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QUASI HAMILTONIAN SEMIGROUPS

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In [10] Pondělíček has determined some properties of semigroups whose subsemigroups are permutable. In the present note we have called *quasi hamiltonian* those semigroups, according to the terminology used for groups, and we have determined their structure, as well as several further properties concerning in particular separative quasi hamiltonian semigroups.

Quasi hamiltonian semigroups are a generalization of quasicommutative semigroups, which the authors studied in [4] and [5]; thus many results contained in those works may be refound as particular cases of the theorems contained in this paper.

Definitions and undefined terms may be found in [8].

1. STRUCTURE OF QUASI HAMILTONIAN SEMIGROUPS

Definition 1.1. A semigroup S is called *quasi hamiltonian* if, for every $a, b \in S$, there exist two positive integers r, s such that $ab = b^r a^s$.

Definition 1.2. A group G is called *quasi hamiltonian* if all subgroups of G are permutable (see [7], p. 718, foot-note (1)).

In the sequel we shall make use of the following propositions, which are either wellknown or however easy to verify:

A) A semigroup is quasi hamiltonian if and only if its subsemigroups are permutable (see [10], Lemma 3).

B) A quasi hamiltonian group is not necessarily a quasi hamiltonian semigroup. The two notions are however equivalent when the group is periodic.

C) The possible idempotents of a quasi hamiltonian semigroup are in the center.

D) A quasi hamiltonian archimedean semigroup is t-archimedean (i.e. left and right archimedean).

E) A quasi hamiltonian semigroup S is a semilattice of t-archimedean semigroups S_α. Throughout we shall call these semigroups the *t-archimedean components* of S.

F) For every element a of a quasi hamiltonian semigroup S the following are equivalent:

i) a is regular,

ii) a is left regular (right regular),

iii) a is in a subgroup of S.

Our aim is now to characterize quasi hamiltonian archimedean semigroups. To this end we state the following.

Lemma 1.3. *A quasi hamiltonian nilsemigroup S is commutative.*

Proof. Let $a, b \in S$ with $ab \neq ba$. There exist four positive integers h, k, r, s such that $ab = b^r a^s = a^{hs} b^{kr}$, with $hs + kr > 2$. Then we have also $ab = a^{\lambda(hs-1)} abb^{\lambda(kr-1)}$ for every positive integer λ [1]. Hence $ab = 0$. Moreover, there exist two positive integers p, q such that $ba = a^p b^q$; thus $ab = 0 = ba$, a contradiction.

Lemma 1.4. *A subgroup of a quasi hamiltonian semigroup is either commutative or periodic.*

Proof. Let G be a non commutative subgroup of a quasi hamiltonian semigroup S. First let us verify that, for every $a, b \in G$, the relation $ab \neq ba$ implies that either a or b has finite order. In fact, let us suppose that both a and b have infinite orders; then there are eight positive integers $p, q, r, s, \alpha, \beta, \gamma, \delta$, with $\alpha s > 1$, $\beta r > 1$, $\gamma q > 1$, $\delta p > 1$, such that $ab = b^r a^s = a^{\alpha s} b^{\beta r}$, $ab^{-1} = b^{-p} a^q = a^{\gamma q} b^{-\delta p}$. From these we obtain $a^{\alpha s - 1} = b^{1-\beta r}$ and $a^{\gamma q - 1} = b^{\delta p - 1}$, whence $b^{(1-\beta r)(\gamma q - 1)} = a^{(\alpha s - 1)(\gamma q - 1)} = b^{(\alpha s - 1)(\delta p - 1)}$. This is a contradiction, since $(1 - \beta r)(\gamma q - 1) < 0$, $(\alpha s - 1) \cdot (\delta p - 1) > 0$, while b is supposed to be of infinite order.

Now, let us show that the elements of G having infinite orders are in the center of G. Let a be an element of infinite order of G. If $b \in G$ and $ab \neq ba$, it follows from the above that b is of finite order. Moreover, there exist four positive integers r, s, α, β such that $ab = b^r a^s = a^{\alpha s} b^{\beta r}$ If $\alpha s > 1$, it follows $a^{\alpha s - 1} = b^{1-\beta r}$, a contradiction, since a is of infinite order, while b has finite order. Thus $\alpha s = 1$, whence $s = 1$ and $ab = b^r a$. Analogously it results $ba = ab^q$ for some positive integer q. So a permutes with all subgroups of G, that is a belongs to the norm of G (see [11], p. 84). Since the norm of a group, when it has elements of infinite order, coincides with the center (see [1], 2, Satz 3), we may conclude that every element of infinite order of G is in the center.

Finally, let us prove that the set T of the elements of G having finite orders is a subgroup of G. In fact, let a, b be two elements of finite order of G. If ab is of infinite order, it results from the above $ab \in Z$, where Z is the center of G. Hence $a = (ab)\, b^{-1} = b^{-1}(ab)$, that is $ba = ab$; this implies that ab is of finite order, a contradiction.

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[1]) We make use of the convention that $x^0 y = y x^0 = y$ for every $x, y \in S$.

That being stated, G turns out to be the set union of its subgroups Z and T, which is a contradiction if both are proper subgroups. Since G is not commutative, it must be $G = T$. Thus G is periodic.

Theorem 1.5. *A semigroup S is non commutative archimedean quasi hamiltonian if and only if S is an ideal extension of a non commutative periodic quasi hamiltonian group by a commutative nilsemigroup.*

Proof. Let S be a non commutative archimedean quasi hamiltonian semigroup. Then S has two elements a, b which do not permute, and therefore there exist six positive integers $h, k, r, s, \alpha, \beta$, with $\alpha s + \beta r > 2$, such that $ab = b^r a^s = a^{\alpha s} b^{\beta r} = a^{\alpha s - 1}(ab) b^{\beta r - 1} = (ab)^h a^{(\alpha s - 1)k} b^{\beta r - 1}$. Let $c = (ab)^{h-1} a^{(\alpha s - 1)k} b^{\beta r - 1}$; then it results $c \in S$ and, S being t-archimedean (Prop. D), there are a positive integer λ and $x \in S$ such that $c^\lambda = xab$. Thus $ab = abc = abc^\lambda = abxab$; so ab is regular and S contains an idempotent. A t-archimedean semigroup with idempotent is an ideal extension of a group G by a nilsemigroup N (see [3], Th. 2). In the present case, N is commutative (Lemma 1.3) and G is a non commutative quasi hamiltonian group, otherwise S itself should be commutative (see [8], III.2.9.3). Thus, by Lemma 1.4, G is periodic.

Conversely, let S be an ideal extension of a non commutative periodic quasi hamiltonian group G by a commutative nilsemigroup N. Then S is a non commutative t-archimedean semigroup (see [3], Th. 2). Moreover, let $a, b \in S$. We must examine the following three cases:

1) $a, b \in G$. Then, G being a periodic quasi hamiltonian group, there are two positive integers r, s such that $ab = b^r a^s$.

2) $a \in G$, $b \in S \setminus G$. Let u be the identity of G. Then it results $ab = abu = (bu)^r a^s = b^r u a^s = b^r a^s$ for some positive integers r, s. In the same way we find $ba = a^p b^q$ for some positive integers p, q.

3) $a, b \in S \setminus G$. Let us denote by $\cdot$ the operation in the commutative nilsemigroup N. Then, if $a \cdot b = b \cdot a = 0$, it results $ab, ba \in G$, and therefore $ab = abu = (au)(bu) = (bu)^r (au)^s = b^r a^s u = b^r a^s$ for some positive integers r, s. If on the contrary $a \cdot b = b \cdot a \neq 0$, it results $ab = a \cdot b = b \cdot a = ba$. Thus S is quasi hamiltonian.

The structure of periodic quasi hamiltonian groups is known (see [7], Th. 4); so Th. 1.5 provides a complete descritpion of archimedean quasi hamiltonian semigroups.

Now we intend to give a characterization of quasi hamiltonian semigroups. To this purpose we list here some wellknown propositions which will be utilized in the sequel.

Definition 1.6. A semigroup H is called a *duo semigroup* if every one sided ideal of H is two sided (see [9], p. 20).

G) A semigroup H is duo if and only if $aH^1 = H^1 a$ for every $a \in H$.

H) An archimedean duo semigroup is t-archimedean.

I) A duo semigroup is a semilattice of t-archimedean semigroups.
J) A quasi hamiltonian semigroup is duo.
K) Every group is a duo semigroup.

Lemma 1.7. *If a duo semigroup H has a kernel K, then K is a subgroup of H.*

Proof. For every $a, b \in K$, there exists an element $c \in H^1$ such that $ba = ac$. Then we have $b(aK) = (ba)K = (ac)K = a(cK) \subseteq aK$, and aK results a left ideal of K. Since aK is also a right ideal of K, and K is simple, it results $aK = K$. In the same way we obtain $Ka = K$; thus K, being left and right simple, is a group.

Lemma 1.8. *Let $H = \langle a, b \rangle$ be a duo semigroup. If ab is in a group, H has a kernel K, and $ab \in K$. Moreover, if H is not archimedean, it results also $ba \in K$.*

Proof. The semigroup H, being generated by two elements, is a semilattice of at most three t-archimedean semigroups H_α, H_β, $H_{\sigma\beta}$ (Prop. I), and it is $ab \in H_{\alpha\beta}$. If ab belongs to a subgroup G of H, it is immediate to find that $G \subseteq H_{\alpha\beta}$. Then $H_{\sigma\beta}$, being t-archimedean, contains an unique maximal subgroup K, which turns to be an ideal of H. Thus K is the kernel of H (see [6], 2.5;5) and obviously $ab \in K$. Moreover, since H is a duo semigroup, there exist $x, y \in H^1$ such that $ba = ax = yb$. If $x = a^h$, and $y = b^k$ $(h, k > 0)$, H results to be archimedean. If H is not archimedean, we may suppose $x = bz$ for some $z \in H^1$, and it results $ba = abz \in K$.

Theorem 1.9. *A semigroup S is quasi hamiltonian if and only if S is a semilattice of archimedean quasi hamiltonian semigroups, and every subsemigroup of S, generated by two elements, is a duo semigroup.*

Proof. The "only if" part of the theorem is immediate. Conversely, let S be a semilattice of archimedean quasi hamiltonian semigroups, and let every subsemigroup of S, generated by two elements, be a duo semigroup. If $a, b \in S$ are in the same archimedean component S_α of S, since S_α is quasi hamiltonian, it results obviously $ab = b^r a^s$ for some positive integers r, s. Then let us suppose $a \in S_\alpha$, $b \in S_\beta$ with $S_\alpha \neq S_\beta$ and $ab \neq ba$. Since $H = \langle a, b \rangle$ is a duo semigroup, there exist $x, y \in H^1$ such that

$$(1) \qquad ab = xa\,, \quad ba = ay\,.$$

Then, either x is a power of a, or we may suppose $x = zb$ with $z \in H^1$. Similarly, either y is a power of a, or we may suppose $y = bw$ with $w \in H^1$. Thus from the (1) we may deduce the following four possible cases (h, k positive integers):

i) $ab = a^h$, $\quad ba = a^k$,
ii) $ab = a^h$, $\quad ba = abw$,
iii) $ab = zba$, $ba = a^k$,
iv) $ab = zba$, $ba = abw$.

First we note that the cases ii) and iii) may be immediately reduced to the case i). In fact, from $ab = a^h$ and $ba = abw$ it follows $ba = a^h w = a^k$ for some positive integer k. Analogously $ba = a^k$ and $ab = zba$ imply $ab = a^h$ for some positive integer h. Thus it remains to examine the cases i) and iv). At this point it is convenient to insert the following remark: *if ab belongs to a subgroup of H, there exist four positive integers p, q, r, s such that*

$$(2) \qquad ab = b^r a^s, \quad ba = a^p b^q$$

In fact, by Lemmas 1.8 and 1.7, ab and ba are in the kernel K of H, and K is a group. Since $K \subseteq S_{\alpha\beta}$, it follows from Lemma 1.4 that K is either periodic quasi hamiltonian or commutative. Then, denoting by u the identity of K, we have $ab = abu = (au)(bu) = (bu)^r (au)^s = b^r a^s u = b^r a^s$ and similarly $ba = bau = (bu)(au) = (au)^p (bu)^q = a^p b^q u = a^p b^q$ for some positive integers p, q, r, s.

That being stated, let us examine the case i). We may assume without loss of generality $h > k$. Then we have $ab = aa^k a^{h-k-1} = aba^{h-k} = aba^{h(h-k)} = (ab)^{h-k+1}$ and ab is in a subgroup of H. Hence the relations (2) hold in the case i). Finally, let us investigate the case iv). It results for some $u, v \in H^1$

$$(3) \qquad ab = zba = bau, \quad ba = abw = vab$$

whence $ab = zvab = abwu$. If $zv \in H_{\alpha\beta}$, since $H_{\alpha\beta} = H \cap S_{\alpha\beta}$ is archimedean quasi hamiltonian, there exist a positive integer t and $g \in H_{\alpha\beta}$ such that $(zv)^t = abg$. Then $ab = zvab = (zv)^t ab = abgab$ and ab, being regular in H, belongs to a subgroup of H. (Prop. F). At the same conclusion we arrive when $wu \in H_{\alpha\beta}$. Thus in both these cases the relations (2) do hold. Therefore it remains to examine the following four possibilities:

I) $z, u, w, v \in \langle a \rangle$,

II) $z, u, w, v \in \langle b \rangle$,

III) $z, v \in \langle a \rangle$; $w, u \in \langle b \rangle$,

IV) $z, v \in \langle b \rangle$; $w, u \in \langle a \rangle$.

In the case I) it follows from the (3) that $ab = ba^i$, $ba = a^j b$ for some positive integers i, j, and a similar result is found in the case II). In the case III) there exist three positive integers l, m, n such that $ab = a^l ba = bab^m$ and $ba = ab^n$. From these it follows also $ab = bab^m = ab^{n+m} = a^l bab^{n+m-1} = a^l b(ab) b^{n+m-2}$ and, since $a^l b \in H_{\alpha\beta}$, we may again conclude that ab is in a subgroup of H. A similar result holds for the case IV). So the relations (2) hold in all the cases examined, and S is quasi hamiltonian.

Remark 1.10. We want to observe that the two conditions contained in the statement of Th. 1.9 are actually independent. This fact is proved by the two following examples:

Example 1. Let S be the semigroup represented by the following multiplication table (see [8], p. 177):

	a	b	c	d
a	a	a	a	a
b	a	a	a	b
c	a	a	a	a
d	a	a	c	d

It is immediate that S is a semilattice of the two commutative archimedean components $\{a, b, c\}$ and $\{d\}$. Nevertheless, S is not quasi hamiltonian, since $bd = b$, while $d^h b^k = db^k = a$ for every positive integers h, k.

Example 2. Let S be the semigroup whose multiplication table is the following (see [8], p. 176):

	a	b	c	d
a	a	a	a	a
b	a	a	a	a
c	a	a	b	b
d	a	a	a	b

It is immediate that S is an archimedean duo semigroup, whose proper subsemigroups are commutative and consequently duo semigroups. However, S is not quasi hamiltonian, since $cd = b$, while $d^h c^k = a$ for every positive integers h, k.

Lemma 1.11. *Let S be a quasi hamiltonian semigroup. For every $a, b \in S$, either $a^2 b^2$ belongs to the kernel K of the subsemigroup $H = \langle a, b \rangle$ or $(ab)^2 = a^2 b^2 = b^2 a^2$.*

Proof. Let $a, b \in S$ with $ab \neq ba$. Then there exist four positive integers p, q, r, s such that

$$ab = b^r a^s, \quad ba = a^p b^q \tag{4}$$

whence $ab = (b^{r-1} a^{p-1})\, ab(b^{q-1} a^{s-1}) = [ab(b^{q-1} a^{s-1})]^m (b^{r-1} a^{p-1})^n$ for some positive integers m, n. Denoting by $H_{\alpha\beta}$ the t-archimedean component of H which contains ab, and putting $c = b^{q-1} a^{s-1} [ab(b^{q-1} a^{s-1})]^{m-1} (b^{r-1} a^{p-1})^n$, if $c \in H_{\alpha\beta}$ there exist a positive integer t and $x \in H_{\alpha\beta}$ such that $c^t = xab$, and it results $ab = abc = abc^t = abxab$. Then ab is in a subgroup of H (Prop. F), H has a kernel K (Lemma 1.8) and, since $ab \in K$, it is also $a^2 b^2 \in K$.

If on the contrary $c \notin H_{\alpha\beta}$, it must be either $p = s = 1$ or $q = r = 1$ and the (4) become respectively

$$ab = b^r a, \quad ba = ab^q \quad (r, q > 1) \tag{5}$$

and

$$ab = ba^s, \quad ba = a^p b \quad (s, p > 1). \tag{6}$$

In the first case, there are four positive integers i, j, h, k such that $a^2b = a(ab) = (ab)^i a^j = (ab)^{i-1} a(ba) a^{j-1} = (ab)^{i-1} a^2bb^{q-1}a^{j-1} = (a^2b)^h (ab)^{(i-1)k}b^{q-1}a^{j-1}$. Let $d = (a^2b)^{h-1} (ab)^{(i-1)k}b^{q-1}a^{j-1}$. Then, if $d \in H_{\alpha\beta}$, there are a positive integer t and $y \in H_{\alpha\beta}$ such that $d^t = ya^2b$; hence $a^2b = a^2bd = a^2bd^t = a^2bya^2b$ and, by Prop. F and Lemma 1.8 it follows that $a^2b \in K$. If $d \notin H_{\alpha\beta}$, it must be $i = j = 1$, whence $a^2b = aba$. In a similar way we find that either $ba^2 \in K$, or $ba^2 = aba$. In the second case, utilizing the (6) instead of the (5), we may deduce as above that either $ab^2 \in K$ or $ab^2 = bab$, as well as either $b^2a \in K$ or $b^2a = bab$. Moreover, since S is quasi hamiltonian, it is immediate that $a^2b \in K \Leftrightarrow ba^2 \in K$ and $ab^2 \in K \Leftrightarrow b^2a \in K$. Now, suppose $a^2b^2 \notin K$ for some $a, b \in S$. Then, if the (5) hold, it results $a^2b, ba^2 \notin K$, whence $a^2b = aba = ba^2$. If on the contrary the (6) hold, it results $ab^2, b^2a \notin K$, whence $ab^2 = bab = b^2a$. In both cases it follows immediately $a^2b^2 = (ab)^2 = b^2a^2$.

Theorem 1.12. *A quasi hamiltonian semigroup S is strongly reversible*[2]).

Proof. Let $a, b \in S$. If the relations $(ab)^2 = a^2b^2 = b^2a^2$ do not hold, it follows from Lemma 1.11 that a^2b^2 belongs to the kernel K of $H = \langle a, b \rangle$. It is immediate that also $b^2a^2 \in K$. Moreover, denoting by u the identity of K, we have obviously $au, bu \in K$. The t-archimedean component $H_{\alpha\beta}$ of H is, by Th. 1.5, either a periodic power joined semigroup[3]) or a commutative semigroup. In the first case there exists an integer $h > 2$ such that $(ab)^h = (ba)^h = (au)^h = (bu)^h = u$ and, by Prop. C, we have $(ab)^h = (au)^h (bu)^h = a^{h-2}a^2b^2ub^{h-2} = a^hb^h$. Similarly, $(ba)^h = b^ha^h$. If $H_{\alpha\beta}$ is commutative, there exists an integer $h > 2$ such that $(ab)^h = (ba)^h \in K$. Thus we have $(ab)^h = (ab)^h u = (aubu)^h = (au)^h (bu)^h = a^{h-2}a^2b^2ub^{h-2} = a^hb^h$. Similarly $(ba)^h = b^ha^h$. So S is strongly reversible.

Remark 1.13. At this point we may note that, since a quasi hamiltonian semigroup is strongly reversible (Th. 1.12) and a quasi hamiltonian archimedean semigroup is t-archimedean (Prop. D), utilizing Th. 4 and Lemma 8 of [3], we may refind the following result due to Pondělíček (see [10], Th. 3):

For a quasi hamiltonian semigroup S the following are equivalent:

i) *All proper subsemigroups of S are archimedean.*

ii) *All proper subsemigroups of S are power joined.*

iii) *S is either power joined or a semilattice of order* 2.

Since quasi hamiltonian semigroups are a generalization of quasicommutative

[2]) A semigroup S is called *strongly reversible* if for every $a, b \in S$ there are three positive integers p, q, r such that $(ab)^p = a^qb^r = b^ra^q$.

[3]) A semigroup S is called *power joined* if for every $a, b \in S$ there are two positive integers m, n such that $a^m = b^n$. It is a well known fact that a periodic power joined semigroup is an ideal extension of a periodic group by a nilsemigroup and conversely.

semigroups[4]), and non-commutative quasicommutative semigroups have always an idempotent, it arises naturally the problem whether also a quasi hamiltonian semigroup has necessarily an idempotent. The question is answered in the negative, as it is shown by the following counterexample.

Let us consider the set $S = \{a^h, b^h, a^h b, ab^2, ab^3, ab^4, ab^5\}$ with $h = 1, 2, 3, \ldots$, and define in S the following product.

$$a^h \cdot a^k = a^{h+k},\ b^h \cdot b^k = b^{h+k},\ (a^h b^k) \cdot (a^r b^s) = a^{h+r} b,$$

$$a^h \cdot b^k = \begin{cases} ab^{[k]} & \text{if } h = 1, \\ a^h b & \text{if } h > 1, \end{cases} \quad b^k \cdot a^h = \begin{cases} ab^{[2k]} & \text{if } h = 1, \\ a^h b & \text{if } h > 1, \end{cases}$$

$$a^h \cdot (a^r b^s) = (a^r b^s) \cdot a^h = a^{h+r} b,$$

$$b^k \cdot (a^r b^s) = \begin{cases} ab^{[s+2k]} & \text{if } r = 1, \\ a^r b & \text{if } r > 1, \end{cases}$$

$$(a^r b^s) \cdot b^k = \begin{cases} ab^{[s+k]} & \text{if } r = 1, \\ a^r b & \text{if } r > 1, \end{cases}$$

where $[m]$ denotes the least residue of m modulo 5. It is easily verifiable that the operation $\cdot$ is associative, consequently $(S, \cdot)$ is a semigroup. It is also immediate that $(S, \cdot)$ does not contain idempotents. It remains to prove that $(S, \cdot)$ is quasi hamiltonian. To this end let us remark that the elements a^h and $a^h b$ with $h > 1$ are in the center of $(S, \cdot)$. Now it is straightforward to verify that

$$a \cdot (ab^k) = (ab^k) \cdot a, \quad (ab^k) \cdot (ab^r) = (ab^r) \cdot (ab^k),$$

$$b^k \cdot b^r = b^r \cdot b^k, \quad a \cdot b^k = b^{3k} \cdot a, \quad b^k \cdot a = a \cdot b^{2k},$$

$$(ab^r) \cdot b^k = b^{3k} \cdot (ab^r), \quad b^k \cdot (ab^r) = (ab^r) \cdot b^{2k}.$$

Thus, we have proved that there are quasi hamiltonian semigroups without idempotents.

2. SEPARATIVE QASI HAMILTONIAN SEMIGROUPS

The present section is devoted to examine separative quasi hamiltonian semigroups[5]), for which we have found results analogous to those already obtained in [5] for separative quasicommutative semigroups. For this reason we judged convenient to omit the proofs which result to be substantially equal to those contained in [5].

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[4]) A semigroup S is called *quasicommutative* if for every $a, b \in S$ there exists a positive integer r such that $ab = b^r a$.

[5]) Remark that here for *separative* we mean a semigroup S where $a^2 = ab = b^2$ imply $a = b$ $(a, b \in S)$; such semigroups are called weakly separative in [8].

Lemma 2.1. *Let S be a separative quasi hamiltonian semigroup. For every $a, b \in S$, the relation $ab \neq ba$ implies that ab and ba are in a quasi hamiltonian subgroup of S.*

Proof. S, being separative and strongly reversible (Th. 1.12), is a semilattice of cancellative semigroups (see [2], Prop. 8). Then its t-archimedean components S_α are cancellative. Let $a \in S_\alpha$, $b \in S_\beta$ with $ab \neq ba$. If $S_{\alpha\beta}$ is commutative, it results $(ab)(aba) = (aba)(ab)$, whence $ab = ba$, a contradiction. Thus, by Th. 1.5, $S_{\alpha\beta}$ is a periodic quasi hamiltonian power joined semigroup which, being cancellative, results to be a quasi hamiltonian group.

Corollary 2.2. *A non commutative separative quasi hamiltonian semigroup contains at least an idempotent.*

Corollary 2.3. *Let S_α, S_β, $S_{\alpha\beta}$ be three t-archimedean components of a separative quasi hamiltonian semigroup S. If $S_{\alpha\beta}$ has no idempotents, each element of S_α permutes with each element of $S_\beta \cup S_{\alpha\beta}$.*

Corollary 2.4. *A cancellative non commutative quasi hamiltonian semigroup S has an identity u. Moreover, for any $a, b \in S$, the relation $ab \neq ba$ implies that a and b are in a same subgroup of S.*

Proof. By Corollary 2.2, S contains an idempotent u which, S being cancellative, results to be the only idempotent and the identity of S. Therefore S contains an unique maximal subgroup G. Since every relation $ab \neq ba$ implies $ab, ba \in G$ (Lemma 2.1), it follows $G \subseteq S_{\alpha\beta}$. Moreover G is an ideal of $S_\alpha \cup S_\beta \cup S_{\alpha\beta}$, consequently we have $a = au \in G$ and $b = bu \in G$.

Lemma 2.5. *A cancellative non commutative quasi hamiltonian semigroup S is a group.*

Proof. S is a semilattice of t-archimedean semigroups S_α, which are either commutative or cancellative periodic power joined semigroups and consequently groups (Prop. E and Th. 1.5). Since S contains an unique idempotent u, which is the identity of S (Corollary 2.4), the only t-archimedean component of S containing u is a group G, while the others are commutative semigroups. Moreover G is not commutative, otherwise S itself should be commutative (see [8], III.7.12; 10). Then, if there exists an $a \in S \setminus G$, for every $b \in G$ we have $ab = ba$ (Corollary 2.4) and $ab \in S \setminus G$, since $ab \in G$ should imply $a = (ab)b^{-1} \in G$. Now, let $c \in G$. Then it results $abc = bac = acb$, whence $bc = cb$, a contradiction, since G is not commutative. Thus $S \setminus G$ is empty, and $S = G$.

At this point, utilizing the above Lemmas and Corollaries, and repeating substantially the proofs of Lemma 2, Th. 3 and Th. 7 of [5], we obtain the following:

Theorem 2.6. *For a semigroup S the following conditions are equivalent:*

i) *S is a semilattice of separative quasi hamiltonian semigroups.*

ii) *S is a semilattice of cancellative quasi hamiltonian semigroups.*

iii) *S is separative quasi hamiltonian.*

iv) *S is embeddable in a quasi hamiltonian inverse semigroup.*

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