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SCHREIER-ZASSENHAUS THEOREM FOR ALGEBRAS II

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The generalization of Schreier-Zassenhaus theorem for algebras consists in the following. There are given two congruence series in an algebra; isomorphic refinements of these series are looked for, i.e. refinements and a bijection of these refinements such that the corresponding congruences are isomorphic as factor algebras (and, of course, existence conditions are examined, too). A number of attempts at such a generalization is known. In the present paper we call attention to two such attempts. O. Borůvka in [2] (see also [1] 17.6) attained one such result and A. Châtelet in [3] (see also [7] Theorem 88) another one. We shall not mention the other results. Both the theorems are algebra generalizations of the Schreier-Zassenhaus theorem for invariant series of subgroups.

A particular attention should be paid to Borůvka's attempt [2] (see also [1] 10.1) at a formulation of an analogous theorem for partition series on a set without operations. The question arises whether or not such a theorem may be applied to algebras. Namely, if we omit algebra operations the isomorphism of the corresponding congruences (partitions) is reduced to a set theoretical equivalence. If the construction of this isomorphism (equivalence) is not known then the theorem cannot be applied to algebras and thus does not represent a generalization of Schreier-Zassenhaus group theorem. O. Borůvka [2], [1] discovered the set theoretical character of the Zassenhaus' construction of the isomorphism of refinements and so ensured that his set theoretical theorem [1] 10.1 is applicable to algebras [1] 17.6. The reader can find more details in Part I of the present paper ([9]). In both the Parts I and II we use Borůvka's idea – the notion of coupled partitions. The purpose of Part I is to find a theorem which is an algebra generalization of Schreier-Zassenhaus group theorem and to prove it under the most general conditions (in some sense necessary and sufficient). The aim of Part II is to find a common generalization of the theorems of Borůvka [2] ([1] 10.1 and 17.6), Châtelet [3] ([7] Theorem 88) and, of course, the Schreier-Zassenhaus group theorem, namely under such conditions which have a formulation as simple as possible. Purposefully, we drop the intention of achieving the greatest generality.

For terminology and denotation, cf. [1], [2], [5] and [9]. Some fundamental notions will be listed in the following.

A partition in a set $\mathfrak{G}$ is a system of nonempty pairwise disjoint subsets of the set $\mathfrak{G}$. The system of all partitions in $\mathfrak{G}$ is clearly in one-to-one correspondence with the system of all symmetric and transitive binary relations in $\mathfrak{G}$. For this reason we shall not distinguish between both notions. If A is any binary relation in $\mathfrak{G}$, $x \in \mathfrak{G}$ and $\emptyset \neq \mathfrak{B} \subseteq \mathfrak{G}$, we define $A(x) = \{y \in \mathfrak{G} : yAx\}$, $A(\mathfrak{B}) = \bigcup\{A(x) : x \in \mathfrak{B}\}$ and $\bigcup A = \bigcup\{A(x) : x \in \mathfrak{G}\} = A(\mathfrak{G})$. If A is a partition and $A(x) \neq \emptyset$, we call the set $A(x)$ a *block of the partition* A and $\bigcup A$ the *domain of the partition* A [5]; if $\bigcup A = \mathfrak{G}$ we speak about the *partition on* $\mathfrak{G}$ or about the *partition of the set* $\mathfrak{G}$. If $\emptyset \neq \mathfrak{B} \subseteq \mathfrak{G}$, $\{\mathfrak{B}\}$ means the partition in $\mathfrak{G}$ with a unique block $\mathfrak{B}$. If A is a partition in $\mathfrak{G}$, $\emptyset \neq \mathfrak{B} \subseteq \mathfrak{G}$, we define $\mathfrak{B} \sqsubset A = \{A^1 \cap \mathfrak{B} : A^1 \in A,\ A^1 \cap \mathfrak{B} \neq \emptyset\}$. This partition is called a *closure of the set* $\mathfrak{B}$ *in the partition* A. If A is a binary relation in $\mathfrak{G}$ and $\emptyset \neq \mathfrak{B} \subseteq \mathfrak{G}$, then the relation $\mathfrak{B} \sqcap A$ is defined as $(\mathfrak{B} \times \mathfrak{B}) \cap A$ and called the *intersection of* A *with* $\mathfrak{B}$. In particular, if A is a partition then $\mathfrak{B} \sqcap A = \{\mathfrak{B}\} \wedge A = \{A^1 \cap \mathfrak{B} : A^1 \in A,\ A^1 \cap \mathfrak{B} \neq \emptyset\}$ [1] 2.3. Two partitions in $\mathfrak{G}$ are called *coupled* if each block of one partition meets exactly one block of the second partition [1] 4.1. The set of all binary relations in $\mathfrak{G}$, $\mathscr{R}(\mathfrak{G})$, is a complete lattice with regard to the set inclusion. The set of all partitions in $\mathfrak{G}$, $\mathscr{P}(\mathfrak{G})$, is a complete lattice with regard to the set inclusion as well, infima in $\mathscr{R}(\mathfrak{G})$ and $\mathscr{P}(\mathfrak{G})$ are meets. Operations in $\mathscr{R}(\mathfrak{G})$ are denoted by $\cap, \cup, \bigcap, \bigcup$, in $\mathscr{P}(\mathfrak{G})$ by $\wedge, \vee, \bigwedge, \bigvee$ ($\vee_{\mathscr{P}}, \bigvee_{\mathscr{P}}$ if necessary). The symbol $\geqq$ denotes the partial order in $\mathscr{R}(\mathfrak{G})$, while $A \sqsupseteq B$, where A and B are partitions in $\mathfrak{G}$, means that each block of B is a block of A. Under the product of two binary relations A and B in $\mathfrak{G}$ we understand the relation $AB = \{(a, b) \in \mathfrak{G} \times \mathfrak{G}$: there exists $c \in \mathfrak{G}$ with $aAcBb\}$. The relations A and B in $\mathfrak{G}$ *commute* if $AB = BA$. Let $(\mathfrak{G}, \Omega)$ be an algebra. Partitions in $\mathfrak{G}$ which are stable binary relations in $(\mathfrak{G}, \Omega)$, are called *congruences* in $(\mathfrak{G}, \Omega)$. The set of all congruences in $(\mathfrak{G}, \Omega)$, $\mathscr{K}(\mathfrak{G}, \Omega)$, is a complete lattice with regard to the set inclusion, its operations are denoted by $\wedge_{\mathscr{K}}, \vee_{\mathscr{K}}, \bigwedge_{\mathscr{K}}, \bigvee_{\mathscr{K}}$. We have $\bigwedge_{\mathscr{K}} = \bigwedge_{\mathscr{P}} = \bigcap$ [5] 1.1. The domain of the congruence in $(\mathfrak{G}, \Omega)$ is a subalgebra of (G, Ω); if $(\mathfrak{G}, \Omega)$ is an Ω-group and $\emptyset \neq A \in \mathscr{K}(\mathfrak{G}, \Omega)$ then $A(0)$ is an ideal of the Ω-subgroup $\bigcup A$ and $A = \bigcup A / A(0)$ [5] I 1.4.

1.

Definition 1.0. ([5] IV 4.8) We say that a set $\emptyset \neq \mathfrak{B} \subseteq \mathfrak{G}$ *respects a partition* A in $\mathfrak{G}$ if $A^1 \in A$, $A^1 \cap \mathfrak{B} \neq \emptyset$ implies $A^1 \subseteq \mathfrak{B}$.

Lemma 1.1. (See Lemma 1.1 [9]) *Let A and B be partitions in a set $\mathfrak{G}$ and $\emptyset \neq \mathfrak{B} \subseteq \mathfrak{G}$. Then $\mathfrak{B} \sqcap (A \vee B) \geqq (\mathfrak{B} \sqcap A) \vee (\mathfrak{B} \sqcap B)$. The equality follows if $\mathfrak{B}$ respects the partitions A and B or if $\mathfrak{B} \sqsupseteq \bigcup A \cap \bigcup B$. An analogous assertion holds for the product. (The symbol $\geqq$ means the order in the lattice of all binary relations in $\mathfrak{G}$.)*

Definition 1.2. A partition A' on a set $\mathfrak{G}$ is said to be an *extension on* $\mathfrak{G}$ of a partition A in $\mathfrak{G}$ if $A = \bigcup A \sqsubset A'$.

Notation 1.3. Let B and C be partitions in a set $\mathfrak{G}$, $e \in \bigcup B \cap \bigcap C$, B_0 a partition on $B(e)$ and C_0 a partition on $C(e)$. As in [9] we define $B_{11} = B_0 \vee (B \wedge C)$, $B_{10} = B_0 \vee (B \wedge C_0)$, $\bar{B}_{11} = B_0(B \wedge C)$, $\bar{B}_{10} = B_0(B \wedge C_0)$, $K = \bar{B}_{11}(e) \sqcap B_{10}$, $\hat{K} = \bar{B}_{11}(e) \sqcap \bar{B}_{10}$. Relations $C_{11}, C_{10}, \bar{C}_{11}, \bar{C}_{10}, L$ and $\hat{L}$ are defined symmetrically with regard to the symbols B and C. Further let us define $\mathfrak{A} = B(e) \cap C(e)$, $M = (\mathfrak{A} \sqcap B_0) \vee (\mathfrak{A} \sqcap C_0) = \mathfrak{A} \sqcap (B_0 \vee C_0)$, $\bar{M} = (\mathfrak{A} \sqcap B_0)(\mathfrak{A} \sqcap C_0) = \mathfrak{A} \sqcap (B_0 C_0)$ and N, $\bar{N}$ symmetrically.

Let $B_0' \leqq B'$, $C_0' \leqq C'$ be extensions on $\mathfrak{G}$ of the partitions B_0, B, C_0 and C, respectively. Analogously, we define $\bar{B}_{11}', \bar{B}_{10}', \ldots, L', \bar{M}', \bar{N}'$ and $\mathfrak{A}'$.

Lemma 1.4. *The relations defined in 1.3 possess the following properties denoted by (1)–(12). Similar properties (1′)–(12′) can be obtained by interchanging B and C.*

By the definition we have $\bar{B}_{11}(e) = \{x \in \mathfrak{G}$: there exists $a \in \mathfrak{G}$ with $xB_0a(B \wedge C)\, e\}$ and so

(1) $\bar{B}_{11}(e) = \bigcup\{B_0(a) : a \in \mathfrak{A}\}$, $B(e) \supseteq \bar{B}_{11}(e) \supseteq \mathfrak{A}$,

(2) $\bar{B}_{11}(e) \sqcap (B \wedge C_0) = \bar{B}_{11}(e) \sqcap C_0$.

We have namely $\bar{B}_{11}(e) \sqcap (B \wedge C_0) = [\bar{B}_{11}(e) \sqcap B] \wedge [\bar{B}_{11}(e) \sqcap C_0] = \{\bar{B}_{11}(e)\} \wedge [\bar{B}_{11}(e) \sqcap C_0] = \bar{B}_{11}(e) \sqcap C_0$ since by (1) $\bar{B}_{11}(e)$ is contained in the block $B(e)$ of the partition B.

(3) $\bar{B}_{11}(e) \cap \bar{C}_{11}(e) = \mathfrak{A}$.

The relation follows directly from (1) and (1′).

(4) $\bar{B}_{11}'(e) = \bar{B}_{11}(e)$.

This follows from the relation $B_0' \sqsupset A' = B_0 \sqsupset \mathfrak{A}$ and (1).

(5) For $y \in \mathfrak{G}$ we have $\hat{K}(y) = \bigcup\{B_0(a) : a \in \mathfrak{A}_y\}$ for some $\mathfrak{A}_y \subseteq \mathfrak{A}$, e.g. $\mathfrak{A}_y = \hat{K}(y) \cap \mathfrak{A}$. Further, $y \in \mathfrak{A}$ implies $y \in \hat{K}(y)$.

Indeed $\hat{K}(y) = \bar{B}_{11}(e) \cap \bigcup\{B_0(a) : a \in (B \wedge C_0)(y)\}$, then by (1), we have the expression for $\hat{K}(y)$. We can choose $\mathfrak{A}_y = \hat{K}(y) \cap \mathfrak{A}$. We have namely $\hat{K}(y) \cap A \supseteq \mathfrak{A}_y$ (for $a \in \mathfrak{A}_y$ implies $a \in B_0(a) \subseteq \hat{K}(y)$), thus $(\mathfrak{C} :=) \bigcup\{B_0(a) : a \in \hat{K}(y) \cap \mathfrak{A}\} \supseteq K(y)$. Conversely, for $a \in \hat{K}(y) \cap \mathfrak{A}$ we have on the one hand $B_0(a) \subseteq \bar{B}_{11}(e)$ and on the other hand $aB_0b(B \wedge C_0)\, y$ for some $b \in \mathfrak{G}$. Hence $B_0(a) = B_0(b) \subseteq \bigcup\{B_0(c) : c \in (B \wedge C_0)(y)\} \cap \bar{B}_{11}(e) = \hat{K}(y)$, then $\mathfrak{C} \subseteq \hat{K}(y)$. The last assertion: If $y \in \mathfrak{A}$ then by (1) $y \in \bar{B}_{11}(e)$ and $yB_0y(B \wedge C_0)\, y$. Hence $y \in \hat{K}(y)$.

(6) The system of sets $T = \{\hat{K}(y) : y \in \mathfrak{A}\}$ covers $\bar{B}_{11}(e)$. If $\hat{K}$ is a partition then $\hat{K} = T$.

The first assertion follows directly from (5) and (1). The second assertion: If $\hat{K}$ is a partition then T is a set of some blocks of the partition $\hat{K}$, i.e. $\hat{K} \supseteq T$. Hence T is a partition. Let $x\hat{K}y$. By (5), $a \in \mathfrak{A}$ exists such that $a \in \hat{K}(y)$. Hence $x \in \hat{K}(y) = \hat{K}(a)$ and thus xTy. Hence $\hat{K} \leqq T$. Finally $\hat{K} = T$.

(7) $\overline{M} = \mathfrak{A} \sqcap (B_0 C_0) = (\mathfrak{A} \sqcap B_0)(\mathfrak{A} \sqcap C_0)$; $\overline{M}' = \overline{M}$.

Then the relation $\overline{M}$ is a partition if and only if the partitions $\mathfrak{A} \sqcap B_0$ and $\mathfrak{A} \sqcap C_0$ commute or equivalently if $\overline{M} = \overline{N}$.

In this case $\overline{M}$ is a partition on $\mathfrak{A}$ and $\overline{M} = \overline{N} = (\mathfrak{A} \sqcap B_0) \vee \vee (\mathfrak{A} \sqcap C_0) = M$.

The first assertion follows from 1.1. The second assertion: If $x\mathfrak{A}' \sqcap B_0' C_0' y$ then $x, y \in \mathfrak{A}'$ and $xB_0' a C_0' y$ for some $a \in \mathfrak{G}$. Hence $eB' \times B'aC'yC'e$, thus $eB'aC'e$, i.e. $a \in \mathfrak{A}'$. Consequently $\overline{M}' \subseteq (\mathfrak{A}' \sqcap B_0')(\mathfrak{A}' \sqcap C_0')$ is proved. But $(\mathfrak{A} \sqcap B_0) . . (\mathfrak{A} \sqcap C_0) = \overline{M}$ is on the right side. Thus $\overline{M}' \subseteq \overline{M}$. The inclusion $\overline{M} \subseteq \overline{M}'$ is evident. The last two assertions follow from the properties of the commuting partitions on a set (on $\mathfrak{A}$) (e.g. [5] 3.1.1(5)).

(8) $\hat{K} = \bar{B}_{11}(e) \sqcap B_0(B \wedge C_0) = [\bar{B}_{11}(e) \sqcap B_0][\bar{B}_{11}(e) \sqcap C_0]$. Then the relation $\hat{K}$ is a partition if and only if the partitions $\bar{B}_{11}(e) \sqcap B_0$ and $\bar{B}_{11}(e) \sqcap C_0$ commute.

The first assertion follows from 1.1, since by (1) $\bar{B}_{11}(e) \supseteq \bigcup B_0 \cap \bigcup (B \wedge C_0) = \mathfrak{A}$. The relation $\hat{K}$ as a product of partitions is a partition if and only if the partitions commute [5] 3.1.

(9) $\overline{K} = \bar{B}_{11}(e) \sqcap B_{10} = [\bar{B}_{11}(e) \sqcap B_0] \vee [\bar{B}_{11}(e) \sqcap C_0]$, $\mathfrak{A} \subseteq \bigcup \overline{K} \subseteq \bar{B}_{11}(e)$ and every block of the partition $\overline{K}$ meets $\mathfrak{A}$.

The first assertion follows from (1), (2) and 1.1 and the last two assertions from (1).

(10) $M = \mathfrak{A} \sqcap (B_0 \vee C_0) = (\mathfrak{A} \sqcap B_0) \vee (\mathfrak{A} \sqcap C_0)$. The relation $\overline{M}$ is a partition if and only if $\overline{M} = M$ or equivalently $\overline{N} = N$ or equivalently $\overline{M} = \overline{N}$.

The representation of M follows from 1.1. As $\mathfrak{A} \sqcap B_0$ and $\mathfrak{A} \sqcap C_0$ are partitions on $\mathfrak{A}$ and $\overline{M}$ is their product (see (7)), the rest of the assertion follows from the properties of commuting partitions on a set (e.g. [5] 3.1.1(5)).

(11) $\overline{M} = \mathfrak{A} \sqcap \hat{K} = C(e) \sqcap \hat{K}$.

We have $C(e) \sqcap \hat{K} = \mathfrak{A} \sqcap \bar{B}_{10} = (\mathfrak{A} \sqcap \bar{B}_{10}) \cap C = \mathfrak{A} \sqcap (\bar{B}_{10} \cap C) = \mathfrak{A} \sqcap \sqcap (B_0 \wedge C)(B \wedge C_0) = [\mathfrak{A} \sqcap (B_0 \wedge C)][\mathfrak{A} \sqcap (B \wedge C_0)] = (\mathfrak{A} \sqcap B_0)(\mathfrak{A} \sqcap C_0) = = \overline{M}$. The second equality follows from the fact that $\mathfrak{A}$ is a subset of a block of the partition C, the fourth from 4.14 [5], the fifth from 1.1 and the last from (7).

(12) If $\overline{M}$ is a partition and $a \in \mathfrak{A}$ then $\hat{K}(a) = \overline{K}(a)$.

For $a \in \mathfrak{A}$ we have $x\overline{K}a \Rightarrow x[\bar{B}_{11}(e) \sqcap (B_0 \vee (B \wedge C_0))]\, a \Rightarrow x \in \bar{B}_{11}(e)$, xA_1x_1 $x_1A_2x_2 \ldots x_{n-1}A_na$, where $A_1, \ldots, A_n$ are by turns equal to B_0 or $B \wedge C_0$. Hence $x_1, \ldots, x_{n-1}, a \in \mathfrak{A}$ and thus

$$(*) \qquad x_1\mathfrak{A} \sqcap A_2x_2 \ldots x_{n-1}\mathfrak{A} \sqcap A_na\,.$$

If $A_1 = B \wedge C_0$, then $x \in \bar{B}_{11}(e) \cap \bigcup C_0 = \mathfrak{A}$ and thus the preceding sequence $(*)$ can be extended at the beginning by the relation $x\mathfrak{A} \sqcap (B \wedge C_0)\, x_1$. Hence $x(\mathfrak{A} \sqcap B_0) \vee (\mathfrak{A} \sqcap C_0)\, a$ (the partitions $\mathfrak{A} \sqcap (B \wedge C_0)$ and $\mathfrak{A} \sqcap C_0$ being evidently the same), thus by (7) $x(\mathfrak{A} \sqcap B_0)(\mathfrak{A} \sqcap C_0)\, a$ and by (1) and (8) $x\hat{K}a$.

Now let $A_1 = B_0$. From $(*)$ it follows that $x_1(\mathfrak{A} \sqcap B_0)(\mathfrak{A} \sqcap C_0)\, a$ and by (1) and (8) $x_1 \hat{K} a$. Then the following relations hold: $x(\bar{B}_{11}(e) \sqcap B_0)\, x_1 \hat{K} a$ and by (8) $x(\bar{B}_{11}(e) \sqcap B_0)\, x_1(\bar{B}_{11}(e) \sqcap B_0)\, y(\bar{B}_{11}(e) \sqcap C_0)\, a$ for some $y \in \mathfrak{G}$, thus $x(\bar{B}_{11}(e) \sqcap B_0)(\bar{B}_{11}(e) \sqcap C_0)\, a$, i.e. $x\hat{K}a$.

Definition 1.5. ([1] 4.1) Two partitions in a set $\mathfrak{G}$ are said to be *coupled* if each block of one partition meets exactly one block of the other partition.

Lemma 1.5a. (Borůvka [1] 4.1, see also 1.4 [9]) *Partitions A and D in a set $\mathfrak{G}$ are coupled if and only if*

(*a*) $\bigcup D \sqcap A = \bigcup A \sqcap D$,

(*b*) *Every block of the partition A meets $\bigcup D$ (or equivalently $\bigcup A \cap \bigcup D$) and symmetrically.*

Evidently, (*a*) is equivalent to

(*a'*) $(\bigcup A \cap \bigcup D) \sqcap A = (\bigcup A \cap \bigcup D) \sqcap D$.

The following Theorem 1.6 follows from [9] 1.6 and 1.8. The special conditions of 1.6 make it possible to give a short direct proof.

Theorem 1.6. *Let B and C be partitions in a set $\mathfrak{G}$, $e \in \bigcup B \cap \bigcup C$, B_0 a partition on $B(e)$ and C_0 a partition on $C(e)$. If the partitions $\mathfrak{A} \sqcap B_0$ $(= \bigcup C_0 \sqcap B_0)$ and $\mathfrak{A} \sqcap C_0$ $(= \bigcup B_0 \sqcap C_0)$ commute then $\bar{K}$, $\bar{L}$ and $\bar{M}$ are pairwise coupled partitions (on $\bar{B}_{11}(e)$, $\bar{C}_{11}(e)$ and $\mathfrak{A}$, respectively). Moreover,*

$$\mathfrak{A} \sqcap \bar{K} = \mathfrak{A} \sqcap \bar{L} = \bar{K} \sqcap \bar{L} = \bar{M} = \bar{N}\,.$$

Proof. The commutativity of the partitions $\mathfrak{A} \sqcap B_0$ and $\mathfrak{A} \sqcap C_0$ implies that the product $(\mathfrak{A} \sqcap B_0)(\mathfrak{A} \sqcap C_0)$ is equal to $\bar{M}$ and is a partition (1.4(7)), so the relation $\mathfrak{A} \sqcap \hat{K}$ $(= \bar{M}$ by 1.4(11)) is a partition, too. This partition is the system of sets $\{\mathfrak{A} \cap \hat{K}(a) : a \in \mathfrak{A}\}$, which is, by 1.4(12), equal to $\{\mathfrak{A} \cap \bar{K}(a) : a \in \mathfrak{A}\} = \mathfrak{A} \sqcap \bar{K}$. $\bar{N} = \mathfrak{A} \sqcap \bar{L}$ is proved analogously. As $\bar{M}$ is a partition, we have $\bar{M} = \bar{N}$ (1.4(10)). By 1.5 and 1.4(9), the partitions $\bar{K}$, $\bar{L}$ and $\bar{M}$ are pairwise coupled. Finally, $\bar{K} \wedge \bar{L} = \mathfrak{A} \sqcap (\bar{K} \wedge \bar{L}) = (\mathfrak{A} \sqcap \bar{K}) \wedge (\mathfrak{A} \sqcap \bar{L}) = \bar{M}$.

We shall introduce some conditions equivalent to the condition of Theorem 1.6.

Lemma 1.7. *Let B_0 be a partition on $B(e)$ and C_0 a partition on $C(e)$. Then the following conditions are equivalent.*

(i) *$\mathfrak{A} \sqcap B_0$ and $\mathfrak{A} \sqcap C_0$ commute,*

(ii) *$\bar{M}$ is a partition,*

(iii) *$T = \{\hat{K}(y) : y \in \mathfrak{A}\}$ is a partition,*

(iv) *$T = \bar{K}$.*

Proof. i $\Leftrightarrow$ ii by 1.4(7).

ii $\Rightarrow$ iii: $x \in \hat{K}(y_1) \cap \hat{K}(y_2) \Rightarrow$ (see 1.4(5)) $x \in B_0(a_1) \cap B_0(a_2)$ for some $a_1, a_2 \in \mathfrak{A} \Rightarrow a_1 \in B_0(a_1) = B_0(x) = B_0(a_2) \subseteq \hat{K}(y_1) \cap \hat{K}(y_2) \Rightarrow \emptyset \neq [\mathfrak{A} \cap \hat{K}(y_1)] \cap$

$\cap [\mathfrak{A} \cap \hat{K}(y_2)]$. Since the intersection of two blocks of the partition $\overline{M} = \mathfrak{A} \sqcap \hat{K}$ (1.4(11)) is nonempty, thus the blocks are the same. By 1.4(5), $\hat{K}(y_1) = \bigcup\{B_0(a) : a \in \hat{K}(y_1) \cap \mathfrak{A}\} = \bigcup\{B_0(a) : a \in \hat{K}(y_1) \cap \mathfrak{A}\} = \hat{K}(y_2)$ holds. Therefore T is a partition.

iii $\Rightarrow$ iv: Suppose xTy. Then there is $z \in \mathfrak{A}$ such that $x, y \in \hat{K}(z)$. Consequently $x, y \in \overline{B}_{11}(e)$, $x\overline{B}_{10}z$, $y\overline{B}_{10}z$. Thus $x\, \overline{B}_{11}(e) \sqcap B_{10}z$, $y\, \overline{B}_{11}(e) \sqcap B_{10}z$ and from the transitivity $x\overline{K}y$. Hence $T \leqq K$. We shall prove $T \geqq \overline{K}$ and so $T = \overline{K}$. By 1.4(1), (5) and (6) we have $T \geqq \overline{B}_{11}(e) \sqcap B_0$. We shall prove $T \geqq \overline{B}_{11}(e) \sqcap (B \wedge C_0)$. Indeed, from the relation $x\, \overline{B}_{11}(e) \sqcap (B \wedge C_0)\, y$ it follows that $x, y \in \overline{B}_{11}(e)$ $(\subseteq B(e) = \bigcup B_0)$, $xB \wedge C_0 y$ and $y \in \overline{B}_{11}(e) \cap C_0(y) \subseteq B(e) \cap C(e) = \mathfrak{A}$. From the first relation, we obtain $xB_0x(B \wedge C_0)\, y$, $x, y \in \overline{B}_{11}(e)$, i.e. $x \in \hat{K}(y)$. The second one implies $y \in \hat{K}(y)$ (1.4(5)) and we have xTy. Consequently, T contains the supremum $[\overline{B}_{11}(e) \sqcap B_0] \vee [\overline{B}_{11}(e) \sqcap (B \wedge C_0)]$, which is equal to $\overline{K}$ by 1.4(9) and 1.4(2).

iv $\Rightarrow$ iii is evident.

iii $\Rightarrow$ ii: If we prove $\mathfrak{A} \sqcap T = \mathfrak{A} \sqcap \hat{K}$ it will be proved that $\overline{M}$ is a partition for by 1.4(11) $\mathfrak{A} \sqcap \hat{K} = \overline{M}$. By definition $\mathfrak{A} \sqcap \hat{K} = \{(x, y) \in \mathfrak{A} \times \mathfrak{A} : x \in \hat{K}(y)\}$ and $\mathfrak{A} \sqcap T = \{(x, y) \in \mathfrak{A} \times \mathfrak{A} : x, y \in \hat{K}(z)$ for some $z \in \mathfrak{A}\}$. In the first case, $y \in \hat{K}(y)$ by 1.4(5) and in the second $\hat{K}(z) = \hat{K}(y)$, since $\hat{K}$ is a partition. Thus $\mathfrak{A} \sqcap T = \mathfrak{A} \sqcap \hat{K}$.

2.

Let us recall some lemmas from [9].

Lemma 2.1. (see Lemma 2.1 [9]) *If A and D are congruences in an algebra $(\mathfrak{G}, \Omega)$ and $\bigcup A \supseteq \bigcup D$ then $A \vee_{\mathscr{K}} D = A \vee_{\mathscr{P}} D$.*

Let $\mathfrak{A}$ and $\mathfrak{D}$ be Ω-subgroups of an Ω-group $(\mathfrak{G}, \Omega)$.

Definition 2.2. (see Definition 2.2 [9]) $\mathfrak{A}$ and $\mathfrak{D}$ are called *Ω-commuting Ω-subgroups* if $[\mathfrak{A}, \mathfrak{D}] = \mathfrak{A} + \mathfrak{D}$, where $[\mathfrak{A}, \mathfrak{D}]$ means the Ω-subgroup of $(\mathfrak{G}, \Omega)$ generated by the set $\mathfrak{A} \cup \mathfrak{D}$.

Clearly, Ω-commuting Ω-subgroups are commuting subgroups.

Lemma 2.3. (see Lemma 2.3 [9]) *Let $\mathfrak{A}$ and $\mathfrak{D}$ be Ω-subgroups of an Ω-group $(\mathfrak{G}, \Omega)$. Then the following conditions* (a) *to* (e) *fulfil $a \Leftrightarrow b \Rightarrow c \Leftrightarrow d \Leftrightarrow e$.*

(a) *$\mathfrak{A}$ and $\mathfrak{D}$ are Ω-commuting,*

(b) $\mathfrak{G}/_r[\mathfrak{A}, \mathfrak{D}] = \mathfrak{G}/_r\mathfrak{A} \,.\, \mathfrak{G}/_r\mathfrak{D}$,

(c) *$\mathfrak{G}/_r\mathfrak{A}$ and $\mathfrak{G}/_r\mathfrak{D}$ commute,*

(d) $\mathfrak{G}/_r\mathfrak{A} \vee_{\mathscr{P}} \mathfrak{G}/_r\mathfrak{D} = \mathfrak{G}/_r\mathfrak{A} \,.\, \mathfrak{G}/_r\mathfrak{D}$,

(e) *$\mathfrak{A}$ and $\mathfrak{D}$ commute.*

Analogous assertions hold for the left-sided decompositions.

Remark 2.4. Theorem 1.6 implies "The general four-group theorem" [1] 23.2 and consequently even the Zassenhaus Lemma (in a strengthened setting).

Corollary 2.5. (Borůvka [1] 23.2) *Let $\mathfrak{B} \supseteq \mathfrak{B}_0$ and $\mathfrak{C} \supseteq \mathfrak{C}_0$ be Ω-subgroups of an Ω-group $(\mathfrak{G}, \Omega)$. Let the following Ω-subgroups be Ω-commuting*:

$$\mathfrak{B} \cap \mathfrak{C} \quad \textit{and} \quad \mathfrak{B} \cap \mathfrak{C}_0 \quad \textit{with} \quad \mathfrak{B}_0\,; \qquad \mathfrak{C} \cap \mathfrak{B} \quad \textit{and} \quad \mathfrak{C} \cap \mathfrak{B}_0 \quad \textit{with} \quad \mathfrak{C}_0\,.$$

Then the following (left- or right-sided) decompositions

$$\text{(1)} \qquad \bar{K} = \mathfrak{B}_0 + \mathfrak{B} \cap \mathfrak{C}/\mathfrak{B}_0 + \mathfrak{B} \cap \mathfrak{C}_0\,, \quad \bar{L} = \mathfrak{C}_0 + \mathfrak{C} \cap \mathfrak{B}/\mathfrak{C}_0 + \mathfrak{C} \cap \mathfrak{B}_0\,,$$
$$\bar{M} = \mathfrak{B} \cap \mathfrak{C}/\mathfrak{B}_0 \cap \mathfrak{C} + \mathfrak{C}_0 \cap \mathfrak{B}$$

are pairwise coupled. All given sums of Ω-subgroups are Ω-subgroups. Further,

$$(\mathfrak{B} \cap \mathfrak{C}) \sqcap \bar{K} = (\mathfrak{B} \cap \mathfrak{C}) \sqcap \bar{L} = \bar{K} \wedge \bar{L} = \bar{M}\,.$$

(Zassenhaus Lemma.) *In particular, if $\mathfrak{B}_0$ and $\mathfrak{C}_0$ are ideals of $\mathfrak{B}$ and $\mathfrak{C}$, respectively, then all the partitions (1) are congruences and the corresponding factor Ω-groups are isomorphic.*

Proof. Let us define $B_0 = \mathfrak{B}/\mathfrak{B}_0$ and $B = \mathfrak{B}/\mathfrak{B}$ (where $/$ means e.g. the right sided decomposition). Similarly C_0 and C are defined. Let $e = 0 =$ the zero element of the additive group $\mathfrak{G}$. Then (in the notation from Theorem 1.6) $\mathfrak{A} = B(0) \cap C(0) = \mathfrak{B} \cap \mathfrak{C}$. The partitions $\mathfrak{A} \sqcap B_0 = (\mathfrak{B} \cap \mathfrak{C}) \sqcap \mathfrak{B}/\mathfrak{B}_0 = \mathfrak{B} \cap \mathfrak{C}/\mathfrak{B}_0 \cap \mathfrak{C}$ and $\mathfrak{A} \sqcap C_0 = \mathfrak{C} \cap \mathfrak{B}/\mathfrak{C}_0 \cap \mathfrak{B}$ commute since the Ω-subgroups $\mathfrak{B}_0 \cap \mathfrak{C}$ and $\mathfrak{C}_0 \cap \mathfrak{B}$ are Ω-commuting (2.3 and [9] Theorem 2.5(iv)). Thus the conditions of Theorem 1.6 are fulfilled. By 1.4(1) $\bar{B}_{11}(0) = \bigcup\{B_0(a) : a \in \mathfrak{A}\} = \mathfrak{B}_0 + \mathfrak{B} \cap \mathfrak{C}$ and by supposition, this set is an Ω-subgroup ([4] III 4.1). By 1.4(9) and (2),

$$\text{(2)} \qquad \bar{K} = (\bar{B}_{11}(0) \sqcap B_0) \vee (\bar{B}_{11}(0) \sqcap (B \wedge C_0)) = (\mathfrak{B}_0 + \mathfrak{B} \cap \mathfrak{C}/\mathfrak{B}_0) \vee$$
$$\vee (\mathfrak{B} \cap \mathfrak{C}/\mathfrak{B} \cap \mathfrak{C}_0),$$

for $B \wedge C_0 = \mathfrak{B} \sqcap \mathfrak{C}/\mathfrak{C}_0 = \mathfrak{B} \cap \mathfrak{C}/\mathfrak{B} \cap \mathfrak{C}_0$ Then (2) can be written in the form

$$\mathfrak{B}_0 + \mathfrak{B} \cap \mathfrak{C}/\mathfrak{B}_0 + \mathfrak{B} \cap \mathfrak{C}_0\,.$$

Inded, this partition (say S) contains both the partitions on the right side of (2), therefore it contains $\bar{K}$ as well and its domain agrees to $\bigcup\bar{K}$. Let R be a partition, $S \geqq R \geqq \bar{K}$ and $a \in \bigcup R(= \mathfrak{B}_0 + \mathfrak{B} \cap \mathfrak{C})$. Then $R(a) - a \supseteq \bar{K}(0) = \{\bar{B}_{11}(0) \sqcap \sqcap [B_0 \vee (B \wedge C_0)]\}(0) \supseteq \mathfrak{B}_0 + \mathfrak{B} \cap \mathfrak{C}_0$ so that $\mathfrak{B}_0 + \mathfrak{B} \cap \mathfrak{C}_0 + a = S(a) \supseteq R(a) \supseteq \mathfrak{B}_0 + \mathfrak{B} \cap \mathfrak{C}_0 + a$; hence $S(a) = R(a)$, thus $S = \bar{K}$. Analogously the representations of $\bar{L}$ and $\bar{M}$ are looked for.

Now, if $\mathfrak{B}_0$ and $\mathfrak{C}_0$ are ideals of $\mathfrak{B}$ and $\mathfrak{C}$, respectively, then by [4] III 4.1 $\mathfrak{B}_0 + \mathfrak{B} \cap \mathfrak{C}$, $\mathfrak{C}_0 + \mathfrak{C} \cap \mathfrak{B}$ are Ω-subgroups of $(\mathfrak{G}, \Omega)$. $\bar{K}$ as $\mathscr{P}$-supremum of congruences in $(\mathfrak{G}, \Omega)$ the domain of the first of which contains that of the other one (see (2)), is a congruence. This completes the proof of (a strengthened form of) Zassenhaus Lemma.

Lemma 2.6. *Let B and C be partitions in a set $\mathfrak{G}$, $e \in \bigcup B \cap \bigcup C$, B_0 a partition on $B(e)$, C_0 a partition on $C(e)$ and let $B_0' \leqq B'$, $C_0' \leqq C'$ be extensions on $\mathfrak{G}$ of the partitions B_0, B, C_0 and C, respectively. If $B'(e) \sqcap B_0'$ and $B'(e) \sqcap C_0'$ commute then $\mathfrak{A} \sqcap B_0$ $(= C(e) \sqcap B_0)$ and $\mathfrak{A} \sqcap C_0$ $(= B(e) \sqcap C_0)$ commute and $\bar{K}' = \hat{K}' = \bar{K}$. Analogous theorem holds if we interchange B and C.*

Proof. First the proof of the relation $\bar{K}' = \hat{K}'$. By 1.4(1), (2), (4) and 1.1 we have

$$\bar{K}' = \bar{B}_{11}'(e) \sqcap [B_0' \vee (B' \wedge C_0')] = \bar{B}_{11}'(e) \sqcap \{B'(e) \sqcap [B_0' \vee (B' \wedge C_0')]\} =$$
$$= \bar{B}_{11}'(e) \sqcap \{[B'(e) \sqcap B_0'] \vee [B'(e) \sqcap C_0']\} ;$$

similarly

$$\hat{K}' = \bar{B}_{11}'(e) \sqcap [B'(e) \sqcap B_0'] [B'(e) \sqcap C_0'] . \tag{1}$$

Since the supremum of two partitions on a set (on $B'(e)$) is equal to their product if (and only if) these partitions commute ([8] 1.1), we have $\bar{K}' = \hat{K}'$.

We shall prove $\hat{K}' = \bar{K}$. (The following statements (2) and (3) hold independently of the suppositions of Lemma.)

$$x\hat{K}'z \Leftrightarrow x\hat{K}z \quad (\text{for } z \in \mathfrak{A}) . \tag{2}$$

Indeed, fix $z \in \mathfrak{A}$. Then (see 1.4(4) and (1)) $x\hat{K}'z \Rightarrow x \in \bar{B}_{11}'(e)$, $x\bar{B}_{10}'z \Rightarrow x \in \bar{B}_{11}(e) \subseteq B(e)$, $xB_0'aC_0'z$, $aB'z$ for some $a \in \mathfrak{G}$. Now, aBz follows from the relations $z \in B(e)$, $aB'z$, consequently $a \in B(z) = B(e)$. From this and from the relation $xB_0'a$ we obtain xB_0a. We have proved xB_0aBz. From the relations $z \in C(e)$, $aC_0'z$ we obtain aC_0z. Together with the preceding result we have $x\bar{B}_{10}z$ and consequently $x\hat{K}z$ because $\bar{B}_{11}'(e) = \bar{B}_{11}(e)$ (see 1.4(4)). With regard to the relation $\hat{K}' \geqq \hat{K}$ we have (2). From (2) it follows immediately that

$$x\hat{K}'z \Rightarrow x\hat{K}z \Rightarrow x\bar{K}z \quad (\text{for } z \in \mathfrak{A}) . \tag{3}$$

Now, let the suppositions of Lemma be satisfied. Then $\hat{K}'$ is a partition which is equal to $\bar{K}'$. From 1.4(4) and (6), it follows that every block of the partition $\hat{K}'$ meets $\mathfrak{A}'$ $(= \mathfrak{A})$. From the relation $x\hat{K}'y$ we conclude the existence of an element $z \in \mathfrak{A}$ with the property $x, y \in \hat{K}'(z)$, therefore $x\bar{K}z$ and $y\bar{K}z$ by (3). Hence $x\bar{K}y$ and thus $\hat{K}' \subseteq \bar{K}$. Further $\bar{K} \subseteq \bar{K}' = \hat{K}'$. Consequently $\hat{K}' = \bar{K}$.

Finally, we shall prove that the partitions $\mathfrak{A} \sqcap B_0$ and $\mathfrak{A} \sqcap C_0$ commute. We have

$$x(\mathfrak{A} \sqcap B_0)(\mathfrak{A} \sqcap C_0)\, y \Rightarrow x(\mathfrak{A} \sqcap B_0')(\mathfrak{A} \sqcap C_0')\, y \Rightarrow$$
$$\Rightarrow x(B'(e) \sqcap B_0')(B'(e) \sqcap C_0')\, y \Rightarrow x(B'(e) \sqcap C_0')(B'(e) \sqcap B_0')\, y .$$

Thus $x, y \in \mathfrak{A}$ and $xC_0'aB_0'y$ for some $a \in \mathfrak{G}$. Relations $x \in C(e)$ and $xC_0'a$ imply

xC_0a, consequently $a \in C_0(x) \subseteq C(x) = C(e)$. Similarly: $y \in B(e)$, $aB'_0y \Rightarrow aB_0y \Rightarrow$ $\Rightarrow a \in B_0(y) \subseteq B(y) = B(e)$. Hence $a \in \mathfrak{A}$, then $x(\mathfrak{A} \sqcap C_0)(\mathfrak{A} \sqcap B_0)\, y$. We have proved

$$(\mathfrak{A} \sqcap B_0)(\mathfrak{A} \sqcap C_0) \subseteq (\mathfrak{A} \sqcap C_0)(\mathfrak{A} \sqcap B_0)\,.$$

The reverse inclusion is proved analogously.

Lemma 2.7. *Let $(\mathfrak{G}, \Omega)$ be an algebra, B and C partitions in the set $\mathfrak{G}$ and $e \in$ $\in \bigcup B \cap \bigcup C$. Let B_0 and C_0 be partitions on $B(e)$ and $C(e)$, respectively. If B_0 and C_0 are congruences in the algebra $(\mathfrak{G}, \Omega)$ then the relations $\bar{K}$ and $\bar{L}$ are congruences on the subalgebras $\bar{B}_{11}(e)$ and $\bar{C}_{11}(e)$, respectively. If $\bar{M}$ is a partition (i.e. if the partitions $\mathfrak{A} \sqcap B_0$ and $\mathfrak{A} \sqcap C_0$ commute), $\bar{M}$ is a congruence on the subalgebra $\mathfrak{A}$ and the congruences $\bar{K}$, $\bar{L}$ and $\bar{M}$ are pairwise coupled (as partitions) and therefore isomorphic (as factor algebras).*

Proof. By supposition, $\bigcup B_0 = B(e)$ and $\bigcup C_0 = C(e)$ are subalgebras of $(\mathfrak{G}, \Omega)$, thus $\mathfrak{A} = B(e) \cap C(e)$ is a subalgebra, too. We shall show that $\bar{B}_{11}(e)$ is a subalgebra. Let $\omega \in \Omega$ be n-ary, $x_1, \ldots, x_n \in \bar{B}_{11}(e)$. By 1.4(1) $a_k \in \mathfrak{A}$ $(\subseteq \bar{B}_{11}(e))$ exist such that $x_kB_0a_k$ $(k = 1, \ldots, n)$. Since B_0 is a congruence, we have $x_1 \ldots x_n\omega B_0 a_1 \ldots a_n\omega$ and because $\mathfrak{A}$ is a subalgebra, we have $a_1 \ldots a_n\omega \in \mathfrak{A}$. Again, by 1.4(1), we have $x_1 \ldots$ $\ldots x_n\omega \in \bar{B}_{11}(e)$, consequently $\bar{B}_{11}(e)$ (and similarly $\bar{C}_{11}(e)$) is a subalgebra. $\bar{B}_{11}(e) \sqcap$ $\sqcap B_0$ and $\bar{C}_{11}(e) \sqcap C_0$ are therefore congruences in $(\mathfrak{G}, \Omega)$. The partition $\bar{K} =$ $= (\bar{B}_{11}(e) \sqcap B_0) \vee_{\mathscr{P}} (\bar{B}_{11}(e) \sqcap C_0)$ (1.4(9)) as the $\mathscr{P}$-supremum of congruences in $(\mathfrak{G}, \Omega)$ whose domains are comparable sets (the domain of the first partition is $\bar{B}_{11}(e)$, that of the other one is obtained in $\mathfrak{A}$ (1.4(1)), is a congruence on the subalgebra $\bar{B}_{11}(e)$ (2.1). Similarly, $\bar{L}$ is a congruence on the subalgebra $\bar{C}_{11}(e)$. Finally $\bar{M} = (\mathfrak{A} \sqcap B_0)$. . $(\mathfrak{A} \sqcap C_0)$ as the product of two congruences is a stable relation in $(\mathfrak{G}, \Omega)$ ([5] 3.2). Hence if $\bar{M}$ is a partition, it is a congruence on the subalgebra $\mathfrak{A}$. The rest follows from 1.6.

In the following Theorem 2.8 a generalization of Châtelet's Theorem (see [3] or [7] Theorem 88) will be deduced as a corollary of Theorem 1.6 and Lemmas 2.6 and 2.7. Information in more detail is given in Remark 2.9 below.

Theorem 2.8. *Let $(\mathfrak{G}, \Omega)$ be an algebra, B and C partitions in the set $\mathfrak{G}$, $e \in$ $\in \bigcup B \cap \bigcup C$, B_0 and C_0 congruences in the algebra $(\mathfrak{G}, \Omega)$, B_0 and C_0 partitions on $B(e)$ and $C(e)$, respectively, B'_0, B', C'_0 and C' extensions on $\mathfrak{G}$ of the partitions B_0, B, C_0 and C, respectively. If the partitions $B'(e) \sqcap B'_0$, $B'(e) \sqcap C'_0$ commute and the partitions $C'(e) \sqcap B'_0$, $C'(e) \sqcap C'_0$ commute, then $\bar{K}' = \hat{K}' = \bar{K}$, $\bar{L}' = \hat{L}' = \bar{L}$, $\bar{M}' = \bar{M}$, the relations $\bar{K}$, $\bar{L}$ and $\bar{M}$ are congruences on the subalgebras $\bar{B}_{11}(e)$, $\bar{C}_{11}(e)$ and $\mathfrak{A}$ of the algebra $(\mathfrak{G}, \Omega)$, respectively, they are pairwise coupled (as partitions) and therefore isomorphic (as factor algebras).*

Proof. By 2.6 $\bar{K}' = \hat{K}' = \bar{K}$ and $\bar{L}' = \hat{L}' = \bar{L}$, by 1.4(7) $\bar{M}' = \bar{M}$, by 2.6 and 2.7 $\bar{K}$, $\bar{L}$ and $\bar{M}$ are congruences on the subalgebras $\bar{B}_{11}(e)$, $\bar{C}_{11}(e)$ and $\mathfrak{A}$, respectively, of the algebra $(\mathfrak{G}, \Omega)$. By 1.6 $\bar{K}, \bar{L}$ and $\bar{M}$ are pairwise coupled (as partitions) and therefore isomorphic (as factor algebras).

Remark 2.9. We shall explain in more detail relations between 2.8 and Châtelet's Theorem ([3], [7] Theorem 88).

1. Denote

$$B_2' = \Phi_{i+1}, \quad B_1' = \Phi_i, \quad B_0' = \Phi_{i-1}, \quad C_2' = \Psi_{j+1}, \quad C_1' = \Psi_j, \quad C_0' = \Psi_{j-1}.$$

In Châtelet's Theorem and in Theorem 2.8, it is supposed that $B_1'(e)$ (denoted by $K(\Phi_i)$ in Theorem 88) is a subalgebra of $(\mathfrak{G}, \Omega)$ and $B_0 = B_1'(e) \sqcap B_0' = K(\Phi_i) \sqcap \sqcap \Phi_{i-1}$ is a congruence on $K(\Phi_i)$. Similarly for Ψ_j. Theorem 88 requires the commutativity of the partitions Φ_i and Φ_{i-1} with both partitions Ψ_j and Ψ_{j-1}. Theorem 2.8 requires only the commutativity of the partitions $B_1'(e) \sqcap B_0' = K(\Phi_i) \sqcap \Phi_{i-1}$, $B_1'(e) \sqcap C_0' = K(\Phi_i) \sqcap \Psi_{j-1}$ and the commutativity of the partitions $C_1'(e) \sqcap B_0' = K(\Psi_j) \sqcap \Phi_{i-1}$, $C_1'(e) \sqcap C_0' = K(\Psi_j) \sqcap \Psi_{j-1}$.

Theorem 88 asserts that the relations (in fact, partitions by hypothesis)

$$\Phi_{ij} = \Phi_{i-1}\Psi_j \wedge \Phi_i = \Phi_{i-1}(\Psi_j \wedge \Phi_i) = B_0'(C_1' \wedge B_1') = \bar{B}_{11}',$$

$$\Psi_{ji} = \Psi_{j-1}\Phi_i \wedge \Psi_j = \Psi_{j-1}(\Phi_i \wedge \Psi_j) = C_0'(B_1' \wedge C_1') = \bar{C}_{11}',$$

$$\Phi_{i,j-1} = B_0' \vee (B_1' \wedge C_0') = B_{10}', \quad \Psi_{j,i-1} = C_0' \vee (C_1' \wedge B_0') = C_{10}'$$

have the following properties

$$K(\Phi_{ij}) \sqcap \Phi_{i,j-1} = \bar{B}_{11}'(e) \sqcap B_{10}' = \bar{K}'$$

is a congruence on the subalgebra $K(\Phi_{ij}) = \bar{B}_{11}'(e)$ (similarly for Ψ) and

$$\bar{K}' = K(\Phi_{ij}) \sqcap \Phi_{i,j-1} \cong K(\Psi_{ji}) \sqcap \Psi_{j,i-1} = \bar{L}'.$$

All these assertions follow from Theorem 2.8.

2. Theorem 2.8 gives a strengthened version of the (essential) part of Châtelet's Theorem (as to the whole Theorem — see 3.4). It supposes instead of the commutativity of the partitions Φ_{i-1}, Ψ_{j-1} only the commutativity of intersections of these partitions with $K(\Phi_i)$ and the commutativity of intersections of these partitions with $K(\Psi_j)$, which is a weaker requirement as we shall show further (point 4). The assertion of Theorem 2.8 is stronger because it proves that the corresponding partitions are coupled.

3. Zassenhaus Lemma (in the formulation of Corollary 2.5) is easily obtained from Theorem 2.8. Using the notation of Corollary 2.5 let us define: $e = 0 =$ the zero element of the group $\mathfrak{G}$,

$$B_0 = \mathfrak{B}/\mathfrak{B}_0, \quad B = \mathfrak{B}/\mathfrak{B}, \quad C_0 = \mathfrak{C}/\mathfrak{C}_0, \quad C = \mathfrak{C}/\mathfrak{C},$$

$$B_0' = \mathfrak{G}/\mathfrak{B}_0, \quad B' = \mathfrak{G}/\mathfrak{B}, \quad C_0' = \mathfrak{G}/\mathfrak{C}_0, \quad C' = \mathfrak{G}/\mathfrak{C},$$

(all decompositions will be understood, e.g. the right sided ones). The partitions B_0, B, C_0 and C are congruences in $(\mathfrak{G}, \Omega)$, B_0', B', C_0' and C' their extensions on $\mathfrak{G}$.

The partitions $B'(e) \sqcap B'_0 = \mathfrak{B} \sqcap \mathfrak{G}/\mathfrak{B}_0 = \mathfrak{B}/\mathfrak{B}_0$ and $B'(e) \sqcap C'_0 = \mathfrak{B} \sqcap \mathfrak{G}/\mathfrak{C}_0 = \mathfrak{B}/\mathfrak{B} \cap \mathfrak{C}_0$ as congruences on the Ω-group $\mathfrak{B}$ commute (2.3). $C'_1(e) \sqcap B'_0$ and $C'_1(e) \sqcap C'_0$ commute for similar reasons. Thus the suppositions of Theorem 2.8 are fulfilled.

Let us change the formulation of the assertion of Theorem 2.8 for our situation. By 1.4(4) and [5] 3.5.5 we have

$$\bar{B}'_{11}(0) = \bar{B}_{11}(0) = [B_0(B \wedge C)](0) =$$
$$= B_0(0) + \bigcup B_0 \cap B(0) \cap C(0) = \mathfrak{B}_0 + \mathfrak{B} \cap \mathfrak{C}\,.$$

By 2.6 $\bar{K}' = \hat{K}' = \bar{K}$. Thus

$$\bar{K} = \bar{K}' = \bar{B}'_{11}(0) \sqcap B'_{10} = (\mathfrak{B}_0 + \mathfrak{B} \cap \mathfrak{C}) \sqcap [\mathfrak{G}/\mathfrak{B}_0 \vee (\mathfrak{G}/\mathfrak{B} \wedge \mathfrak{G}/\mathfrak{C}_0)] =$$
$$= (\mathfrak{B}_0 + \mathfrak{B} \cap \mathfrak{C}) \sqcap \mathfrak{G}/(\mathfrak{B}_0 + \mathfrak{B} \cap \mathfrak{C}_0)$$

(by 2.3 $(a \Rightarrow b, d)$ – since $\mathfrak{B}_0$ is an ideal of $\mathfrak{B}$). Hence

$$\bar{K} = \mathfrak{B}_0 + \mathfrak{B} \cap \mathfrak{C}/\mathfrak{B}_0 + \mathfrak{B} \cap \mathfrak{C}_0\,.$$

Similarly we obtain $\bar{L}$ and $\bar{M}$. Now, the assertions of the Zassenhaus Lemma (in the formulation of Corollary 2.5) follow from 2.8.

4. Example for which the suppositions of Theorem 2.8 are fulfilled and the commutativity of the partitions B'_0, C'_0 (supposed in Theorem 88) fails.

Let $(\mathfrak{G}, \Omega)$ be an Ω-group and let the Ω-subgroups $\mathfrak{B}_0$, $\mathfrak{B}$, $\mathfrak{C}_0$ and $\mathfrak{C}$ be chosen as in point 3. If $\mathfrak{B}_0$ and $\mathfrak{C}_0$ do not commute the partitions $B'_0 = \mathfrak{G}/\mathfrak{B}_0$ and $C'_0 = \mathfrak{G}/\mathfrak{C}_0$ do not commute (2.3). On the other hand, by 2.3 and [4] III 4.1, the partitions $B'(0) \sqcap B'_0 = \mathfrak{B} \sqcap \mathfrak{G}/\mathfrak{B}_0 = \mathfrak{B}/\mathfrak{B}_0$ and $B'(0) \sqcap C'_0 = \mathfrak{B} \sqcap \mathfrak{G}/\mathfrak{C}_0 = \mathfrak{B}/\mathfrak{B} \cap \mathfrak{C}_0$ commute and the partitions $C'(0) \sqcap B'_0 = \mathfrak{C}/\mathfrak{C} \cap \mathfrak{B}_0$ and $C'(0) \sqcap C'_0 = \mathfrak{C}/\mathfrak{C}_0$ commute as well.

Another consequence of Theorem 1.6 and Lemmas 2.6 and 2.7 is Theorem [1] 10.8 or its algebra version 17.6.

Corollary 2.10. *Let $B'_0 \leqq B'$ and $C'_0 \leqq C'$ be partitions on a set $\mathfrak{G}$, $e \in \mathfrak{G}$, let the partitions $B'(e) \sqcap B'_0$ and $B'(e) \sqcap C'_0$ commute and let the partitions $C'(e) \sqcap B'_0$ and $C'(e) \sqcap C'_0$ commute. Denote by B and C partitions with the unique block $B'(e)$ and $C'(e)$, respectively, $B_0 = B'(e) \sqcap B'_0$ and $C_0 = B'(e) \sqcap C'_0$. Then the partitions $\bar{K}'$, $\bar{L}'$ and $\bar{M}'$ are pairwise coupled and satisfy $\bar{K}' = \hat{K}' = \bar{K}$, $\bar{L}' = \hat{L}' = \bar{L}$, $\bar{M}' = \bar{M}$.*

If $(\mathfrak{G}, \Omega)$ is an algebra and if the partitions B'_0, B', C'_0 and C' are congruences on $(\mathfrak{G}, \Omega)$ then $\bar{K}'$, $\bar{L}'$ and $\bar{M}'$ are isomorphic congruences on $(\mathfrak{G}, \Omega)$.

Proof. The conditions of Lemma 2.6 are fulfilled so that $\bar{K}' = \hat{K}' = \bar{K}$, $\bar{L}' = \hat{L}' = \bar{L}$ and by 1.4(7) $\bar{M}' = \bar{M}$. In virtue of 2.6 one can use Theorem 1.6, therefore the partitions $\bar{K}$, $\bar{L}$, $\bar{M}$ and thus even the partitions $\bar{K}'$, $\bar{L}'$, $\bar{M}'$ are pairwise coupled. Hence the algebra version follows trivially.

Remark 2.11. If we suppose the conditions of 2.10 to be satisfied for every $e \in \mathfrak{G}$, we obtain from 2.10 the essential part of Theorem 10.8 [1] (as to the whole Theorem – see Remark 3.3). In more detail: Let

$$B_1 \leqq B_2 \leqq \ldots \leqq B_r, \quad C_1 \leqq C_2 \leqq \ldots \leqq C_s$$

be two partition series on a set $\mathfrak{G}$ (see Definition 3.1 below). In Theorem 10.8 [1], under the supposition of commutativity of every partition B_i with every C_j, refinements of these series are constructed which are co-basally joint; this result can be formulated in the following way. Partitions

$$(1) \qquad \begin{array}{ll} [B_{i-1}(B_i \wedge C_j)](e) \sqcap [B_{i-1}(B_i \wedge C_{j-1})] & \text{and} \\ [C_{j-1}(C_j \wedge B_i)](e) \sqcap [C_{j-1}(C_j \wedge B_{i-1})] & \text{are coupled.} \end{array}$$

This is, however, the assertion of 2.10 if we put

$$B'_0 = B_{i-1}, \quad B' = B_i, \quad C'_0 = C_{j-1}, \quad C' = C_j.$$

Using this notation the assertion (1) reads as follows

$$\hat{K}' \quad \text{and} \quad \hat{L}' \quad \text{are coupled partitions.}$$

With regard to the commutativity of B'_0 and B' with both the partitions C'_0 and C' we have (see point 2)

$$\hat{K}' = \bar{K}', \quad \hat{L}' = \bar{L}',$$

so that (1) is the assertion of Corollary 2.10.

2. Let us compare the conditions of Theorem 10.8 [1] and those of Corollary 2.10: The commutativity of B'_0 and C'_0 implies the commutativity of the partitions $B'(e) \sqcap B'_0$ and $B'(e) \sqcap C'_0$ and the commutativity of the partitions $C'(e) \sqcap B'_0$ and $C'(e) \sqcap C'_0$ for every $e \in \mathfrak{G}$. (This is true since $(B'(e) \sqcap B'_0)(B'(e) \sqcap C'_0) = (B'(e) \sqcap B'_0)(B'(e) \sqcap (B' \wedge C'_0)) = B'(e) \sqcap [B'_0(B' \wedge C'_0)] = B'(e) \sqcap (B'_0C'_0)$ – see 1.1. We obtain the last equality as follows. Evidently $B'(e) \sqcap (B'_0C'_0) \geqq B'(e) \sqcap [B'_0(B' \wedge C'_0)]$. To obtain the converse inequality, put $x\, B'(e) \sqcap (B'_0C'_0)\, y$. Then $x, y \in B'(e)$, $xB'_0aC'_0y$ for an $a \in \mathfrak{G}$. It suffices to prove $aB'y$. Indeed, $x \in B'(e)$ and xB'_0a implies $eB'xB'_0a$, and $B'_0 \leqq B'$ implies $eB'xB'a$, i.e. $a \in B'(e)$. Then the relations $a, y \in B'(e)$ give $aB'y$.)

The converse assertion about the commutativity is not true as the following example shows.

3. Example where the conditions of Theorem 2.10 are satisfied (even the conditions of Theorem 2.8) for every $e \in \mathfrak{G}$, and in spite of this B'_0 and C'_0 do not commute. This example is presented by an arbitrary Ω-group $(\mathfrak{G}, \Omega)$ in which non-commuting Ω-subgroups $\mathfrak{B}$ and $\mathfrak{C}$ and their ideals $\mathfrak{B}_0$ and $\mathfrak{C}_0$, respectively, are given.

If we denote the (e.g. right sided) decompositions $\mathfrak{G}/\mathfrak{B}$, $\mathfrak{G}/\mathfrak{C}$, $\mathfrak{G}/\mathfrak{B}_0$ and $\mathfrak{G}/\mathfrak{C}_0$ by B', C', B'_0 and C'_0, respectively, then the partitions B'_0 and C'_0 do not commute since the subgroups $\mathfrak{B}_0$ and $\mathfrak{C}_0$ do not commute (2.3), while the partitions

$$B'(e) \sqcap B'_0 = (\mathfrak{B} + e) \sqcap \mathfrak{G}/\mathfrak{B}_0 \quad \text{and}$$

$$B'(e) \sqcap C'_0 = (\mathfrak{B} + e) \sqcap \mathfrak{G}/\mathfrak{C}_0 = (\mathfrak{B} + e) \sqcap \mathfrak{G}/\mathfrak{B} \cap \mathfrak{C}_0$$

commute. To prove it we consider first the following:

$$(B'(e) \sqcap B'_0) \vee (B'(e) \sqcap C'_0) =^{(1)} [(\mathfrak{B} + e) \sqcap \mathfrak{G}/\mathfrak{B}_0] \vee$$
$$\vee [(\mathfrak{B} + e) \sqcap \mathfrak{G}/\mathfrak{B} \cap \mathfrak{C}_0] =^{(2)} (\mathfrak{B} + e) \sqcap (\mathfrak{G}/\mathfrak{B}_0 \vee \mathfrak{G}/\mathfrak{B} \cap \mathfrak{C}_0) =^{(3)}$$
$$=^{(3)} (\mathfrak{B} + e) \sqcap [(\mathfrak{G}/\mathfrak{B}_0)(\mathfrak{G}/\mathfrak{B} \cap \mathfrak{C}_0)] =^{(4)} [(\mathfrak{B} + e) \sqcap \mathfrak{G}/\mathfrak{B}_0] \cdot$$
$$\cdot [(\mathfrak{B} + e) \sqcap \mathfrak{G}/\mathfrak{B} \cap \mathfrak{C}_0] =^{(5)} (B'(e) \sqcap B'_0)(B'(e) \sqcap C'_0) .$$

The second and fourth equalities are true since $\mathfrak{B} + e$ respects the partitions $\mathfrak{G}/\mathfrak{B}_0$ and $\mathfrak{G}/\mathfrak{B} \cap \mathfrak{C}_0$ (1.1). The third equality follows from 2.3 (since $\mathfrak{B}_0$ and $\mathfrak{B} \cap \mathfrak{C}_0$ commute). The obtained result says that the supremum of partitions is their product; consequently, the partitions commute. Similarly it can be proved that the partitions $C'(e) \sqcap B'_0$ and $C'(e) \sqcap C'_0$ commute.

Lemma 2.12. *Let $B'_0 \leqq B'$ and $C'_0 \leqq C'$ be partitions on a set $\mathfrak{G}$ and $e \in \mathfrak{G}$. Let the partitions $B'(e) \sqcap B'_0$ and $B'(e) \sqcap C'$ commute. Then $\bar{B}'_{11}(e) = B'_{11}(e)$.*

Proof. Let $xB'_{11}e$. Since evidently $B'_{11}(e) \subseteq B'(e)$ we have $x = x_0A_1x_1 \ldots \ldots x_{n-1}A_nx_n = e$ for some $x_1, \ldots, x_{n-1} \in B'(e)$, where $A_1, \ldots, A_n$ are alternately equal to B'_0 and $B' \wedge C'$. By supposition the partitions $B'(e) \sqcap B'_0$ and $B'(e) \sqcap \sqcap (B' \wedge C') = B'(e) \sqcap C'$ commute, therefore $x_{k-1}A_kA_{k+1}x_{k+1} \Rightarrow x_{k-1}(B'(e) \sqcap \sqcap A_k)(B'(e) \sqcap A_{k+1})\, x_{k+1} \Rightarrow x_{k-1}(B'(e) \sqcap A_{k+1})(B'(e) \sqcap A_k)\, x_{k+1} \Rightarrow x_{k-1}A_{k+1} \cdot \cdot A_kx_{k+1}$. Hence $x\bar{B}'_{11}e$ and consequently $\bar{B}'_{11}(e) = B'_{11}(e)$.

3.

Definition 3.1. ([1] I 10.1) A finite chain

$$(*) \qquad A_1 \leqq A_2 \leqq \ldots \leqq A_n$$

of partitions in a set $\mathfrak{G}$ is said to be a *partition series (from A_1 to A_n) in the set $\mathfrak{G}$*.

([1] I 10.2) Let $e \in \bigcup A_1$. A *local chain* of a partition series (*) is the partition chain $\{A_1(e)\} \leqq A_2(e) \sqcap A_1 \leqq A_3(e) \sqcap A_2 \leqq \ldots \leqq A_n(e) \sqcap A_{n-1}$. We speak also about the *e-chain*.

([1] I 10.6) We say that two partition series are *e-joint* if there exists a bijection of the e-chain of one series on the e-chain of the other one such that the corresponding partitions are coupled.

Let

$$B'_1 \leqq B'_2 \leqq \ldots \leqq B'_r, \quad C'_1 \leqq C'_2 \leqq \ldots \leqq C'_s$$

be two partition series on a set $\mathfrak{G}$ and $e \in \mathfrak{G}$.

Define

$$(1)\qquad \bar{B}'_{ij} = B'_{i-1}(B'_i \wedge C'_j)\,, \quad B'_{ij} = B'_{i-1} \vee (B'_i \wedge C'_j)\,,$$
$$\bar{K}'_{ij} = \bar{B}'_{ij}(e) \sqcap B'_{i,j-1}\,, \quad \hat{K}'_{ij} = \bar{B}'_{ij}(e) \sqcap \bar{B}'_{i,j-1}\,,$$
$$\bar{M}_{ij} = (C_j(e) \sqcap B_{i-1})(B_i(e) \sqcap C_{j-1})\,, \quad \bar{K}_{ij} = \bar{B}_{ij}(e) \sqcap B_{i,j-1}\,,$$

where

$$B_{i-1} = B'_i(e) \sqcap B'_{i-1}\,, \quad C_{j-1} = C'_j(e) \sqcap C'_{j-1}\,.$$

Relations $\bar{C}'_{ji}, C'_{ji}, \bar{L}'_{ji}, \hat{L}'_{ji}, \bar{L}_{ji}, \bar{N}_{ji}$ are defined symmetrically (by interchanging B and C).

Theorem 3.2. I. *Let*

$$(a)\qquad B'_1 \leqq B'_2 \leqq \ldots \leqq B'_r\,, \quad C'_1 \leqq C'_2 \leqq \ldots \leqq C'_s$$

be two partition series on a set $\mathfrak{G}$ *and* $e \in \mathfrak{G}$. *Put*

$$B'_{r+1} = B'_r \vee C'_s = C'_{s+1} \quad \text{and} \quad B'_0 = B'_1 \wedge C'_1 = C'_0\,.$$

If for $2 \leqq i \leqq r + 1,\ 2 \leqq j \leqq s + 1$

$$(b)\qquad B'_i(e) \sqcap B'_{i-1} \quad \text{and} \quad B'_i(e) \sqcap C'_{j-1} \quad \text{commute, and}$$
$$C'_j(e) \sqcap C'_{j-1} \quad \text{and} \quad C'_j(e) \sqcap B'_{i-1} \quad \text{commute}$$

then there exist e-joint refinements of the series (a).

These refinements are (2) *and* (3) (see proof) *and do not depend on the element e; the members* $\bar{K}'_{ij}$ *and* $\bar{L}'_{ji}$ *of the e-chains* (4) *and* (5), *respectively, of these refinements are coupled.*

II. *Let* $(\mathfrak{G}, \Omega)$ *be an algebra. Define as in* (1) B_{i-1} *and* C_{j-1} *for* $2 \leqq i \leqq r + 1$, $2 \leqq j \leqq s + 1$. *If for some pair* (i, j) B_{i-1} *and* C_{j-1} *are congruences in* $(\mathfrak{G}, \Omega)$ *then* $\bar{K}_{ij}, \bar{L}_{ji}$ *and* $\bar{M}_{ij}$ (defined in (1)) *are congruences on the subalgebras* $B_{ij}(e)$, $C_{ji}(e)$ *and* $B_i(e) \cap C_j(e)$ *of the algebra* $(\mathfrak{G}, \Omega)$, *respectively, they are pairwise coupled (as partitions) and hence isomorphic (as factor algebras).*

Note. Since for $i = 1$, $1 \leqq j \leqq s + 2$ and $2 \leqq i \leqq r + 1$, $j = 1, s + 2$, $B'_i(e) \sqcap B'_{i-1}$ and $B'_i(e) \sqcap C'_{j-1}$ as comparable partitions commute (and symmetrically), then (b) implies

$$(b')\qquad B'_i(e) \sqcap B'_{i-1} \quad \text{and} \quad B'_i(e) \sqcap C'_{j-1} \quad \text{commute for} \quad 1 \leqq i \leqq r + 1\,,\ 1 \leqq j \leqq s + 2\,,$$

$$C'_j(e) \sqcap C'_{j-1} \quad \text{and} \quad C'_j(e) \sqcap B'_{i-1} \quad \text{commute for} \quad 1 \leqq i \leqq r + 2\,,\ 1 \leqq j \leqq s + 1\,.$$

Proof. I. The refinements of the series (a) can be chosen as follows:

$$(2) \qquad B'_0(=B'_{10}) \leqq B'_{11} \leqq \dots \leqq B'_{1,s+1}(=B'_1=B'_{20}) \leqq$$
$$\leqq B'_{21} \leqq \dots \leqq B'_{r,s+1}(=B'_r=B'_{r+1,0}) \leqq B'_{r+1,1} \leqq \dots \leqq B'_{r+1,s+1}(=B'_{r+1}),$$
$$(3) \qquad C'_0(=C'_{10}) \leqq C'_{11} \leqq \dots \leqq C'_{1,r+1}(=C'_1=C'_{20}) \leqq$$
$$\leqq C'_{21} \leqq \dots \leqq C'_{s,r+1}(=C'_s=C'_{s+1,0}) \leqq C'_{s+1,1} \leqq \dots \leqq C'_{s+1,r+1}(=C'_{s+1}).$$

The equalities are evident since

$$B'_{i0} = B'_{i-1} \vee (B'_i \wedge C'_0) = B'_{i-1} \vee B'_0 = B'_{i-1},$$
$$B'_{i,s+1} = B'_{i-1} \vee (B'_i \wedge C'_{s+1}) = B'_{i-1} \vee B'_i = B'_i$$

and analogously for C'_{j0} and $C'_{j,r+1}$. The inequalities are clear.

The e-chains are the following ones (see also (7) and (8))

$$(4) \quad \{B'_0(e)\} \leqq K'_{11} \leqq \dots \leqq K'_{1,s+1} \leqq K'_{21} \leqq \dots \leqq K'_{r+1,1} \leqq \dots \leqq K'_{r+1,s+1},$$
$$(5) \quad \{C'(e)\} \leqq L'_{11} \leqq \dots \leqq L'_{1,r+1} \leqq L'_{21} \leqq \dots \leqq L'_{s+1,1} \leqq \dots \leqq L'_{s+1,r+1}.$$

All inequalities are evident up to $K'_{i,s+1} \leqq K'_{i+1,1}$:

$$K'_{i,s+1} = B'_{i,s+1}(e) \sqcap B'_{i,s} = B'_{i+1,0}(e) \sqcap B'_{i,s} \leqq B'_{i+1,1}(e) \sqcap B'_{i,s+1} =$$
$$= B'_{i+1,1}(e) \sqcap B'_{i+1,0} = K'_{i+1,1}.$$

By 2.12, $K'_{ij} = \bar{K}'_{ij}$ if $B'_i(e) \sqcap B'_{i-1}$ and $B'_i(e) \sqcap C'_j$ commute. Then by the first part of (b) (see also (b′) in Note), we have

$$(7) \qquad K'_{ij} = \bar{K}'_{ij} \text{ for all } K'_{ij} \text{ from (4) and } L'_{ji} = \bar{L}'_{ji} \text{ for all } L'_{ji} \text{ from (5)}.$$

Define as in (1)

$$B_{i-1} = B'_i(e) \sqcap B'_{i-1} \quad \text{and} \quad C_{j-1} = C'_j(e) \sqcap C'_{j-1} \quad \text{for}$$
$$1 \leqq i \leqq r+1, \quad 1 \leqq j \leqq s+1.$$

Then B'_{i-1} and C'_{j-1} are extensions on $\mathfrak{G}$ of B_{i-1} and C_{j-1}, and also, B_{i-1} and C_{j-1} are partitions on $B_i(e)$ and $C_j(e)$, respectively. By 2.6

$$(8) \qquad \bar{K}'_{ij} = \bar{K}_{ij} \quad \text{and} \quad \bar{L}'_{ji} = \bar{L}_{ji} \quad \text{for all } \bar{K}'_{ij} \text{ and } \bar{L}'_{ji} \text{ from (7)}$$

and the partitions $\bigcup C_{j-1} \sqcap B_{i-1}$ and $\bigcup B_{i-1} \sqcap C_{j-1}$ commute for $1 \leqq i \leqq r+1$, $1 \leqq j \leqq s+1$. By 1.6 $\bar{K}_{ij}, \bar{L}_{ji}$ and $\bar{M}_{ij}$ are pairwise coupled partitions for $1 \leqq i \leqq r+1$, $1 \leqq j \leqq s+1$. Thus the chains (4) and (5) are e-joint.

II. Now, if for some (i, j) $(2 \leqq i \leqq r+1, 2 \leqq j \leqq s+1)$ the partitions B_{i-1} and C_{j-1} (defined above) are congruences in an algebra $(\mathfrak{G}, \Omega)$ then the partitions B_0 and C_0 are also congruences in $(\mathfrak{G}, \Omega)$ and by Theorem 2.8, $\bar{K}_{ij}$, $\bar{L}_{ji}$ and $\bar{M}_{ij}$ are congruences on the subalgebras $B_{ij}(e)$, $C_{ji}(e)$ and $B_i(e) \cap C_j(e)$ of the algebra $(\mathfrak{G}, \Omega)$, respectively $(1 \leqq i \leqq r+1,\ 1 \leqq j \leqq s+1)$, they are pairwise coupled (as partitions) and hence isomorphic (as factor algebras). This completes the proof of Theorem.

Remark 3.3. As consequences of Theorem 3.2 we obtain Theorems 10.8 and 17.6 [1] having the following form.

Let

$$\text{(c)} \qquad B'_1 \leqq B'_2 \leqq \ldots \leqq B'_r, \quad C'_1 \leqq C'_2 \leqq \ldots \leqq C'_s$$

be two partition (congruence) series on a set $\mathfrak{G}$ (on an algebra $(\mathfrak{G}, \Omega)$). Let

$$B'_i \text{ and } C'_j \text{ commute for } \quad 1 \leqq i \leqq r, \quad 1 \leqq j \leqq s.$$

Then refinements of the series (c) exist such that for arbitrary $e \in \mathfrak{G}$ these refinements are e-joint. In the algebra case, elements of the refinements are congruences in $(\mathfrak{G}, \Omega)$ and the corresponding congruences of these (e-joint) refinements are isomorphic (as factor algebras).

Proof. We prove that the condition (b) of Theorem 3.2 is satisfied. It is seen that

$$(A =)\ B'_i(e) \sqcap B'_{i-1} \quad \text{and} \quad (D =)\ B'_i(e) \sqcap C'_{j-1}$$

commute for $2 \leqq i \leqq r+1$, $2 \leqq j \leqq s+1$ and symmetrically (where $B'_{r+1} = B'_r \vee C'_s = C'_{s+1}$).

Indeed, we have $xADy \Rightarrow xB'_{i-1}C'_{j-1}y \Rightarrow xC'_{j-1}aB'_{i-1}y$ for some $a \in \mathfrak{G}$. We have also $y \in B'_i(e)$, consequently $B'_i(y) = B'_i(e)$ and further $a \in B'_{i-1}(y) \subseteq B'_i(y) = B'_i(e)$ and $x \in B'_i(e)$, therefore $xDAy$. This completes the proof.

In Remark 2.11 it has been proved that the conditions of Theorem 3.2 are weaker than the conditions of Theorem 10.8 [1].

The following generalization of Châtelet's Theorem ([7] Theorem 88) is another corollary of Theorem 3.2.

Corollary 3.4. *Let $(\mathfrak{G}, \Omega)$ be an algebra, $e \in \mathfrak{G}$, and let*

$$(1) \qquad B'_1 \leqq B'_2 \leqq \ldots \leqq B'_r, \quad C'_1 \leqq C'_2 \leqq \ldots \leqq C'_s$$

be two partition series on a set $\mathfrak{G}$. Let the partitions belonging to the e-chains of these series be congruences in the algebra $(\mathfrak{G}, \Omega)$.

If for $2 \leqq i \leqq r$, $2 \leqq j \leqq s$

(2) *the partitions* $B'_i(e) \sqcap B'_{i-1}$, $B'_i(e) \sqcap C'_{j-1}$ *commute, and*

the partitions $C'_j(e) \sqcap C'_{j-1}$, $C'_j(e) \sqcap B'_{i-1}$ *commute,*

then there exist e-joint refinements of the series (1) and the partitions belonging to their e-chains are congruences in the algebra $(\mathfrak{G}, \Omega)$. Indeed, the coupled members of the e-chains are isomorphic algebras.

Note 3.5. The results of the present paper suggest a simple sufficient condition under which Theorem 3.5 [9] is true. This condition reads

$$(*)\qquad \bigcup B_i \sqcap C_{j-1} \text{ and } \bigcup C_{j-1} \sqcap B_i \text{ commute for } 1 \leqq i \leqq r,\ 1 \leqq j \leqq s,$$
$$\bigcup C_j \sqcap B_{i-1} \text{ and } \bigcup B_{i-1} \sqcap C_j \text{ commute for } 1 \leqq i \leqq r,\ 1 \leqq j \leqq s.$$

This follows from Proposition 1.8 [9]. With respect to the supposition $B_0 = C_0$ and $B_r = C_s$ of Theorem 3.5 [9], the partitions $\bigcup B_i \sqcap C_0$, $\bigcup C_0 \sqcap B_i$ commute for $1 \leqq i \leqq r$ and the partitions $\bigcup C_j \sqcap B_0$, $\bigcup B_0 \sqcap C_j$ commute for $1 \leqq j \leqq s$, thus condition $(*)$ is equivalent to the following one

$$(**)\qquad \bigcup B_i \sqcap C_j \text{ and } \bigcup C_j \sqcap B_i \text{ commute for } 1 \leqq i \leqq r-1,$$
$$1 \leqq j \leqq s-1.$$

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