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TRANSLATIONAL HULLS OF POLYNOMIALLY RELATED SEMIGROUPS

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Hewitt and Zuckerman [1] defined an equivalence relation, which we shall denote $\mathscr{P}$, among semigroups defined on the same set: semigroups $(S, \cdot)$ and $(S, \circ)$ are $\mathscr{P}$-related if they have the same ternary multiplication polynomial, i.e., $xyz = x \circ y \circ z$ for all $x, y, z \in S$. Under the assumption that one of a pair of $\mathscr{P}$-related semigroups is weakly reductive and globally idempotent, Hewitt and Zuckerman proved that the semigroups are isomorphic and asked whether either hypothesis could be omitted.

We shall answer the Hewitt-Zuckerman question in the negative (Examples 8, 9, 10). Moreover, for a given weakly reductive, globally idempotent semigroup we shall determine, by means of a subgroup of the translational hull, all semigroups that are $\mathscr{P}$-related to the given one (Theorem 6). We shall also show that under either hypothesis the translational hulls of a pair of $\mathscr{P}$-related semigroups coincide (Theorems 2 and 5). In the globally idempotent case, $\mathscr{P}$-related semigroups even have the same left, right, and inner translations and therefore the same left, right, and two-sided ideals and congruences (Theorem 5). Thus our negative solutions to the Hewitt-Zuckerman question provide examples of non-isomorphic semigroups having a number of basic structures in common.

We begin by introducing the concept of strong interassociativity (so called because it is more restrictive than Zupnik's [4] interassociativity): semigroups $(S, \cdot)$ and $(S, \circ)$ are *strongly interassociative* if $x(y \circ z) = x \circ (yz) = (xy) \circ z = (x \circ y) z$ for all $x, y, z \in S$. This definition has an immediate and useful consequence in the case of $\mathscr{P}$-related semigroups:

Lemma 1. *If $(S, \cdot)$ and $(S, \circ)$ are strongly interassociative and $\mathscr{P}$-related, then $(x \circ y)(u \circ v) = xyuv$ and $(xy) \circ (uv) = x \circ y \circ u \circ v$ for all $x, y, u, v \in S$.*

We deal first with weakly reductive semigroups.

Theorem 2. *Let $(S, \cdot)$ and $(S, \circ)$ be $\mathscr{P}$-related semigroups with $(S, \cdot)$ weakly reductive. Then $(S, \circ)$ is also weakly reductive, and the two semigroups are strongly interassociative and have the same translational hull.*

Proof. If $a, b \in S$ induce the same inner bitranslation of $(S, \circ)$, then $a \circ x \circ y = b \circ x \circ y$ and $y \circ a \circ x = y \circ b \circ x$ for all $x, y \in S$. The $\mathscr{P}$-relation yields the same equation in $(S, \cdot)$, whence weak reductivity gives $ax = bx$, and a dual argument gives $xa = xb$. By weak reductivity $a = b$, and therefore $(S, \circ)$ is weakly reductive.

To prove strong interassociativity, let $a, b, c, x \in S$ and note that $[a(b \circ c)]\, x = a \circ b \circ c \circ x = [(a \circ b)\, c]\, x$ and likewise $x[a(b \circ c)] = x[(a \circ b)\, c]$, whence weak reductivity yields $a(b \circ c) = (a \circ b)\, c$. Interchanging the roles of the two operations and using the weak reductivity of $(S, \circ)$, we obtain $a \circ (bc) = (ab) \circ c$. Now, using the two identities just established, we have $[(a \circ b)\, c] \circ x = (a \circ b) \circ (cx) = abcx = [a \circ (bc)] \circ x$ and $x \circ [(a \circ b)\, c] = x \circ [a(b \circ c)] = (xa) \circ (b \circ c) = xabc = x \circ [a \circ (bc)]$. Weak reductivity of $(S, \circ)$ now implies $(a \circ b)\, c = a \circ (bc)$, as desired.

Now let $(\lambda, \varrho) \in \Omega(S, \cdot)$. For all $a, b, x \in S$ strong interassociativity gives $[\lambda(a \circ b)\, x = \lambda[(a \circ b)\, x] = \lambda[a(b \circ x)] = (\lambda a)\, (b \circ x) = [(\lambda a) \circ b]\, x$, and dually $x[(a \circ b)\, \varrho] = x[a \circ (b\varrho)]$. Taking advantage of linkage, we further have $x[\lambda(a \circ b)] = (x\varrho)\, (a \circ b) = [(x\varrho)\, a] \circ [x(\lambda a)] \circ b = x[(\lambda a) \circ b]$, and dually $[(a \circ b)\, \varrho]\, x = [a \circ (b\varrho)]\, x$. Thus, by weak reductivity λ is a left translation, and ϱ a right translation, of $(S, \circ)$.

Finally, $[a \circ (\lambda b)]\, x = [a(\lambda b)] \circ x = [(a\varrho)\, b] \circ x = [(a\varrho) \circ b]\, x$, and likewise $x[a \circ (\lambda b)] = x[(a\varrho) \circ b]$, whence weak reductivity implies that λ and ϱ are linked as translations of $(S, \circ)$. Thus $\Omega(S, \cdot) \subset \Omega(S, \circ)$. Because $(S, \circ)$ is weakly reductive, the roles of the two semigroups may be interchanged, whereupon $\Omega(S, \cdot) = \Omega(S, \circ)$.

As a quick corollary we obtain a theorem of Hewitt and Zuckerman which implies that the $\mathscr{P}$-relation is equality for such diverse semigroups as bands, groups of odd order, and abelian groups lacking elements of order two. (The generalization to all groups lacking central involutions is a consequence of Corollary 7 below.)

Corollary 3. (Hewitt and Zuckerman [1], Theorem 23). *Let $(S, \cdot)$ be a semigroup in which the function $x \to x^2$ is one-to-one. If $(S, \circ)$ is $\mathscr{P}$-related to $(S, \cdot)$, then $(S, \circ) = (S, \cdot)$.*

Proof. Because $(S, \cdot)$ is weakly reductive, we can apply Lemma 1 to obtain $(a \circ b)\, (a \circ b) = abab$ for all $a, b \in S$, whence the hypothesis implies $a \circ b = ab$.

Corollary 4. *Let $(S, \cdot)$ be a subsemigroup of the multiplicative semigroup of an integral domain D, and let $(S, \circ)$ be a semigroup $\mathscr{P}$-related to $(S, \cdot)$. Then either $(S, \circ) = (S, \cdot)$ or $x \circ y = -xy$ for all $x, y \in S$.*

Proof. As $(S, \cdot)$ is clearly reductive, Lemma 1 gives $(a \circ b)(a \circ b) = abab$ for all $a, b \in S$. It follows that $a \circ b = \pm ab$ because the implication $x^2 = y^2 \Rightarrow x = \pm y$ holds in every integral domain. Thus it suffices to show that there cannot exist elements a, b, c, d of S for which $a \circ b = ab \neq -ab$ and $c \circ d = -cd \neq cd$. If such elements did exist they would satisfy $0 \neq -abcd = (a \circ b)\, (c \circ d)$, which is equal to $abcd$ by Lemma 1. If the characteristic of D is not two, this would imply that $abcd = 0$, a contradiction, while if the characteristic of D is two the assumption $ab \neq -ab$ is untenable to begin with.

We now turn to the globally idempotent case.

Theorem 5. *Let $(S, \cdot)$ and $(S, \circ)$ be $\mathscr{P}$-related semigroups with $(S, \cdot)$ globally idempotent. Then $(S, \circ)$ is also globally idempotent, and the two semigroups are strongly interassociative and have the same translational hull. Moreover, they have the same left, right, and inner translations and hence the same left, right, and two-sided ideals and congruences.*

Proof. That $(S, \circ)$ is globally idempotent is readily observable, and is Theorem 20 of Hewitt and Zuckerman [1].

By the symmetry of the $\mathscr{P}$-relation and the global idempotence of $(S, \circ)$ it will suffice to show that every (left, right, inner, bi-) translation of $(S, \cdot)$ is a translation (of the same kind) of $(S, \circ)$. Strong interassociativity will largely be a consequence of the preservation of left and right translations. Nothing further need be said about the ideals or congruences, as they are determined by the inner translations.

For the remainder of the proof, we fix elements $a, a_1, a_2, b, b_1, b_2, c$ of S such that $a = a_1 a_2$ and $b = b_1 \circ b_2$.

If λ is a left translation of $(S, \cdot)$ then $\lambda(a \circ b) = \lambda(a \circ b_1 \circ b_2) = \lambda(ab_1b_2) = (\lambda a)\, b_1 b_2 = (\lambda a) \circ b_1 \circ b_2 = (\lambda a) \circ b$, and so λ is a left translation of $(S, \circ)$. We omit the analogous argument for right translations.

To show that linkage is preserved, let $(\lambda, \varrho) \in \Omega(S, \cdot)$. Then $a \circ (\lambda b) = a \circ \lambda(b_1 \circ b_2) = a \circ (\lambda b_1) \circ b_2 = a(\lambda b_1)\, b_2 = (a\varrho)\, b_1 b_2 = (a\varrho) \circ b_1 \circ b_2 = (a\varrho) \circ b$.

To verify strong interassociativity, note that $a(b \circ c) = (ab) \circ c$ and $(a \circ b)\, c = a \circ (bc)$ because the left (respectively right) inner translation induced by a (respectively c) in $(S, \cdot)$ is a left (respectively right) translation of $(S, \circ)$. Finally, $a(b \circ c) = a(b_1 \circ b_2 \circ c) = ab_1b_2c = (a \circ b_1 \circ b_2)\, c = (a \circ b)\, c$, as desired.

As for inner translations, we have $ac = a_1 a_2 c = (a_1 \circ a_2) \circ c$ and likewise $ca = c \circ (a_1 \circ a_2)$. Thus, the inner (left, right, bi-) translation of $(S, \cdot)$ induced by a is an inner (left, right, bi-) translation of $(S, \circ)$.

We note in passing that the result of Hewitt and Zuckerman quoted in the first paragraph follows from Theorems 2 and 5: if $(S, \cdot)$ is weakly reductive and globally idempotent and $\mathscr{P}$-related to $(S, \circ)$, then the semigroups are isomorphic because each is isomorphic to the inner part of its translational hull and these inner parts coincide.

Our next goal is to determine all semigroups that are $\mathscr{P}$-related to a given weakly reductive, globally idempotent semigroup. First we need some terminology.

By a *central involutorial bitranslation* (briefly, c.i.b.) of a semigroup S we mean a bitranslation (λ, ϱ) in the center of $\Omega(S)$ satisfying $\lambda^2 = \varrho^2 = \mathrm{id}_S$ (the identity function on S). Petrich [2, Proposition V.1.5] shows that for a weakly reductive or globally idempotent semigroup S, a bitranslation (λ, ϱ) lies in the center of $\Omega(S)$ if and only if $\lambda x = x\varrho$ for all $x \in S$. We note, however, that the assumption of linkage is superfluous: if a left translation λ and a right translation ϱ satisfy $\lambda x = x\varrho$ for all x, then $(\lambda, \varrho) \in \Omega(S)$ because $x(\lambda y) = x(y\varrho) = (xy)\,\varrho = \lambda(xy) = (\lambda x)\,y = (x\varrho)\,y$ for all $x, y \in S$.

Given a weakly reductive, globally idempotent semigroup S, we now exhibit a one-to-one correspondence between the group of all c.i.b.'s of S and the set of all semigroups that are $\mathscr{P}$-related to S.

Theorem 6. *Let (λ, ϱ) be a c.i.b. of a weakly reductive or globally idempotent semigroup $(S, \cdot)$, and define an operation $\circ$ on S by setting $x \circ y = \lambda(xy)$ for all $x, y \in S$. Then $(S, \circ)$ is a semigroup $\mathscr{P}$-related to $(S, \cdot)$ and:*

(1) *λ is an isomorphism of each semigroup onto the other;*

(2) *distinct c.i.b.'s give rise to distinct semigroups;*

(3) *if $(S, \cdot)$ is both weakly reductive and globally idempotent, then every semigroup that is $\mathscr{P}$-related to $(S, \cdot)$ can be obtained from a c.i.b. in the prescribed manner.*

Proof. For $a, b, c \in S$ we have $(a \circ b) \circ c = \lambda[\lambda(ab)\,c] = [\lambda^2(ab)]\,c = abc$. Because $\lambda x = x\varrho$ for all x, the dual argument gives $a \circ (b \circ c) = abc$, whereupon $(S, \circ)$ is a semigroup $\mathscr{P}$-related to $(S, \cdot)$.

Because $\lambda^2 = \mathrm{id}_S$, to prove (1) it suffices to show that λ is a homomorphism of $(S, \cdot)$ into $(S, \circ)$. In the globally idempotent case we need only note that for all $a, b, c, d \in S$, $\lambda(abcd) = (ab) \circ (cd) = a \circ b \circ c \circ d = \lambda(ab) \circ \lambda(cd)$, where Theorem 5 has allowed the application of Lemma 1. In the weakly reductive case it is enough to verify that $[\lambda(ab)]\,x = [(\lambda a) \circ (\lambda b)]\,x$ and $x[(ab)\,\varrho] = x[(a\varrho) \circ (b\varrho)]$ for all $a, b, x \in S$. As a dual argument will serve to prove the latter, we prove only the former: $[(\lambda a) \circ (\lambda b)]\,x = \{\lambda[(\lambda a)\,(\lambda b)]\}\,x = (\lambda^2 a)\,(\lambda b)\,x = a(\lambda b)\,x$; since λ commutes with every inner left translation we further have $a(\lambda b)\,x = [\lambda(ab)]\,x$ as desired.

To prove (2), let (λ', ϱ') be a c.i.b. distinct from (λ, ϱ). As λ and ϱ are in fact the same function, we have $\lambda \neq \lambda'$. In the globally idempotent case there exist $a, b \in S$ with $\lambda(ab) \neq \lambda'(ab)$, whence the induced semigroups are distinct. In the weakly reductive case, choose $a \in S$ such that $\lambda a \neq \lambda' a$. Then there is some $b \in S$ such that either $(\lambda a)\,b \neq (\lambda' a)\,b$ or $b(\lambda a) \neq b(\lambda' a)$. In the former case $\lambda(ab) \neq \lambda'(ab)$ as desired, while centrality (as in the above paragraph) reduces the latter case to $\lambda(ba) \neq \lambda'(ba)$.

To prove (3), given a semigroup $(S, \circ)$ that is $\mathscr{P}$-related to a weakly reductive, globally idempotent semigroup $(S, \cdot)$, define (λ, ϱ) by setting $\lambda(xy) = (xy)\,\varrho = x \circ y$ for all $x, y \in S$. By global idempotence λ and ϱ are defined on all of S. To see that they are well defined, note that $ab = cd$ implies $(ab) \circ x = (cd) \circ x$ for all $x \in S$, whence strong interassociativity gives $(a \circ b)\, x = (c \circ d)\, x$, and likewise $x(a \circ b) = x(c \circ d)$. By weak reductivity $a \circ b = c \circ d$ as desired.

Now, for all $a, b, c \in S$ we have, by strong interassociativity, $\lambda(abc) = a \circ (bc) = (a \circ b)\, c = [\lambda(ab)]\, c$ and $(abc)\, \varrho = (ab) \circ c = a(b \circ c) = a[(bc)\, \varrho]$. Hence global idempotence implies that λ and ϱ are respectively a left and right translation of $(S, \cdot)$. Finally $\lambda^2(abc) = \lambda[a \circ (bc)] = \lambda[(a \circ b)\, c] = a \circ b \circ c = abc$, and so $\lambda^2 = \varrho^2 = \mathrm{id}_S$ by global idempotence. In view of the remarks preceding the theorem, we have now shown that (λ, ϱ) is a c.i.b.

Corollary 7. *Let $(S, \cdot)$ be a semigroup with identity, 1. For each t in the center of $(S, \cdot)$ satisfying $t^2 = 1$, the operation $x \circ y = xty$ defines a semigroup $\mathscr{P}$-related to $(S, \cdot)$, and the map $x \to tx$ is an isomorphism of each semigroup onto the other. Conversely, every semigroup $\mathscr{P}$-related to $(S, \cdot)$ is obtained in this way, and distinct choices of t yield distinct semigroups.*

We now present examples answering the Hewitt-Zuckerman question in the negative; by Theorem 6 these also serve as examples of semigroups satisfying one of our basic hypotheses but not the other, and having $\mathscr{P}$-relatives that are not induced by c.i.b.'s.

The semigroups of the first two examples are commutative and reductive.

Example 8. The set $S = \{0, 2, 4, 6, 8, 9\}$ is, under multiplication modulo 12, a reductive subsemigroup of Z_{12}. With $\circ$ defined by $x \circ y = 5xy \pmod{12}$, $(S, \circ)$ is a semigroup and is $\mathscr{P}$-related to $(S, \cdot)$ because $5^2 = 1 \pmod{12}$.

To see that the two semigroups are not isomorphic, first note that $S \cdot S = \{0, 4, 6, 8, 9\}$, whence $S \circ S = \{5x \mid x \in S \cdot S\} = \{0, 8, 6, 4, 9\} = S \cdot S = S \setminus \{2\}$. It follows that any isomorphism f of $(S, \cdot)$ onto $(S, \circ)$ must fix 2. But then $f(8) = f(2) \circ f(2) \circ f(2) = 2 \circ 2 \circ 2 = 2 \cdot 2 \cdot 2 = 8 = 2 \circ 2 = f(2) \circ f(2) = f(4)$, which is impossible because f is one-to-one.

Example 9. The set $I = (-\infty, -2) \cup [2, \infty)$, as a subsemigroup of the real numbers under multiplication, is reductive. As mandated by Corollary 4, we define $x \circ y = -xy$ for all $x, y \in I$ to obtain a semigroup $\mathscr{P}$-related to $(I, \cdot)$.

Suppose $f\colon (I, \cdot) \to (I, \circ)$ is an isomorphism. We first show that $f(2) = 2$. Set $u = f^{-1}(2)$. If $u \neq 2$, then $-u \in I$ and $[f(-u)]^2 = -[f(-u) \circ f(-u)] = -f[(-u)^2] = -f(u^2) = -[f(u) \circ f(u)] = [f(u)]^2$, and so $f(-u) = \pm f(u) = \pm 2$. As $-2 \notin I$, we have $f(-u) = 2 = f(u)$. But then $-u = u$, whence $u = 0$, which is impossible because $0 \notin I$. Thus $u = 2$, and so $f(2) = 2$ as desired.

It now follows, as in the above example, that $f(8) = 8$. However, $f(8) = f(\sqrt{8} \cdot \sqrt{8}) = f(\sqrt{8}) \circ f(\sqrt{8}) = -[f(\sqrt{8})]^2 < 0$, a clear contradiction.

In our final example we exhibit a pair of commutative, globally idempotent, $\mathscr{P}$-related semigroups that are not isomorphic.

Example 10. The set $T = (-\infty, -1) \cup (1, \infty)$ is, under multiplication, a globally idempotent subsemigroup of the reals, and contains as an ideal the set I of the above example. As I is also an ideal of the $\mathscr{P}$-related semigroup $(T, \circ)$, defined by $x \circ y = -xy$, we have two Rees quotient semigroups defined on the set $S = T/I$. To avoid confusion we shall use the symbol $*$ to denote the operation in S arising from $(T, \cdot)$, but we shall retain the symbol $\circ$ for the operation in S derived from $(T, \circ)$. Because the universal algebra $(S, *, \circ)$ is a homomorphic image of $(T, \cdot, \circ)$, equations valid in the latter hold in the former, so in particular $(S, *)$ is $\mathscr{P}$-related to $(S, \circ)$. The global idempotence of these semigroups is of course also a consequence of the homomorphism.

We claim that whenever a member x of S satisfies $x \circ x = I \neq x$, then $|x| > \sqrt{2}$. Indeed, for such an x we have $-x^2 \in I$. Since $-x^2$ cannot be 2, it follows that $|-x^2| > 2$, whence $|x| > \sqrt{2}$.

Now let f be an isomorphism of $(S, *)$ onto $(S, \circ)$. As the zero of both semigroups, I is fixed by f. Because $\sqrt{2} * \sqrt{2} = (-\sqrt{2}) * (-\sqrt{2}) = I$, it follows that $f(\sqrt{2})$ and $f(-\sqrt{2})$ both satisfy the hypotheses of the above claim, whereupon $|f(-\sqrt{2}) \cdot f(\sqrt{2})| > 2$. Hence $f(-\sqrt{2}) \circ f(\sqrt{2}) = I$. Because -2, as the unique non-zero annihilator in both $(S, *)$ and $(S, \circ)$, must be fixed by f, we arrive at a contradiction: $I = f(-\sqrt{2}) \circ f(\sqrt{2}) = f(-\sqrt{2} * \sqrt{2}) = f(-2) = -2$.

As the semigroups in the above examples are all commutative, two remarks are in order: (1) It is readily verified that if a weakly reductive or globally idempotent semigroup is commutative, then any semigroup $\mathscr{P}$-related to it is also commutative. (2) By a result of B. M. Schein [3], if a finite, globally idempotent semigroup is commutative, then it is reductive.

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