{G} \setminus X$, $T_K(G') \in a(T_K(G))$, then $X \vee T_K(G') \in a(X)$. By applying this remark and using a construction analogous to that of X_0 the following proposition can be proved (the detailed proof will be omitted):

7.5. Proposition. *Let $X \in \mathscr{R}_K$. There exists $Y \in \mathscr{R}_K$ such that*

(i) *$X \leqq Y$;*

(ii) *$a(X) = \emptyset$;*

(iii) *if $Z \in \mathscr{R}_K$, $X \leqq Z$ and $a(Z) = \emptyset$, then $Y \leqq Z$.*

Let X_A, X_B, X_C and S_t be as in Introduction. Now we shall prove that $a(X) \neq \emptyset$ for each $X \in \{X_A, X_B, X_C, S_t\}$.

7.6. Proposition. *$a(X_A)$ is a proper class.*

Proof. Let G be a nonzero archimedean linearly ordered group. For each infinite cardinal α let $G_{[\alpha]}$ be as in Sec. 5. Put $Y_\alpha = X_A \vee T_K(G_{[\alpha]})$. We have $G_{[\alpha]} \notin X_A$. Hence in view of 5.1 we infer that $Y_\alpha \in a(X_A)$. Now 5.4 implies that $a(X_A)$ is a proper class.

Let us denote by F the additive group of all real functions f defined on the set M of all rational numbers $x \in (0, 1)$ which have the following property: there are irrational numbers $\alpha_1, \ldots, \alpha_n \in (0, 1)$ with $\alpha_1 < \alpha_2 < \ldots < \alpha_n$ (depending on f) such that f is a constant on each of the sets $(0, \alpha_1) \cap M, (\alpha_1, \alpha_2) \cap M, \ldots, (\alpha_n, 1) \cap M$.

For $f_1, f_2 \in F$ we put $f_1 \leqq f_2$ if $f_1(x) \leqq f_2(x)$ is valid for each $x \in M$. Then F is a lattice ordered group.

It follows immediately from the construction of F that each nonzero convex l-subgroup of F is isomorphic to F. Hence in view of 4.3 we obtain:

7.7. Lemma. *$T_K(F)$ is an atom of $\mathscr{R}_K$.*

7.8. Proposition. *$a(X_B) \neq \emptyset$.*

Proof. If $\{0\} \neq G \in X_B$, then there is a nonzero convex l-subgroup H of G such that $K(H)$ is a chain (cf. 3.2). If $G \in c(F)$, then G does not have the mentioned property; hence according to 2.9 we have $X_B \wedge T_K(F) = R_0$. Therefore in view of 7.7, $X_B \vee T_K(G)$ covers X_B.

Also, no nonzero convex l-subgroup of F is completely distributive; hence by analogous reasoning as in the proof of 7.8 we infer:

7.9. Proposition. $a(X_C) \neq \emptyset$.

7.10. Lemma. (Cf. [1], p. 191.) *Let $G \in \mathcal{G}$. Then G belongs to S_t if and only if each polar of G is complemented in $K(G)$.*

Let $G_{[\alpha]}$ be as above. Then 7.10 implies that $G_{[\alpha]}$ does not belong to S_t.

7.11. Lemma. $T_K(G_{[\alpha]}) \wedge S_t = T_K(G)$.

Proof. We have $T_K(G) < T_K(G_{[\alpha]})$ and in view of 7.10, $T_K(G) \leqq S_t$; hence we have to verify that $T_K(G_{[\alpha]}) \wedge S_t \leqq T_K(G)$.

Let $H \in T_K(G_{[\alpha]}) \wedge S_t$. Then there are H_j $(j \in J)$ in $K(H) \cap \text{Lat Sub}\,\{G_{[\alpha]}\}$ such that $H = \bigvee_{j \in J} H_j$. Hence there exist $H'_j \in \text{Sub}\,\{G_{[\alpha]}\}$ such that $K(H_j)$ is isomorphic to $K(H'_j)$ for each $j \in J$. Because of $H \in S_t$ we have $H_j \in S_t$ and $H'_j \in S_t$ for each $j \in J$. Thus $H'_j \neq G_{[\alpha]}$ and therefore $H'_j \subseteq \prod_{i \in I} G_i$ for each $j \in J$ (the denotations are as in Sec. 5). Hence $H'_j \in T_K(G)$ and so $H_j \in T_K(G)$ for each $j \in J$, implying $H \in T_K(G)$.

7.12. Proposition. *$a(S_t)$ is a proper class.*

Proof. From 7.11 and 5.1 we infer that $T_K(G_{[\alpha]}) \vee S_t$ belongs to $a(S_t)$. Therefore in view of 5.4, $a(S_t)$ is a proper class.

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