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EDGE-DOMATIC NUMBER OF A GRAPH

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With help of the concept of a dominating set, E. J. Cockayne and S. T. Hedetniemi [1] have defined the domatic number of a graph. Here we shall introduce the edge analogue of this concept and prove some assertions concerning it.

Let G be an undirected graph without loops and multiple edges. Two edges e_1, e_2 of G are called adjacent, if they have an end vertex in common. The degree of an edge e in G is the number of edges of G which are adjacent to e.

An independent set of edges of a graph G is a subset of the edge set of G with the property that no two edges of this set are adjacent. A set A of edges of a graph G is said to cover a set B of vertices of G, if each vertex of B is an end vertex of at least one edge of A.

An edge-dominating set [3] in G is a subset D of the edge set $E(G)$ of G with the property that for each edge $e \in E(G) - D$ there exists at least one edge $f \in D$ adjacent to e. An edge-domatic partition of G is a partition of $E(G)$, all of whose classes are edge-dominating sets in G. The maximum number of classes of an edge-domatic partition of G is called the edge-domatic number of G and denoted by $ed(G)$.

Note that the edge-domatic number of G is equal to the domatic number [1] of the line-graph of G.

First we shall determine edge-domatic numbers of complete graphs and complete bipartite graphs.

Proposition 1. *Let K_n be the complete graph with n vertices, $n \geqq 2$. If n is even, then $ed(K_n) = n - 1$; if n is odd, then $ed(K_n) = n$.*

Proof. Let n be even. Then it is well-known that K_n can be decomposed into $n - 1$ pairwise edge-disjoint linear factors. The edge set of each of these factors is evidently an edge-dominating set in K_n. Hence $ed(K_n) \geqq n - 1$. Suppose that $ed(K_n) \geqq n$. Consider an edge-domatic partition of K_n with n classes. As the number of edges of K_n is $\frac{1}{2}n(n-1)$, the mean value of the cardinalities of these classes is $\frac{1}{2}(n-1)$. This implies that at least one of the classes has at most $[\frac{1}{2}(n-1)] = \frac{1}{2}n - 1$ edges. But then this set C of edges covers at most $n = 2$ vertices. There are two vertices which are incident to no edge of C and the edge joining these vertices is adjacent to no edge of C, which is a contradiction with the assumption that C is an edge-dominating set. We have proved that $ed(K_n) = n - 1$ for n even.

Now let n be odd. Denote the vertices of K_n by $u_1, \ldots, u_n$. In the sequel all subscripts will be taken modulo n. For each $i = 1, \ldots, n$ let E_i be the set of all edges $u_{i+j}u_{i-j+1}$, where $j = 1, \ldots, \frac{1}{2}(n - 1)$. The reader may verify himself that the sets $E_1, \ldots, E_n$ form a partition of the edge set of K_n. Each set E_i covers all vertices of K_n except one. Each edge of K_n not belonging to E_i is incident with at least one vertex covered by E_i and thus adjacent to at least one edge of E_i; the sets $E_1, \ldots, E_n$ form a domatic partition of K_n and $ed(K_n) \geqq n$. Suppose that $ed(K_n) \geqq n + 1$. Then we analogously prove that there exists an edge-domatic partition of G, one of whose classes has at most $\frac{1}{2}(n - 3)$ edges; this set covers at most $n - 3$ vertices and it is not an edge-dominating set, which is a contradiction. Therefore $ed(K_n) = n$ for n odd.

Proposition 2. *Let $K_{m,n}$ be a complete bipartite graph. Then $ed(K_{m,n}) = \max(m, n)$.*

Proof. Without loss of generality let $m \geqq n$, i.e. $\max(m, n) = m$. Let $K_{m,n}$ be the bipartite graph on the vertex sets A, B such that $|A| = m$, $|B| = n$. Then for each $u \in A$ the set $E(u)$ of edges which are incident with u is an edge-dominating set in $K_{m,n}$; it covers all vertices of B and each edge of $K_{m,n}$ has one end vertex in B. Therefore the sets $E(u)$ for all $u \in A$ form an edge-domatic partition of $K_{m,n}$ with m classes. We have proved that $ed(K_{m,n}) \geqq m$. Now suppose that $ed(K_{m,n}) \geqq m + 1$ and consider an edge-domatic partition of $K_{m,n}$ with $m + 1$ classes. As $K_{m,n}$ has mn edges, there exists at least one class C of this partition which contains less than n edges. Then this set C covers neither A nor B. There exists a vertex of A and a vertex of B which are incident with no edge of C and the edge joining them is adjacent to no edge of C. The set C is not edge-dominating, which is a contradiction. Hence $ed(K_{m,n}) = m = \max(m, n)$.

Proposition 3. *Let C_n be a circuit of the length n. If n is divisible by 3, then $ed(C_n) = 3$, otherwise $ed(C_n) = 2$.*

Proof. A circuit is isomorphic to its own line-graph, therefore its edge-domatic number is equal to its domatic number and for it this assertion was proved in [1].

Now we shall prove two theorems.

Theorem 1. *For each finite undirected graph G we have*

$$\delta(G) \leqq ed(G) \leqq \delta_e(G) + 1 ,$$

where $ed(G)$ is the edge-domatic number of G, $\delta(G)$ is the minimum degree of a vertex of G and $\delta_e(G)$ is the minimum degree of an edge of G. These bounds cannot be improved.

Proof. The number $\delta_e(G)$ is equal to the minimum degree of a vertex of the line-graph of G. According to [1], the domatic number of this line-graph cannot be greater than $\delta_e(G) + 1$; this domatic number is equal to the edge-domatic number of G. Hence $ed(G) \leqq \delta_e(G) + 1$.

Now we shall prove that $\delta(G) \leqq ed(G)$. By induction we shall prove the following assertion: If the degree of each vertex of G is greater than oı equal to k (where k is an arbitrary positive integer), then there exists an edge-domatic partition of G with k classes. For $k = 1$ the assertion is true; the required partition consists of one class equal to the whole $E(G)$ which is evidently an edge-dominating set in G. Now let $k_0 \geqq 2$ and suppose that the assertion is true for $k = k_0 - 1$. Consider a graph G in which the degree of each vertex is at least k_0. Let E_0 be a maximal (with respect to the set inclusion) independent set of edges of G. This set is edge-dominating; otherwise an edge could be added to it without violating the independence, which would be a contradiction with the maximality of E_0. Let G_0 be the graph obtained from G by deleting all edges of E_0. Each vertex of G is incident at most with one edge of E_0, therefore each vertex of G_0 has the degree at least $k_0 - 1$. According to the induction hypothesis there exists an edge-domatic partition $\mathscr{P}$ of G_0 with $k_0 - 1$ classes. Then $\mathscr{P} \cup \{E_0\}$ is an edge-domatic partition of G with k_0 classes, which was to be proved. The proved assertion implies $ed(G) \geqq \delta(G)$. If G is a circuit C_n and n is divisible by 3, then $ed(G) = \delta_e(G) + 1$. If G is a circuit C_n and n is not divisible by 3, then $ed(G) = \delta(G)$. (See Proposition 3.)

Theorem 2. *Let T be a tree, let $\delta_e(T)$ be the minimal degree of an edge of T. Then $ed(T) = \delta_e(T) + 1$.*

Proof. Let us have the colours $1, \ldots, \delta_e(T) + 1$; we shall colour the edges of T by them. First we choose a terminal edge e_0 of T and colour it by the colour 1. Now let us have an edge e or T with the end vertices u, v; suppose that all edges incident with v are already coloured. Moreover, if the number of these edges is less than $\delta_e(T) + 1$, we suppose that they are coloured by pairwise differed colours; in the opposite case we suppose that all colours $1, \ldots, \delta_e(T) + 1$ occur among the colours of these edges. Now we shall colour the edges incident with u and distinct from e. We colour them in the following way. If there are colours by which no edge incident with v is coloured, we use all of them. (This must be always possible according to the assumption.) If the number of edges to be coloured is less than $\delta_e(T) + 1$, we colour them by pairwise distinct colours; in the opposite case we colour them by using all the colours $1, \ldots, \delta_e(T) + 1$ (some of them may repeat). The result is a colouring of edges of T by the colours $1, \ldots, \delta_e(T) + 1$ with the property that each edge is adjacent to edges of all colours different from its own one. If C_i for $i = 1, \ldots, \delta_e(T) + 1$ is the set of all edges of T coloured by the colour i, then the sets $C_1, \ldots, C_{\delta_e(T)+1}$ form an edge-domatic partition of T with $\delta_e(T) + 1$ classes and $ed(T) \geqq \delta_e(T) + 1$. According to Theorem 1 it cannot be greater, therefore $ed(T) = \delta_e(T) + 1$.

Corollary 1. *The edge-domatic number of a path is equal to 2.*

Corollary 2. *The edge-domatic number of a star is equal to the number of its edges.*

Remark. As we have just seen, any tree is an example of a graph G for which $ed(G) = \delta_e(G) + 1$. Another example is the odd graph O_k for any integer k such that $k \geqq 2$; it was defined in [2]. It is a graph whose vertex set is the set of all subsets of the number set $\{1, \ldots, 2k + 1\}$ having the cardinality k and in which two vertices are adjacent if and only if their intersection (as of sets) is empty. In [4] it is proved that $ed(O_k) = 2k + 1$, while the degree of any edge of O_k is $2k$. Every complete graph K_n with n even is an example of a graph G for which $ed(G) = \delta(G)$ holds.

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