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NOTES ON COMPACT RINGS WITH OPEN RADICAL

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1. INTRODUCTION

By a *topological ring* R we mean an (associative) ring which is also a Hausdorff space such that $a - b$ and ab are continuous functions of a and b, where a and b are elements of R. If the space R is compact, we call R a *compact ring*. Jans investigated a compact ring R in which N^2 is open, where N is the Jacobson radical of R, and showed that in such a ring the powers N^r form a base for the neighborhood filter at zero ([1; Theorem 2]).

We shall treat in this paper a compact ring with identity and give several conditions which are equivalent to the assumption that N^2 is open (Theorem 11, below). As an application of this theorem we obtain a result (Theorem 12, below) concerning compact semilocal rings that is an improvement of the result of Warner [5; Theorem 4].

2. PRELIMINARIES

In what follows R will denote a compact ring with identity. Unless otherwise stated, *ideal* means two-sided ideal and *radical* always means Jacobson radical. An ideal P of R is said to be *maximal* if (i) $P \neq R$ and (ii) there is no proper ideal of R properly containing P. If A and B are ideals of R, AB denotes the *ideal product* of A and B, that is, AB is the ideal of R generated by the set of all products of the form ab, $a \in A$ and $b \in B$. Similarly, if $A_1, \ldots, A_n$ are a finite number of ideals of R, $A_1 \ldots A_n$ denotes the ideal product of the ideals $A_1, \ldots, A_n$. In particular, A^n denotes the ideal product of n copies of an ideal A. We denote by $\bar{S}$ the *topological closure* of a subset S of R.

Now we list some known results, concerning compact rings, which will be used later.

(I) If C is the connected component of 0 in R, then $C \cdot R = 0$, where $C \cdot R$ is the set of all products of the form cx, $c \in C$ and $x \in R$ (cf. [2; Lemma 10 and Theorem 8]).

From this it results that R is totally disconnected, because R has an identity. Hence R has a fundamental system of neighborhoods of 0 each of which is a compact ideal of R(cf. [2; Lemma 9]).

(II) If R is semisimple, then it is isomorphic and homeomorphic to a complete direct sum (with the product topology) of finite discrete simple rings (cf. [2; Theorem 16]).

(III) The radical N of R is closed (cf. [2; Corollary to Theorem 13]) and it is a topological nilpotent ideal, i.e., for any neighborhood U of 0 in R there exists a positive integer m such that $n > m$ implies $N^n \subset U$ (cf. [2; Theorem 15] and (I)).

The residue class ring R/N is a compact semisimple ring.

(IV) The radical N of R is the intersection of all open maximal ideals of R (cf. [4; Lemma 3.7]).

3. COMPACT RINGS WITH OPEN RADICAL

Proposition 1. *Let P be a maximal ideal of R. Then P is open if and only if it is closed.*

Proof. In the theory of topological groups it is well-known that open subgroups are closed, so the "only if" part of the proposition is obvious.

Assume that P is closed. Let $\{V_\alpha\colon \alpha \in \Lambda\}$ be a fundamental system of ideal neighborhoods of 0. If P is not open, then $V_\alpha \not\subset P$ for every $\alpha \in \Lambda$. As P is a maximal ideal, $P + V_\alpha = R$ for every $\alpha \in \Lambda$, and so $\bigcap_\alpha(P + V_\alpha) = R$. On the other hand, it is obvious that $\bar{P} = \bigcap_\alpha(P + V_\alpha)$. Therefore it follows that $P = R$, since P is closed, and this contradicts the fact that P is a maximal ideal of R. Hence P must be open.

Lemma 2. *Let R be semisimple, and let $R = \Sigma_i \oplus R_i$ be an expression of R as the complete direct sum of finite (discrete) simple rings R_i, $i \in I$ (cf. (II)). If the number of the component rings R_i appearing in the expression $R = \Sigma_i \oplus R_i$ is infinite, then R contains a maximal ideal which is everywhere dense in the space R.*

Proof. Let A be the subset of $\Sigma_i \oplus R_i$ consisting of all elements $a = (a_i)$ with the property that at most a finite number of the components a_i are different from zero. Since the number of the component rings R_i is infinite, it is easily verified that A is a proper ideal of R. Furthermore, it is not difficult to show that A is everywhere dense in the space R. Let P be a maximal ideal containing A; then P is the desired maximal ideal.

The following result is fairly easy to prove, so we will omit the proof.

Lemma 3. *Let $R = \Sigma_i \oplus R_i$, $i \in I$ be as in Lemma 2. If the number of the component rings R_i appearing in the expression $R = \Sigma_i \oplus R_i$ is infinite, then R has infinitely many maximal ideals.*

The next lemma is a well-known result and is valid for any ring with identity.

Lemma 4. *Every maximal ideal of R contains the radical of R.*

We can now prove the following theorem.

Theorem 5. *Let R be a compact ring with identity. Then all of the following conditions are equivalent*:

(1) *Every maximal ideal of R is open.*

(2) *Every maximal ideal of R is closed.*

(3) *The radical N of R is open.*

(4) *There exist only a finite number of maximal ideals in R.*

Proof. The equivalence of (1) and (2) has been proved in Proposition 1.

(2) ⇒ (3): We shall first show that every maximal ideal of the residue class ring R/N is closed. Let θ be the natural homomorphism of R onto R/N. It is well-known that the map θ is continuous and open. Suppose that P^* is maximal ideal of R/N, and let P be the complete inverse image of P^* by θ, i.e. $P = \{x \in R: \theta(x) \in P^*\}$. Since every maximal ideal of R contains N, it is easy to see that P is a maximal ideal of the ring R. By the assumption P is closed, so it is compact. Since θ is continuous, $P^* = \theta(P)$ is also compact, therefore it is closed in R/N. Namely, every maximal ideal of R/N is closed.

In view of Lemma 2, we see that R/N is isomorphic with a complete direct sum of a finite number of finite simple rings. That is, R/N is a finite ring. Hence $\{0^*\}$ (0^* denotes the zero element of the ring R/N) is an open set in R/N, so that N is open in R as the complete inverse image of $\{0^*\}$ by θ.

(3) ⇒ (4): Since every maximal ideal of R contains N, there is a one-to-one correspondence between the set of maximal ideals of R and the set of maximal ideals of R/N. As N is open, R/N is discrete, therefore it must be finite, because it is compact. Hence R/N has only a finite number of maximal ideals, so that R also has only a finite number of maximal ideals.

(4) ⇒ (1): Suppose that R has only a finite number of maximal ideals; then R/N also has only a finite number of maximal ideals. In view of Lemma 3, R/N is isomorphic to a complete direct sum of a finite number of finite simple rings. Therefore R/N is a finite ring, whence N is open. As maximal ideals of R contain N, they must be open.

4. COMPACT RINGS IN WHICH N^2 IS OPEN

In this section we shall establish three theorems concerning a compact ring with N^2 open, N the radical of the ring, which are the main results of this paper. Before proving the theorems we give several lemmas that will be needed for the proofs of the theorems.

The next lemma is the same as Lemma 2 in [5]. But for the sake of completeness, we state a proof of the lemma.

(To avoid repetition we remark once and for all that the term "left ideal (or left module)" may be replaced by the term "right ideal (or right module)"; we state only one of such pairs of leammas or theorems.)

Lemma 6. *Every finitely generated left ideal of R is compact, so it is closed in R.*

Proof. Let L be a left ideal of R with a finite number of generators $a_1, \ldots, a_n$: $L = Ra_1 + \ldots + Ra_n$. The sets Ra_i are compact as continuous images of the compact space R. Therefore $Ra_1 + \ldots + Ra_n$ is compact, since it is a continuous image of the compact space $Ra_1 \times \ldots \times Ra_n$.

To prove the following two lemmas compactness of R is not needed.

Lemma 7. *Let A and B be two ideals of R. If A and B are finitely generated as left R-modules, then AB is also finitely generated as a left R-module.*

Proof. Let $A = Ra_1 + \ldots + Ra_m$ and $B = Rb_1 + \ldots + Rb_n$. Then it follows that

$$\begin{aligned} AB &= A(Rb_1 + \ldots + Rb_n) = Ab_1 + \ldots + Ab_n = \\ &= (Ra_1 + \ldots + Ra_m)\, b_1 + \ldots + (Ra_1 + \ldots + Ra_m)\, b_n = \\ &= \sum_i \sum_j Ra_i b_j \,. \end{aligned}$$

Hence AB is finitely generated as a left R-module.

Lemma 8. *If $P_1, \ldots, P_n$ are mutually distinct maximal ideals of R, then we have $P_1 \cap \ldots \cap P_n = \sum_\pi P_{\pi(1)} \ldots P_{\pi(n)}$, where π ranges over all the permutations of the set $\{1, \ldots, n\}$.*

Proof. For any permutation π of the set $\{1, \ldots, n\}$, it is evident that $P_{\pi(1)} \ldots \ldots P_{\pi(n)} \subset P_1 \cap \ldots \cap P_n$. Therefore we have $\sum_\pi P_{\pi(1)} \ldots P_{\pi(n)} \subset P_1 \cap \ldots \cap P_n$.

We assert that the reverse inclusion also holds. To show this we shall apply the mathematical induction on n.

For $n = 1$ the assertion is certainly true. Assume that the assertion is true in case of $n = k - 1$, and we will show that the assertion is also true in case of $n = k$.

From the assumption it follows that $P_1 \cap \ldots \cap P_{k-1} = \sum_\tau P_{\tau(1)} \ldots P_{\tau(k-1)}$, where τ ranges over all the permutations of the set $\{1, \ldots, k - 1\}$. Since two ideals $P_1 \cap \ldots \ldots \cap P_{k-1}$ and P_k are relatively prime, we have

$$\begin{aligned} &P_1 \cap \ldots \cap P_{k-1} \cap P_k = \\ &= (P_1 \cap \ldots \cap P_{k-1})\, P_k + P_k(P_1 \cap \ldots \cap P_{k-1}) \,. \end{aligned}$$

This implies that

$$\begin{aligned} &P_1 \cap \ldots \cap P_{k-1} \cap P_k = \\ &= \Big(\sum_\tau P_{\tau(1)} \ldots P_{\tau(k-1)}\Big) P_k + P_k\Big(\sum_\tau P_{\tau(1)} \ldots P_{\tau(k-1)}\Big) \subset \sum_\pi P_{\pi(1)} \ldots P_{\pi(k)} \,. \end{aligned}$$

Thus the proof of the lemma is complete.

Lemma 9. *Let $P_1, \ldots, P_n$ be mutually distinct maximal ideals of R and let $M = P_1 \cap \ldots \cap P_n$. If $P_1, \ldots, P_n$ are finitely generated as left R-modules, then M^r is an open ideal of R for any positive integer r.*

Proof. Let π be any permutation of the set $\{1, \ldots, n\}$. In view of Lemma 7, the ideal $P_{\pi(1)} \ldots P_{\pi(n)}$ is finitely generated as a left R-module. Therefore, by Lemma 8, the ideal M is finitely generated as a left R-module. Using Lemma 7 again, we can conclude that M^r is finitely generated as a left R-module. Consequently M^r/M^{r+1} is a finitely generated left R/M-module.

In view of Proposition 1 and Lemma 6, the maximal ideals P_i are open, so that M is open. Hence the ring R/M is finite, because it is compact and discrete. Therefore M^r/M^{r+1} must be finite as a finitely generated left module over the finite ring R/M. This means that M^{r+1} is open in M^r. But M is open in R, so that M^2 is open in R and so on.

(It should be noticed that M^{r+1} is compact, so it is closed in M^r.)

We are now ready to prove the following theorems.

Theorem 10. *Let R be a compact ring with identity. If every maximal ideal of R is finitely generated as a left R-module, then the powers N^r of the radical N form a base for the neighborhood filter at zero.*

Proof. By Lemma 6 every maximal ideal of R is closed, therefore, in view of Theorem 5, there exist only a finite number of maximal ideals in R.

Let $P_1, \ldots, P_n$ be the maximal ideals of R; then $N = P_1 \cap \ldots \cap P_n$ (cf. (IV)). Of course, it is assumed that $P_i \neq P_j$ for $i \neq j$. From Lemma 9, we see that N^r is open for every positive integer r. Using this fact and (III), we can conclude that the powers N^r form a base for the neighborhood filter at zero.

Theorem 11. *Let R be a compact ring with identity and N its radical. Then all of the following conditions are equivalent:*

(1) *N^2 is open.*

(2) *N is open and is finitely generated as a left R-module.*

(3) *Every maximal ideal of R is finitely generated as a left R-module.*

Proof. (1) $\Rightarrow$ (2): It is clear that N is open, since N^2 is open.

We shall show that N is finitely generated as a left R-module. The ring N/N^2 is finite, because it is compact and discrete. Let $a_1, \ldots, a_s$ be any representatives in N of these cosets, and let $A = Ra_1 + \ldots + Ra_s$. Then it is clear that $A \subset N$. We assert that $N = A$. From $N = A + N^2$ it follows that for any positive integer i

$$N^i = N^{i-1}A + N^{i+1} .$$

Therefore, we have

$$N = A + NA + \ldots + N^{i-1}A + N^{i+1} = A + N^{i+1} .$$

Since N is topologically nilpotent, it follows that $N = A$.

(2) ⇒ (3): Let P be an arbitrary maximal ideal of R; then it follows from Lemma 4 that $P \supset N$. As N is open, P is also open, so P is closed. Because the ring P/N is finite, there exist a finite number of elements, say $b_1, \ldots, b_t$, in P such that

$$P = Rb_1 + \ldots + Rb_t + N .$$

Since N is finitely generated as a left R-module, P is also finitely generated as a left R-module.

(3) ⇒ (1): The proof of this implication is part of the proof of Theorem 10.

A *semilocal ring* is a commutative Noetherian ring (with identity) having only a finite number of maximal ideals. If A is a semilocal ring with radical N, one can introduce a topology on A by taking the family $\{N^r: r = 1, 2, \ldots\}$ to be a base of neighborhoods of zero. The topology just introduced on A is a T_2-topology, and moreover it is compatible with the ring structure of A. Hence A becomes a topological ring under this topology, which we call the *natural topology* of A.

Now we can state the following theorem.

Theorem 12. *Let R be a compact commutative ring with identity. If R satisfies one (and hence all) of the conditions in Theorem* 11, *then R is a semilocal ring and its given topology is the natural topology of R.*

Proof. Let N be the radical of R. By Theorem 10, the powers N^r form a base for the neighborhood filter at 0, therefore the original topology of R coincides with the natural topology of R. Moreover, every finitely generated ideal of R is closed by Lemma 6 and maximal ideals of R have finite bases. Hence, in view of Theorem 31.8 in [3; p. 110], we can conclude that R is a semilocal ring.

Corollary 13. *Let R be a compact commutative ring with identity satisfying one of the conditions in Theorem* 11. *Then every ideal of R is closed.*

Proof. By the above theorem R is a Noetherian ring; therefore every ideal of R is finitely generated. Hence, the conclusion of the corollary results from Lemma 6.

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