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Z-GROUP WREATH PRODUCTS

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In a recent paper [5], Konyndyk proved the following theorem. „If A and G are nontrivial locally nilpotent groups, then $A \operatorname{wr} G$ is residually central if and only if (1) G is torsion-free, or (2) for some prime p, all elements of G and of A of finite order have p-power order." We show that these groups are actually Z-groups. We also show that if A is a Z-group and G is a torsion-free locally nilpotent group, then $A \operatorname{wr} G$ is a Z-group.

A Z-group is a group with a central series. Hickin and Phillips in [4] proved that a group G is a Z-group if and only if for each non-trivial finitely generated subgroup K of G, $K \nleqq [K, G]$. If A and G are groups, the standard restricted wreath product of A and G, denoted $A \operatorname{wr} G$, is the semidirect product of $\bar{A}$ by G where $\bar{A}$ is the set of all functions from G into A with only finitely many non-1 values. If $\alpha \in \bar{A}$, $g \in G$, $g^{-1} \alpha g = \alpha^g \in \bar{A}$ such that $\alpha^g(x) = \alpha(xg^{-1})$ for all $x \in G$. $\bar{A}$ is called the *base group* of $A \operatorname{wr} G$. If A_1 is a subgroup of A, we let $\bar{A}_1 = \{\alpha \in \bar{A} \mid \alpha(g) \in A_1 \text{ for all } g \in G\}$.

Lemma 1. *Let A and G be non-trivial Z-groups. Then $W = A \operatorname{wr} G$ is a Z-group if and only if for every finitely generated subgroup $K \neq 1$ of $\bar{A}$, $K \nleqq [K, W]$.*

Proof. If W is a Z-group, the condition holds by [4]. Conversely, if $K \nleqq [K, W]$ for all non-trivial finitely generated subgroups K of $\bar{A}$, let $L = \langle w_1, \ldots, w_n \rangle \leqq W$. If $L \nleqq \bar{A}$, $W/\bar{A} \cong G$, a Z-group and $L\bar{A}/\bar{A}$ is a non-trivial finitely generated subgroup of $W/\bar{A}$. Hence, $L\bar{A}/\bar{A} \nleqq [L\bar{A}/\bar{A}, W/\bar{A}] = [L, W]\,\bar{A}/\bar{A}$ and so $L \nleqq [L, W]$. If $L \leqq \bar{A}$, $L \nleqq [L, W]$ by assumption. Thus, W is a Z-group. □

Theorem 1. *Suppose that A and G are locally nilpotent groups. Then $W = A \operatorname{wr} G$ is a Z-group if and only if*

(1) *G is torsion-free, or*

(2) *for some prime p, all elements of G and of A of finite order have order a power of p.*

Proof. If W is a Z-group, then W is residually central and (1) and (2) follow from Theorem 3 of [5].

Suppose that (1) or (2) holds and that $W = A \operatorname{wr} G$ is not a Z-group. Since the class of Z-groups is a local class [7], there is a finitely generated subgroup L of W which is not a Z-group. Hence, there are finitely generated subgroups A_1 of A and G_1 of G such that L can be embedded in A_1 wr G_1. Thus, A_1 wr G_1 is not a Z-group and A_1 and G_1 satisfy (1) or (2). Therefore, we may assume that A and G are finitely generated nilpotent groups.

By Lemma 1, there is a finitely generated subgroup K of $\bar{A}$ such that $K \leqq [K, W]$, $K \neq 1$. Since K is finitely generated and A is nilpotent, there exists an integer s such that $K \leqq \zeta_s(\bar{A})$ but $K \nleqq \zeta_{s-1}(\bar{A})$. If σ is the natural homomorphism of W onto $W/\zeta_{s-1}(\bar{A})$, $\sigma(K) \neq 1$ and $\sigma(K) \leqq \sigma([K, W]) = [\sigma(K), \sigma(W)]$ so that $W/\zeta_{s-1}(\bar{A})$ is not a Z-group. $\sigma(K) \leqq \sigma(\zeta_s(\bar{A})\, G)$ and $\sigma(K) \leqq [\sigma(K), \sigma(W)] = [\sigma(K), \sigma(\zeta_s(\bar{A})\, G)]$ so that $\sigma(\zeta_s(\bar{A})\, G)$ is not a Z-group. If we set $A_1 = \zeta_s(A)/\zeta_{s-1}(A)$, $\sigma(\zeta_s(\bar{A})\, G) \cong$ $\cong A_1$ wr G so that A_1 wr G is not a Z-group. By Corollary 2.11 in Baumslag [1], if A and G satisfy (2), A_1 and G also satisfy (2). Thus, we may assume that A is abelian.

If (1) holds, G is a finitely generated torsion-free nilpotent group and so is residually a finite q-group for all primes q [2]. Thus, by Theorem B2 of Hartley [3], W is residually nilpotent and hence a Z-group, a contradiction. If (2) holds, by Theorem 2.1 of Gruenberg [2], A and G are residually of order a power of p. Hence, by Theorem B1 of Hartley [3], W is residually a nilpotent p-group of finite exponent and so is a Z-group, again a contradiction.

Lemma 2. *Let A and G be nontrivial Z-groups. If A* wr *G is not a Z-group, then there exists a finitely generated Abelian group A_2 and a finitely generated subgroup G_1 of G such that A_2* wr *G_1 is not a Z-group.*

Proof. Let A and G be Z-groups, $A \neq 1 \neq G$, and assume that A wr G is not a Z-group. Since the Z-groups form a local class and are subgroup closed, there exist finitely generated subgroups A_1 of A and G_1 of G with $W = A_1$ wr G_1 not a Z-group. Hence there exists a finitely generated subgroup K of $\bar{A}_1$, $K \neq 1$, such that $K \leqq [K, W]$. Since A_1 has a central series, $S = \{(V_\sigma, \Lambda_\sigma) \mid \sigma \in \Sigma\}$, and K is finitely generated, there exists a $\sigma \in \Sigma$ with $K \subseteq \bar{\Lambda}_\sigma$ but $K \nsubseteq \bar{V}_\sigma$. Now, $\bar{V}_\sigma \lhd W$ and $W/\bar{V}_\sigma \cong$ $\cong (A_1/V_\sigma)$ wr G_1. Also, $1 \neq K\bar{V}_\sigma/\bar{V}_\sigma \leqq ([K, W]\, \bar{V}_\sigma)/\bar{V}_\sigma = [K\bar{V}_\sigma/\bar{V}_\sigma, W/\bar{V}_\sigma]$ so that $W/\bar{V}_\sigma$ is not a Z-group. Since $K\bar{V}_\sigma/\bar{V}_\sigma \leqq \bar{\Lambda}_\sigma/\bar{V}_\sigma \leqq \zeta(\bar{A}_1/\bar{V}_\sigma)$, $[(K\bar{V}_\sigma/\bar{V}_\sigma), (W/\bar{V}_\sigma)] =$ $= [(K\bar{V}_\sigma/\bar{V}_\sigma), (\bar{\Lambda}_\sigma G_1/\bar{V}_\sigma)]$ and so, $\bar{\Lambda}_\sigma G_1/\bar{V}_\sigma \cong (\Lambda_\sigma/V_\sigma)$ wr G_1 is not a Z-group.

Theorem 2. *Let A be a Z-group and G a torsion-free locally nilpotent group. Then A* wr *G is a Z-group.*

Proof. Assume that A wr G is not a Z-group. Thus there exists a finitely generated Abelian group A_1 and a finitely generated torsion-free nilpotent subgroup G_1 of G with A_1 wr G_1 not a Z-group. However, by our earlier theorem A_1 wr G_1 is a Z-group. Hence A wr G is a Z-group.

Phillips and Roseblade [6] have constructed examples of residually central groups which are not Z-groups. They obtain their examples by starting with a residually nilpotent torsion-free polycyclic group G with an abelian normal subgroup of finite index and $[G : G'] < \infty$. If p is a prime which does not divide $[G : G']$ and K is the field with p elements, their groups are split extensions of the group algebra, KG, by G with elements of G inducing automorphisms of KG by right multiplications. This group is isomorphic to K wr G and so gives examples of wreath products which are residually central but not Z-groups.

References

[1] *G. Baumslag* Lecture Notes on Nilpotent Groups Amer. Math. Soc., (Regional Conference Series in Mathematics, No. 2), Providence, RI, 1971.

[2] *K. W. Gruenberg:* Residual properties of infinite soluble groups, Proc. London Math. Soc., (3) *7* (1957), 29—62.

[3] *B. Hartley:* The residual nilpotence of wreath products, Proc. London Math. Soc., (3) *20* (1970), 365—392.

[4] *K. K. Hickin* and *R. E. Phillips:* On classes of groups defined by systems of subgroups, Arch. Math., *24* (1973), 346—350.

[5] *R. D. Konyndyk:* Residually central wreath products, Pac. J. Math., *68* (1977), 99—103.

[6] *R. E. Phillips* and *J. E. Roseblade:* A residually central group that is not a Z-group, Mich. Math. J., *25* (1978), 233—234.

[7] *D. J. S. Robinson:* Finiteness Conditions and Generalized Soluble Groups, Part II, Springer-Verlag, Berlin, (1972).

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