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ALMOST PRODUCT RIEMANNIAN MANIFOLDS

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Introduction and notation. An almost product manifold is a smooth m-manifold M with a smooth tensor field F of type (1,1) such that $F^2 = I_m$. If M has a positive definite Riemannian metric g such that $g(\bar{X}, \bar{Y}) = g(X, Y)$, (where X, Y are arbitrary vector fields and $\bar{X} = FX$ throughout the paper) then M is called an *almost product Riemannian manifold* or *almost para Hermite manifold*, [1], [2], [5]. If $(D_X F)\, Y = 0$ for all X, Y then M is called *para Kahler*. D is the natural metric connection. The manifold M will be called *para Hermite* if $(D_X F)\, Y \neq 0$ and there exist real constants $K_1, K_2, \ldots, K_7$, not all zeros such that:

$$(*) \qquad (D_X F)\, Y + K_1(D_{\bar{X}} F)\, Y + K_2(D_X F)\, \bar{Y} + K_3(D_{\bar{X}} F)\, \bar{Y} + K_4(D_Y F)\, X + K_5(D_Y F)\, \bar{X} + K_6(D_{\bar{Y}} F)\, X + K_7(D_{\bar{Y}} F)\, \bar{X} = 0\,.$$

$(*)$ will be called a *para Hermite condition.*

In this paper we will determine all classes of para Hermite manifolds and discuss inclusions among them. In the last section we discuss the relation between integrability and parallelism of distributions on M as related to par Hermite structures. I would like to thank professor R. S. Mishra for the useful discussions.

Proposition 1. *Let (M, g, F) be an almost product Riemannian manifold. If $F'(X, Y) = g(\bar{X}, Y)$ then:*

i) $F'(X, Y) = F'(Y, X) = F'(\bar{X}, \bar{Y})$,
ii) $(D_X F')(Y, Z) = g((D_X F)\, Y, Z) = g(Y, (D_X F)\, Z)$,
iii) $(D_X F')(Y, Z) = -(D_X F')(\bar{Y}, \bar{Z})$,
iv) $(D_X F')(\bar{Y}, Z) = -(D_X F')(Y, \bar{Z})$,
v) $(D_X F')(Y, Z) = (D_X F')(Z, Y)$

where X, Y, Z are arbitrary vector fields.

The proof of this proposition is easy and will be omitted.

Theorem 1. *The following is a para Hermite condition:*

$$(P_1) \qquad (D_X F)\, Y - (D_Y F)\, X = 0\,.$$

A para Hermite manifold satisfying (P_1) will be called P_1-Hermite.

Proof. Define $I = (D_X F)\, Y$, $\sigma = (D_Y F)\, X$, then $\{I, \sigma\}$ admits the following table of multiplication:

Table 1

	I	σ
I	I	σ
σ	σ	I

Thus the set G_1 of all linear combinations of I, σ is an infinite commutative ring under addition and multiplication as defined in table 1. Now in G_1 we have:

$$I = \sigma^2 , \quad (I - \sigma)(I + \sigma) = 0 .$$

Thus we have one of the following possibilities:

i) $I - \sigma = 0$, $I + \sigma \neq 0$, then $(D_X F)\, Y - (D_Y F)\, X = 0$ and the manifold is P_1-Hermite.

ii) $I - \sigma \neq 0$, $I + \sigma = 0$, then $(D_X F)\, Y + (D_Y F)\, X = 0$ or $(D_X F)\, Y = -(D_Y F)\, X$.

Using Proposition 1 we get:

$$\begin{aligned}(D_X F')(Y, Z) &= -(D_Y F')(X, Z) = -(D_Y F')(Z, X) = \\ &= (D_Z F')(X, Y) = -(D_X F')(Y, Z) .\end{aligned}$$

Thus $(D_X F')(Y, Z) = 0$. Since Z is arbitrary then using Proposition 1 again we get $(D_X F)\, Y = 0$ and $I + \sigma = 0$ is equivalent to the para Kähler condition, i.e. it is not a para Hermite condition.

iii) $I - \sigma = 0$, $I + \sigma = 0$, is not a para Hermite condition as can be seen from (ii) above.

iv) $I - \sigma \neq 0$, $I + \sigma \neq 0$, the manifold is not para Hermite.

Since $I - \sigma = 0$ is a para Hermite condition then $(I - \sigma)^n = 0$, $n \in N$ will give para Hermite conditions. But $(I - \sigma)^2 = 0 \Leftrightarrow I - 2\sigma + I = 0 \Leftrightarrow 2(I - \sigma) = 0$ which is the P_1-Hermite condition. So is $(I - \sigma)^n = 0$, $n \in N$. Thus we conclude that in G_1, the only para Hermite condition is the one obtained.

Theorem 2. *The following are para Hermite conditions*:

(P_2) $\quad (D_X F)\, Y - (D_{\bar X} F)\, Y = 0 ,$ $\quad$ *P_2-Hermite* ,

(P_3) $\quad (D_X F)\, Y + (D_{\bar X} F)\, Y = 0 ,$ $\quad$ *P_3-Hermite* ,

(P_4) $\quad (D_X F)\, Y - (D_X F)\, \bar Y = 0 ,$ $\quad$ *P_4-Hermite* ,

(P_5) $\quad (D_X F)\, Y + (D_X F)\, \bar Y = 0 ,$ $\quad$ *P_5-Hermite* ,

(P_6) $\quad (D_X F)\, Y - (D_{\bar X} F)\, \bar Y = 0 ,$ $\quad$ *P_6-Hermite* ,

(P_7) $\quad (D_X F)\, Y + (D_{\bar X} F)\, \bar Y = 0 ,$ $\quad$ *P_7-Hermite* ,

(P_8) $\quad (D_X F)\, Y - (D_{\bar X} F)\, Y - (D_X F)\, \bar Y + (D_{\bar X} F)\, \bar Y = 0 ,$ $\quad$ *P_8-Hermite* ,

$$(P_9) \qquad (D_X F)\, Y - (D_{\bar{X}} F)\, Y + (D_X F)\, \bar{Y} - (D_{\bar{X}} F)\, \bar{Y} = 0\,, \qquad P_9\text{-}Hermite\,,$$
$$(P_{10}) \qquad (D_X F)\, Y + (D_{\bar{X}} F)\, Y - (D_X F)\, \bar{Y} - (D_{\bar{X}} F)\, \bar{Y} = 0\,, \qquad P_{10}\text{-}Hermite\,,$$
$$(P_{11}) \qquad (D_X F)\, Y + (D_{\bar{X}} F)\, Y + (D_X F)\, \bar{Y} + (D_{\bar{X}} F)\, \bar{Y} = 0\,, \qquad P_{11}\text{-}Hermite\,.$$

Proof. Let $I = (D_X F)\, Y$, $\alpha = (D_{\bar{X}} F)\, Y$, $\beta = (D_X F)\, \bar{Y}$, $\gamma = (D_{\bar{X}} F)\, \bar{Y}$. Then I, α, β, γ admit the following table of multiplication:

Table 2

	I	α	β	γ
I	I	α	β	γ
α	α	I	γ	β
β	β	γ	I	α
γ	γ	β	α	I

The set G_2 of all linear combinations of I, α, β, γ will be an infinite commutative ring under addition and multiplication as in table 2. Now in G_2 we have:

$$I = \alpha^2\,, \qquad (I - \alpha)(I + \alpha) = 0\,,$$
$$I = \beta^2\,, \qquad (I - \beta)(I + \beta) = 0\,,$$
$$I = \gamma^2\,, \qquad (I - \gamma)(I + \gamma) = 0\,,$$

Arguments similar to those in theorem 1 will give P_i-Hermite, $i = 2, \ldots, 7$. Since products of para Hermite conditions give para Hermite conditions, we take products of conditions (2) to (7) on the following table:

Table 3

	P_2	P_3	P_4	P_5	P_6	P_7	P_8	P_9	P_{10}	P_{11}
P_2	P_2	0	P_8	P_9	P_9	P_8	P_8	P_9	0	0
P_3		P_3	P_{10}	P_{11}	P_{10}	P_{11}	0	0	P_{10}	P_{11}
P_4			P_4	0	P_{10}	P_8	P_8	0	P_{10}	0
P_5				P_5	P_9	P_{11}	0	P_9	0	P_{11}
P_6					P_6	0	0	P_9	P_{10}	0
P_7						P_7	P_8	0	0	P_{11}
P_8							P_8	0	0	0
P_9								P_9	0	0
P_{10}									P_{10}	0
P_{11}										P_{11}

Thus from table 3 we see that multiplications of P_2 to P_7 yield

$$(P_8) \qquad (I - \alpha)(I - \beta) = 0\,, \quad P_8\text{-}Hermite\,,$$
$$(P_9) \qquad (I - \alpha)(I + \beta) = 0\,, \quad P_9\text{-}Hermite\,,$$

(P_{10}) $\qquad (I+\alpha)(I-\beta)=0\,, \quad P_{10}$*-Hermite*,

(P_{11}) $\qquad (I+\alpha)(I+\beta)=0\,, \quad P_{11}$*-Hermite*

and that P_2 to P_{11} are the only structures that can be obtained in G_2.

Now, in order to get more structures, we should be able to multiply elements of G_1 by those of G_2. This can be done if G_1, G_2 are subrings of a larger ring G.

G can be constructed in four different ways according to how we interpret the operations α, β, γ, σ.

Case 1. Let α, β, γ bar X, Y, X and Y respectively and σ switches slots. In this case the set G_3 of all linear combinations of $I, \alpha, \beta, \gamma, \sigma, \sigma\alpha, \sigma\beta, \sigma\gamma$ is a commutative ring with multiplication defined by tables 1, 2 and the following:

(A) $\qquad \sigma\alpha = \alpha\sigma = (D_Y F)\overline{X}\,, \quad \sigma\beta = \beta\sigma = (D_{\overline{Y}}F)X\,, \quad \sigma\gamma = \gamma\sigma = (D_{\overline{Y}}F)\overline{X}\,.$

Theorem 3. *The following are para Hermite conditions*:

(P_{12}) $\quad (D_X F)\,Y - (D_Y F)X - (D_{\overline{X}}F)\,Y + (D_Y F)\overline{X} = 0\,,$ $\qquad P_{12}$*-Hermite*,

(P_{13}) $\quad (D_X F)\,Y - (D_Y F)X + (D_{\overline{X}}F)\,Y - (D_Y F)\overline{X} = 0\,,$ $\qquad P_{13}$*-Hermite*,

(P_{14}) $\quad (D_X F)\,Y - (D_Y F)X - (D_X F)\,\overline{Y} + (D_{\overline{Y}}F)X = 0\,,$ $\qquad P_{14}$*-Hermite*,

(P_{15}) $\quad (D_X F)\,Y - (D_Y F)X + (D_X F)\,\overline{Y} - (D_{\overline{Y}}F)X = 0\,,$ $\qquad P_{15}$*-Hermite*,

(P_{16}) $\quad (D_X F)\,Y - (D_Y F)X - (D_{\overline{X}}F)\,\overline{Y} + (D_{\overline{Y}}F)\overline{X} = 0\,,$ $\qquad P_{16}$*-Hermite*,

(P_{17}) $\quad (D_X F)\,Y - (D_Y F)X + (D_{\overline{X}}F)\,\overline{Y} - (D_{\overline{Y}}F)\overline{X} = 0\,,$ $\qquad P_{17}$*-Hermite*,

(P_{18}) $\quad (D_X F)\,Y - (D_{\overline{X}}F)\,Y - (D_X F)\,\overline{Y} + (D_{\overline{X}}F)\,\overline{Y} - (D_Y F)X +$
$\qquad + (D_Y F)\overline{X} + (D_{\overline{Y}}F)X - (D_{\overline{Y}}F)\overline{X} = 0\,,$ $\qquad P_{18}$*-Hermite*,

(P_{19}) $\quad (D_X F)\,Y - (D_{\overline{X}}F)\,Y + (D_X F)\,\overline{Y} - (D_{\overline{X}}F)\,\overline{Y} - (D_Y F)X +$
$\qquad + (D_Y F)\overline{X} - (D_{\overline{Y}}F)X + (D_{\overline{Y}}F)\overline{X} = 0\,,$ $\qquad P_{19}$*-Hermite*,

(P_{20}) $\quad (D_X F)\,Y + (D_{\overline{X}}F)\,Y - (D_X F)\,\overline{Y} - (D_{\overline{X}}F)\,\overline{Y} - (D_Y F)X -$
$\qquad - (D_Y F)\overline{X} + (D_{\overline{Y}}F)X + (D_{\overline{Y}}F)\overline{X} = 0\,,$ $\qquad P_{20}$*-Hermite*,

(P_{21}) $\quad (D_X F)\,Y + (D_{\overline{X}}F)\,Y + (D_X F)\,\overline{Y} + (D_{\overline{X}}F)\,\overline{Y} - (D_Y F)X -$
$\qquad - (D_Y F)\overline{X} - (D_{\overline{Y}}F)X - (D_{\overline{Y}}F)\overline{X} = 0\,,$ $\qquad P_{21}$*-Hermite*.

Proof. Conditions 12 to 21 are given respectively by:

$$(I-\sigma)(I-\alpha),\ (I-\sigma)(I+\alpha),\ (I-\sigma)(I-\beta),\ (I-\sigma)(I+\beta),$$
$$(I-\sigma)(I-\gamma),\ (I-\sigma)(I+\gamma),\ (I-\sigma)(I-\alpha-\beta+\gamma),$$
$$(I-\sigma)(I-\alpha+\beta-\gamma),\ (I-\sigma)(I+\alpha-\beta-\gamma),\ (I-\sigma)(I+\alpha+\beta+\gamma)\,.$$

From the above construction and theorems 1 and 2 we see that these are the only structures in G_3.

Case 2. In this case we let α, β, γ bar first, seconds, first and second slots respectively and σ switches vectors. The set G_4 of all linear combinations of I, α, β, γ, σ, $\sigma\alpha$, $\sigma\beta$, $\sigma\gamma$ is an infinite commutative ring with G_1, G_2 as subrings. In G_4 product is given by Tables 1, 2 and the following:

$$\text{(B)} \qquad \sigma\alpha = \alpha\sigma = (D_{\bar{Y}}F)\,X\,, \quad \sigma\beta = \beta\sigma = (D_Y F)\,\bar{X}\,, \quad \sigma\gamma = \gamma\sigma = (D_{\bar{Y}}F)\,\bar{X}\,.$$

Theorem 4. *The following are para Hermite conditions*

$$(P_{22}) \qquad (D_X F)\,Y - (D_Y F)\,X - (D_{\bar{X}}F)\,Y + (D_{\bar{Y}}F)\,X = 0\,, \qquad P_{22}\text{-}Hermite\,,$$
$$(P_{23}) \qquad (D_X F)\,Y - (D_Y F)\,X + (D_{\bar{X}}F)\,Y - (D_{\bar{Y}}F)\,X = 0\,, \qquad P_{23}\text{-}Hermite\,,$$
$$(P_{24}) \qquad (D_X F)\,Y - (D_Y F)\,X - (D_X F)\,\bar{Y} + (D_Y F)\,\bar{X} = 0\,, \qquad P_{24}\text{-}Hermite\,,$$
$$(P_{25}) \qquad (D_X F)\,Y - (D_Y F)\,X + (D_X F)\,\bar{Y} - (D_Y F)\,\bar{X} = 0\,, \qquad P_{25}\text{-}Hermite\,.$$

Proof. These are respectively given by $(I - \sigma)(I - \alpha)$, $(I - \sigma)(I + \alpha)$, $(I - \sigma)\,.$ $.\,(I - \beta)$, $(I - \sigma)(I + \beta)$.

The rest of the products give structures already given in theorem 3.

Case 3. Let α, β, γ bar vectors and σ switches vectors. Here we have:

$$\text{(C)} \qquad \sigma\alpha = \beta\sigma = (D_{\bar{Y}}F)\,X\,, \quad \alpha\sigma = \sigma\beta = (D_Y F)\,\bar{X}\,, \quad \sigma\gamma = \gamma\sigma = (D_{\bar{Y}}F)\,\bar{X}\,.$$

If we take G_5 to be the set of all linear combinations of I, α, β, γ, σ, $\sigma\alpha$, $\alpha\sigma$, $\sigma\gamma$, then G_5 will be an infinite non commutative ring with G_1, G_2 as subrings.

To get new structures in G_5 we have to multiply on both sides. It can be seen easily from A, B and C that if we multiply in one direction, we obtain the structures of theorem 3, and if we multiply in the other direction we get those of theorem 4. Thus in this case no new structures are obtained.

Case 4. Here α, β, γ bar slots and σ switches slots and we get:

$$\alpha\sigma = \sigma\beta = (D_{\bar{Y}}F)\,X\,, \quad \sigma\alpha = \beta\sigma = (D_Y F)\,\bar{X}\,, \quad \sigma\gamma = \gamma\sigma = (D_{\bar{Y}}F)\,\bar{X}$$

and as in case 3 nothing new is obtained.

Inclusions. Now, let us study inclusions among these classes of para Hermite structures. The class of P_i-Hermite manifolds is contained in the class of P_j-Hermite manifolds $(P_i \subset P_j)$ if

$$(P_i) \Rightarrow (P_j)$$

where (P_i), (P_j) represent the para Hermite conditions.

Theorem 5. *In the class of para Hermite manifolds the following inclusion relations hold*:

$$P_1 \subset P_6\,, \quad P_2, P_4, P_7 \subset P_8\,, \quad P_2, P_5, P_6 \subset P_9\,,$$
$$P_3, P_4, P_6 \subset P_{10}\,, \quad P_3, P_5, P_7 \subset P_{11}\,,$$

$$
\begin{aligned}
&P_1, P_2 \subset P_{12}\,, \quad P_1, P_2 \subset P_{22}\,,\\
&P_1, P_3 \subset P_{13}\,, \quad P_1, P_3 \subset P_{23}\,,\\
&P_1, P_4 \subset P_{14}\,, \quad P_1, P_4 \subset P_{24}\,,\\
&P_1, P_5 \subset P_{15}\,, \quad P_1, P_5 \subset P_{25}\,,\\
&P_1, P_6 \subset P_{16}\,, \quad P_1, P_7 \subset P_{17}\,, \quad P_1, P_8 \subset P_{18}\,,\\
&P_1, P_9 \subset P_{19}\,, \quad P_1, P_{10} \subset P_{20}\,, \quad P_1, P_{11} \subset P_{21}\,.
\end{aligned}
$$

Proof. We will only prove that $P_1 \subset P_6$. The rest follow directly from the definitions of the structures.

Since $(D_X F^2) = (D_X F)\,F + F(D_X F)$, and $F^2 = I_m$ we get:

(a) $$(D_X F)\,\overline{Y} = -\overline{(D_X F)\,Y},$$

for all vector fields X, Y and hence

(b) $$(D_{\overline{X}} F)\,\overline{Y} = -\overline{(D_{\overline{X}} F)\,Y},$$

if (P_1) holds, i.e. $(D_X F)\,Y - (D_Y F)\,X = 0$, then from (b) we get:

$$(D_{\overline{X}} F)\,\overline{Y} = -\overline{(D_{\overline{Y}} F)\,\overline{X}} = \overline{\overline{(D_Y F)\,X}} = (D_Y F)\,X = (D_X F)\,Y$$

and $(D_X F)\,Y - (D_{\overline{X}} F)\,\overline{Y} = 0$ which is (P_6).

Distributions. It is well known that the existence of an almost product structure F on a Riemannian manifold (M, g) is equivalent to the existence of two complementary distributions P, Q on M. That is, starting with F, then P and Q can be respectively defined by the following projections:

$$L_1 = \tfrac{1}{2}[I_m + F]\,, \quad L_2 = \tfrac{1}{2}[I_m - F]\,.$$

Conversely, starting with two complementary distributions P, Q on M, then F can be defined in a natural way as follows: $F(X + Y) = X - Y$ where $X + Y$ is any vector field. In fact, we can start with one distribution P and take Q its orthogonal complement.

In the following theorems, which give a geometric explanation of some of the para Hermite structures found, F is always constructed as above. The proofs of these theorems are direct applications of the results of Walker [3] and Willmore [4] and will be omitted.

Theorem 6. *Let P be a distribution on a Riemannian manifold (M, g). If (M, g, F) is P_{11}-Hermite then P is a semi-parallel.*

Theorem 7. *Let P, Q be two complementary distributions on a Riemannian manifold (M, g). If (M, g, F) is P_9-Hermite then one of the distributions is parallel along the other.*

Theorem 8. *Let P, Q be two complementary distributions on a Riemannian manifold (M, g). If (M, g, F) is P_7-Hermite then both distributions are semi parallel.*

Theorem 9. *Let P be a distribution on a Riemannian manifold (M, g). If (M, g, F) is P_5-Hermite then P is parallel and hence integrable.*

Theorem 10. *Let P, Q be two complementary distributions on a Riemannian manifold (M, g). If (M, g, F) is para Kähler then both distributions are parallel and hence integrable.*

References

[1] *R. S. Mishra:* Structures in a differentiable manifold, INSA Scientific report series No. 3, New Delhi, 1978.

[2] *S. Tachibana:* Some theorems on locally product Riemannian manifolds, Tohoku Math. J. *12* (1960), 281—292.

[3] *A. G. Walker:* Connexions for parallel distributions in the large I, II, Quart. J. Math. Oxford (2), *6*, (1955), 301—308 and *9*, (1958), 221—231.

[4] *T. J. Willmore:* Connexion for a system of parallel distributions, Quart. J. Math. Oxford (2), *7*, (1956), 209—276.

[5] *K. Yano:* Differential geometry on complex and almost complex spaces, Pergamon Press, (1965).

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