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Czechoslovak Mathematical Journal, Vol. 33 (1983), No. 1, 7–26

Persistent URL: http://dml.cz/dmlcz/101850

Czechoslovak Mathematical Journal, 33 (108) 1983, Praha

TORSION GROUPOIDS

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(Received February 13, 1980)

1. PRELIMINARIES

For every groupoid G we define a binary relation t_G on G as follows: $(x, y) \in t_G$ iff $ax = ay$ and $xa = ya$ for all $a \in G$. It is evident that t_G is a congruence of G; moreover, every equivalence which is contained in t_G is a congruence of G.

Let G be a groupoid. For every ordinal number i we define a binary relation $t_{G,i}$ on G as follows:

(1) $t_{G,0} = \mathrm{id}_G$;
(2) $(x, y) \in t_{G,i+1}$ iff $(ax, ay) \in t_{G,i}$ and $(xa, ya) \in t_{G,i}$ for all $a \in G$;
(3) if i is a limit ordinal then $(x, y) \in t_{G,i}$ iff $(x, y) \in t_{G,j}$ for some ordinal $j < i$.

It is easy to see that $t_{G,i}$ is a congruence of G for any i and if $i \leqq j$ then $t_{G,i} \subseteq t_{G,j}$. Evidently, $t_G = t_{G,1}$; for any i, $t_{G,i+1}$ is the only congruence of G with $t_{G,i+1} \supseteq t_{G,i}$ and $t_{G,i+1}/t_{G,i} = t_{G/t_{G,i}}$. We could define the congruences $t_{G,i}$ equivalently as follows: $t_{G,0} = \mathrm{id}_G$; if $i \neq 0$ then $(x, y) \in t_{G,i}$ iff there exists an ordinal $j < i$ such that $(ax, ay) \in t_{G,j}$ and $(xa, ya) \in t_{G,j}$ for all $a \in G$.

For every groupoid G we denote by $\bar{t}_G$ the union of the chain formed by the congruences $t_{G,i}$ (where i runs over all ordinals). Thus $\bar{t}_G$ is a congruence of G.

G is said to be a *torsion groupoid* if $\bar{t}_G = G \times G$.

For every groupoid G, the least ordinal i such that $t_{G,i} = t_{G,i+1}$ is called the length of G; it is just the least ordinal such that $t_{G,i} = \bar{t}_G$. The length of G will be denoted by $l(G)$.

A groupoid G is said to be semifaithful if $t_G = \mathrm{id}_G$; evidently, G is semifaithful iff $\bar{t}_G = \mathrm{id}_G$; also, G is semifaithful iff $l(G) = 0$.

For every groupoid G, the groupoid $G/\bar{t}_G$ is semifaithful.

For every ordinal number i we denote by $\mathscr{T}_i$ the class of torsion groupoids of length at most i. Further, let $\mathscr{T}$ denote the class of all torsion groupoids.

1.1. Lemma. *The following assertions are true*:
(1) *If H is a subgroupoid of G then $t_{G,i} \mid H \subseteq t_{H,i}$ for any ordinal i.*
(2) *If G, H are groupoids and $f : G \to H$ is a surjective homomorphism then $f(t_{G,i}) \subseteq t_{H,i}$ for any ordinal i.*

(3) *Let* G_p $(p \in P)$ *be a family of groupoids and* G *be its cartesian product; let* $a, b \in G$ *and let* i *be an ordinal. Let either* P *or* i *be finite. Then* $(a, b) \in t_{G,i}$ *iff* $(a(p), b(p)) \in t_{G_p,i}$ *for all* $p \in P$.

Proof is easy.

1.2. Proposition. *The classes* $\mathcal{T}_i$ *(for any ordinal number* i*) and* $\mathcal{T}$ *are closed under subgroupoids, homomorphic images and finite cartesian products.*

Proof follows from 1.1.

A groupoid G is said to be

– *trivial* if it contains only one element,
– *a semigroup with zero multiplication* if it satisfies the identity $xy = uv$,
– *medial* if it satisfies the identity $xy \, . \, uv = xu \, . \, yv$,
– *a left unar* if it satisfies the identity $xy = xz$,
– *a right unar* if it satisfies the identity $yx = zx$,
– *regular* if the following is true for all $a, b, c \in G$: if $ca = cb$ then $xa = xb$ for all $x \in G$; if $ac = bc$ then $ax = bx$ for all $x \in G$.

For every groupoid G we define two equivalences p_G and q_G on G as follows: $(x, y) \in p_G$ iff $xa = ya$ for all $a \in G$; $(x, y) \in q_G$ iff $ax = ay$ for all $a \in G$. We have $t_G = p_G \cap q_G$.

1.3. Lemma. *Let* G *be a regular groupoid such that* Card $(GG) = n$ *for some finite ordinal* n. *Then* Card $(G/p_G) \leqq n$, Card $(G/q_G) \leqq n$ *and* Card $(G/t_G) \leqq n^2$.

Proof is easy.

2. THE VARIETIES $\mathcal{T}_n$

Let G be a groupoid, $a_0, \ldots, a_k$ (where $k \geqq 0$ is an integer) elements of G and $e_1, \ldots, e_k$ elements of $\{1, 2\}$. Then we define an element $[a_0, e_1, a_1, \ldots, e_k, a_k]$ of G as follows:

if $k = 0$ then $[a_0, e_1, a_1, \ldots, e_k, a_k] = a_0$;
if $k \neq 0$ and $e_k = 1$ then $[a_0, e_1, a_1, \ldots, e_k, a_k] = [a_0, e_1, a_1, \ldots, e_{k-1}, a_{k-1}] \, . \, a_k$;
if $k \neq 0$ and $e_k = 2$ then $[a_0, e_1, a_1, \ldots, e_k, a_k] = a_k \, . \, [a_0, e_1, a_1, \ldots, e_{k-1}, a_{k-1}]$.

2.1. Proposition. *Let* n *be a non-negative integer. Then* $\mathcal{T}_n$ *is a variety; it is determined by the identities*

$$[x, e_1, x_1, \ldots, e_n, x_n] = [y, e_1, x_1, \ldots, e_n, x_n]$$

where $e_1, \ldots, e_n$ *is an arbitrary* n*-termed sequence whose all members belong to* $\{1, 2\}$.

Proof is easy.

If W is an absolutely free groupoid over a set X, then for every $a \in W$ we define the length $\lambda(a)$ of a in this way: $\lambda(x) = 1$ for all $x \in X$; if $a = bc$ then $\lambda(a) = \lambda(b) + \lambda(c)$.

2.2. Lemma. *Let W be an absolutely free groupoid over a set X and let n be a finite ordinal. Then for every $a \in W$ there exists an element $b \in W$ such that the identity $a = b$ is satisfied in $\mathscr{T}_n$ and $\lambda(b) \leqq 2^n$.*

Proof. Let $a \in W$ and let $b \in W$ be an element of minimal length such that the identity $a = b$ is satisfied in $\mathscr{T}_n$. Suppose $\lambda(b) > 2^n$. Define elements $b_0, \ldots, b_n \in W$ such that $\lambda(b_i) > 2^{n-i}$ as follows: $b_0 = b$; if $0 \leqq i < n$ and b_i is already defined, then $b_i \notin X$, $b_i = c_i d_i$ for some $c_i, d_i \in W$ and either $\lambda(c_i) > 2^{n-i-1}$ or $\lambda(d_i) > 2^{n-i-1}$; put $b_{i+1} = c_i$ if $\lambda(c_i) > 2^{n-i-1}$ and $b_{i+1} = d_i$ otherwise. We have $b = [b_n, e_1, b_{n-1}, \ldots, e_n, b_0]$ for some $e_1, \ldots, e_n \in \{1, 2\}$ and $\lambda(b_n) > 2^0 = 1$. If x is an arbitrary element of X and $c = [x, e_1, b_{n-1}, \ldots, e_n, b_0]$, then $\lambda(c) < \lambda(b)$ and the identity $b = c$ is satisfied in $\mathscr{T}_n$ by 2.1, a contradiction with the minimality of $\lambda(b)$.

2.3. Propostion. *Let n be a finite ordinal. Then the variety $\mathscr{T}_n$ is locally finite (i.e. every finitely generated groupoid from $\mathscr{T}_n$ is finite).*

Proof. It follows from 2.2 that for any finite set X the free groupoid in $\mathscr{T}_n$ over X is finite. Consequently, $\mathscr{T}_n$ is locally finite.

2.4. Proposition. *Let n be a finite ordinal. Then $\mathscr{T}_n$ has only a finite number subvarieties.*

Proof. It follows from 2.2 that there exists a finite set I of identites such that any identity is equivalent in $\mathscr{T}_n$ to some identity from I.

2.5. Example. $\mathscr{T}_0$ is the trivial variety.

2.6. Example. $\mathscr{T}_1$ is the variety of semigroups with zero multiplication.

2.7. Example. $\mathscr{T}_2$ is the variety determined by the identities

$$xy \,.\, z = uv \,.\, z \,, \quad z \,.\, xy = z \,.\, uv \,.$$

Especially, every groupoid from $\mathscr{T}_2$ is medial.

2.8. Example. It is easy to describe the lattice of subvarieties of $\mathscr{T}_2$. The lattice has exactly 24 elements and its picture is given in Fig. 1. The subvarieties $V_1, \ldots, V_{24}$ of $\mathscr{T}_2$ are determined by the identities of $\mathscr{T}_2$ together with the following identities (where 0 stands for $xx.xx$):

V_1: $x = x$
V_2: $x0 = 0x$
V_3: $xx = 0$

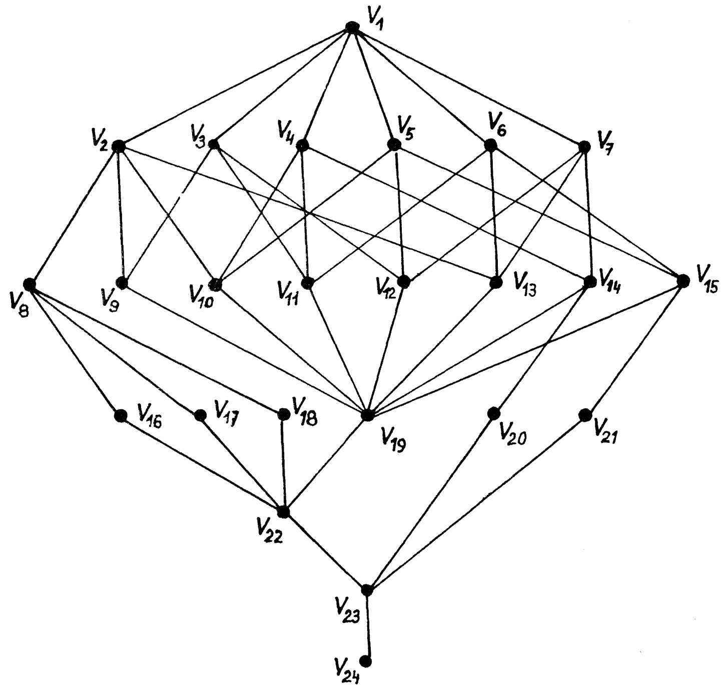

Fig. 1.

V_4: $x0 = 0$
V_5: $0x = 0$
V_6: $x0 = xx$
V_7: $0x = xx$
V_8: $xy = yx$
V_9: $xx = 0,\ x0 = 0x$
V_{10}: $x0 = 0x = 0$
V_{11}: $xx = x0 = 0$
V_{12}: $xx = 0x = 0$
V_{13}: $x0 = 0x = xx$
V_{14}: $x0 = 0,\ 0x = xx$
V_{15}: $0x = 0,\ x0 = xx$
V_{16}: $xx = 0,\ xy = yx$
V_{17}: $x0 = 0,\ xy = yx$
V_{18}: $x0 = xx,\ xy = yx$
V_{19}: $x0 = 0x = xx = 0$
V_{20}: $yx = 0x$
V_{21}: $xy = x0$

V_{22}: $xx = x0 = 0,\ xy = yx$
V_{23}: $xy = 0$
V_{24}: $x = y$

3. BASIC PROPERTIES OF TORSION GROUPOIDS

3.1. Lemma. *Let G be a finitely generated groupoid and R a congruence of G such that G/R is finite. Then R is a finitely generated congruence of G.*

Proof. There exist a finite subset M of G generating G and a finite subset N of G such that for every $a \in G$ there exists a $b \in N$ with $(a, b) \in R$. Denote by K the set of all elements of G that either belong to $M \cup N$ or can be expressed as ab for some elements $a, b \in M \cup N$. Evidently, K is a finite subset of G. Denote by S the congruence of G generated by the pairs (a, b) such that $a, b \in K$ and $(a, b) \in R$. Hence S is a finitely generated congruence and $S \subseteq R$. It is enough to prove $R \subseteq S$. Denote by H the set of all elements $a \in G$ such that whenever $b \in N$ and $(a, b) \in R$ then $(a, b) \in S$.

Let us prove that H is a subgroupoid of G. Let $a_1, a_2 \in H$; let $b \in N$ and $(a_1a_2, b) \in R$. There exist elements $b_1, b_2 \in N$ with $(a_1, b_1) \in R$ and $(a_2, b_2) \in R$. Since $a_1, a_2 \in H$, we have $(a_1, b_1) \in S$ and $(a_2, b_2) \in S$. Hence $(a_1a_2, b_1b_2) \in S \subseteq R$ and so $(b_1b_2, b) \in R$; since b_1b_2 and b belong to K, we have $(b_1b_2, b) \in S$ by the definition of S. We get $(a_1a_2, b) \in S$ and so $a_1a_2 \in H$.

Let us prove $M \subseteq H$. Let $a \in M$; let $b \in N$ and $(a, b) \in R$. Since a, b belong both to K, we have $(a, b) \in S$ by the definition of S. Hence $a \in H$.

We have proved that H is a subgroupoid of G containing the generating subset M. Consequently, $H = G$.

Let $(a, b) \in R$. There is an element $c \in N$ with $(a, c) \in R$. Since $a \in H$ and $b \in H$, we have $(a, c) \in S$ and $(b, c) \in S$ by the definition of H. Hence $(a, b) \in S$. This proves $R \subseteq S$.

3.2. Lemma. *Let i, j be two ordinal numbers and let G be a torsion groupoid of length $i + j$. Then $G/t_{G,i}$ is a torsion groupoid of length j.*

Proof is easy.

3.3. Proposition. *Every finitely generated torsion groupoid is finite.*

Proof. Suppose that there exists an infinite finitely generated torsion groupoid G. By 2.3, $l(G)$ is an infinite ordinal and so $l(G) = i + n$ for some limit ordinal $i \neq 0$ and some finite ordinal n. By 3.2, $G/t_{G,i}$ is a torsion groupoid of length n; moreover, it is finitely generated and so it is finite by 2.3. By 3.1, the congruence $t_{G,i}$ is finitely generated. However, $t_{G,i}$ is the union of the chain formed by the pairwise different congruences $t_{G,j}$ $(j < i)$; we get a contradiction.

3.4. Lemma. *Let G be a groupoid with zero 0; let H be a subgroupoid of G such that $xy = yx = 0$ for all $x \in H$ and $y \in G \setminus H$. Then $t_{G,i} \mid H = t_{H,i}$ for any ordinal i.*

Proof. It is easy.

3.5. Proposition. *For every ordinal number i there exists a commutative torsion groupoid G with zero such that $l(G) = i$.*

Proof. We shall proceed by induction on i. For $i = 0$, every trivial groupoid has the desired properties. Let $i = j + 1$ for some ordinal j and let H be a commutative torsion groupoid with zero 0 such that $l(H) = j$. For each ordinal $k < j$ there are elements $a_k, b_k \in H$ such that $(a_k, b_k) \notin t_{H,k}$. Put $G = H \cup \{a, b\} \cup \{c_k;\ k < j\}$ where a, b, c_k are pairwise different elements not belonging to H, and define a multiplication on G as follows: H is a subgroupoid of G; $ac_k = c_k a = a_k$ and $bc_k = c_k b = b_k$ for all $k < j$; $xy = 0$ in the remaining cases. Evidently, G is a commutative groupoid and 0 is the zero of G. Moreover, $GG \subseteq H$ and thus (xz, yz) and (zx, zy) belong to $t_{H,j}$ for all $x, y, z \in G$. It follows from 3.4 that (xz, yz) and (zx, zy) belong to $t_{G,j}$ for all $x, y, z \in G$. Consequently, $(x, y) \in t_{G,i}$ for all $x, y \in G$ and G is a torsion groupoid of length $\leqq i$. Now it suffices to show that $(a, b) \notin t_{G,j}$. Suppose $(a, b) \in t_{G,j}$. Then $j \neq 0$, since $a \neq b$; there exists a $k < j$ such that $(ax, bx) \in t_{G,k}$ for all $x \in G$; for $x = c_k$ we get $(a_k, b_k) \in t_{G,k}$, so that $(a_k, b_k) \in t_{H,k}$, a contradiction.

Now let $i \neq 0$ be a limit ordinal; for every ordinal $k < i$ let G_k be a commutative torsion groupoid with zero 0 such that $l(G_k) = k$. We can assume that $G_{k_1} \cap G_{k_2} = \{0\}$ for all $k_1, k_2 < i$ such that $k_1 \neq k_2$. Denote by G the union of the sets G_k $(k < i)$ and define a multiplication on G so that G_k be subgroupoids of G for all $k < i$ and $xy = 0$ in the remaining cases. Evidently, G is a commutative groupoid with zero 0. Let $a, b \in G$; we shall show that $(a, b) \in t_{G,i}$. If $a, b \in G_k$ for some $k < i$ then $(a, b) \in t_{G_k,k}$ and so $(a, b) \in t_{G,k} \subseteq t_{G,i}$ by 3.4. Let $a \in G_k$ and $b \in G_j$ where $k, j < i$ and $k \neq j$. If $c \in G_k$ then $ac \in G_k$, $bc = 0 \in G_k$ and so $(ac, bc) \in t_{G_k,k} \subseteq t_{G,k}$. If $c \in G_j$ then $(ac, bc) \in t_{G,j}$ similarly. If $c \in G \setminus (G_k \cup G_j)$ then $ac = bc = 0$. Thus $(ac, bc) \in t_{G,\mathrm{Max}(k,j)}$ for all $c \in G$; hence $(a, b) \in t_{G,\mathrm{Max}(k,j)+1} \subseteq t_{G,i}$. We have proved $t_{G,i} = G \times G$ and so G is a torsion groupoid of length $\leqq i$. If $k < i$, then there are elements $a, b \in G_{k+1}$ such that $(a, b) \notin t_{G_{k+1},k}$; we have $(a, b) \notin t_{G,k}$ and so $l(G) > k$.

3.6. Lemma. *Let G be a groupoid and i an ordinal number. Suppose that a block H of $t_{G,i}$ is a subgroupoid of G. Then H is a torsion groupoid and $l(H) \leqq i$.*

Proof follows from 1.1(1).

3.7. Lemma. *Let G be a torsion groupoid and $l(G) = i + 1$ for some ordinal i. Then $G/t_{G,i}$ is a non-trivial semigroup with zero multiplication. There exists exactly one block H of $t_{G,i}$ such that H is a subgroupoid of G; H is a torsion groupoid of length $\leqq i$ and we have $GG \subseteq H$.*

Proof is easy.

3.8. Proposition. *Every torsion groupoid contains exactly one idempotent.*

Proof. Let G be a torsion groupoid. First we shall show that G contains at least one idempotent. Denote by i the least ordinal such that $(a, aa) \in t_{G,i}$ for some $a \in G$. Clearly, $i \leqq l(G)$. Suppose $i \neq 0$. Then i is not a limit ordinal, $i = j + 1$ for some j, $(a, aa) \in t_{G,i}$, $(aa, a \,.\, aa) \in t_{G,j}$, $(a \,.\, aa, aa \,.\, aa) \in t_{G,j}$, $(aa, aa \,.\, aa) \in t_{G,j}$, a contradiction with the minimality of i. Hence $i = 0$ and $a = aa$ for some $a \in G$. Now we are going to prove that G contains at most one idempotent. We shall proceed by induction on $l(G)$. If $l(G) = 0$, there is nothing to prove. Let $l(G) \geqq 1$ and let a, b be two idempotents of G. Denote by i the least ordinal with $(a, b) \in t_{G,i}$. Obviously, $i \leqq l(G)$. If $i = l(G)$ then i is not a limit ordinal, $(a, b) \notin t_{G,i-1}$ and two different blocks of $t_{G,i-1}$ are subgroupoids of G, a contradiction with 3.7. Thus $i < l(G)$. Let H be the block of $t_{G,i}$ containing a. Then H is a subgroupoid of G; by 3.6, H is a torsion groupoid of length $\leqq i < l(G)$; since $a, b \in H$, we get $a = b$ by the induction assumption.

3.9. Proposition. *Let G be a torsion groupoid such that $GG = G$. Then $l(G)$ is a limit ordinal.*

Proof follows from 3.7.

3.10. Example. Let $G(+) = C(2^\infty)$ be the quasicyclic Prüffer 2-group. Define a multiplication on G by $xy = 2x + 2y$ for all $x, y \in G$. It is easy to verify that G is a commutative torsion division groupoid and $l(G) = \omega_0$.

3.11. Lemma. *Let G be a groupoid; let A_x $(x \in G)$ be pairwise disjoint non-empty sets; let f be a mapping of $G \times G$ into the set $H = \bigcup\{A_x;\, x \in G\}$ such that $f(x, y) \in A_{xy}$ for all $x, y \in G$. Define a multiplication on H as follows: if $x, y \in G$, $a \in A_x$ and $b \in A_y$ then $ab = f(x, y)$. Hence H is a groupoid. The following assertions are true:*

(1) *There is a congruence r of H such that $r \subseteq t_H$ and G is isomorphic to H/r.*
(2) *If G is a torsion groupoid then H is a torsion groupoid, too.*
(3) *Suppose that $x = y$ whenever $x, y \in G$ are such that $f(x, z) = f(y, z)$ and $f(z, x) = f(z, y)$ for all $z \in G$. Then G is isomorphic to H/t_H.*
(4) *The groupoid H is regular iff the following two conditions are satisfied:*
   (i) *if $x, y, z \in G$ are such that $f(x, z) = f(y, z)$ then $f(x, u) = f(y, u)$ for every $u \in G$;*
   (ii) *if $x, y, z \in G$ are such that $f(z, x) = f(z, y)$ then $f(u, x) = f(u, y)$ for every $u \in G$.*
(5) *If f is injective then H is regular and G is isomorphic to H/t_H.*

Proof is evident.

3.12. Proposition. *For every torsion groupoid G there exists a regular torsion groupoid H such that $G \simeq H/t_H$ and H is finite if G is finite. Moreover, for every non-trivial torsion groupoid G there exists a non-regular torsion groupoid K such that $G \simeq K/t_K$ and K is finite if G is finite.*

Proof follows from 3.11.

3.13. Corollary. *Let n be a positive integer and let f be a mapping of $\{0, \ldots, n\}$ into $\{0, 1\}$ such that $f(n-1) = f(n) = 1$. Then there exists a finite torsion groupoid G of length n such that for every $i \in \{0, \ldots, n\}$, the groupoid $G/t_{G,i}$ is regular iff $f(i) = 1$.*

A groupoid G is said to be *strongly regular* if $G/t_{G,n}$ is regular for any finite ordinal n. Evidently, every strongly regular groupoid is regular.

3.14. Proposition. *Let G be a strongly regular torsion groupoid. Then $l(G) \leqq \omega_0$.*

Proof. Let $(a, b) \in t_{G,\omega_0+1}$; it is enough to prove $(a, b) \in t_{G,\omega_0}$. Take an arbitrary element $c \in G$. We have $(ca, cb) \in t_{G,\omega_0}$ and $(ac, bc) \in t_{G,\omega_0}$ and so $(ca, cb) \in t_{G,n}$ and $(ac, bc) \in t_{G,n}$ for some finite n. Since $G/t_{G,n}$ is regular, $(xa, xb) \in t_{G,n}$ and $(ax, bx) \in \in t_{G,n}$ for all $x \in G$. Hence $(a, b) \in t_{G,n+1} \subseteq t_{G,\omega_0}$.

3.15. Lemma. *Let H be a subgroupoid of a strongly regular torsion groupoid G. Then $t_{H,n} = t_{G,n} \mid H$ for every finite n. Consequently, every subgroupoid of a strongly regular torsion groupoid is strongly regular.*

Proof is easy.

3.16. Lemma. *Let G be a non-trivial strongly regular torsion groupoid such that $l(GG) = n$ is finite. Then $l(G) = n + 1$.*

Proof. Proceeding by induction on n, we shall show that $l(G) = n + 1$. If $n = 0$ then GG is trivial, G is a non-trivial semigroup with zero multiplication and $l(G) = 1$. Let $n \geqq 1$. Denote by f the natural homomorphism of G onto the non-trivial strongly regular torsion groupoid $H = G/t_G$. Then $f(GG) = HH$. By 3.15, $t_{GG} = t_G \mid GG$ and so HH is isomorphic to GG/t_{GG}. We get $l(HH) = l(GG/t_{GG}) = n - 1$. By the induction hypothesis, $l(H) = n$ and so $l(G) = n + 1$.

3.17. Proposition. *Let G be a strongly regular torsion groupoid. Denote by 0 the only idempotent of G; for every ordinal $i \leqq l(G)$ denote by A_i the block of $t_{G,i}$ containing 0. Then $\{0\} = A_0 \subset A_1 \subset A_2 \subset \ldots \subset A_{l(G)} = G$ are subgroupoids of G; for every $i \leqq l(G)$ we have $l(A_i) = i$; for every $i < l(G)$ we have $A_{i+1}A_{i+1} \subseteq \subseteq A_i$; if $l(G) = \omega_0$ then $G = \bigcup_{i=0}^{\infty} A_i$.*

Proof. By 3.14, we have $l(G) \leqq \omega_0$. Consider first the case $l(G) = n < \omega_0$. It is clear that $\{0\} = A_0 \subseteq A_1 \subseteq A_2 \subseteq \ldots \subseteq A_n = G$ are subgroupoids of G and

$A_{i+1}A_{i+1} \subseteq A_i$ for all $i < n$; it remains to prove $l(A_i) = i$ for all $i \leqq n$. Suppose $l(A_i) \neq i$ for some i, so that $l(A_i) < i$ and $i < n$. By 3.16 we have $l(A_{i+1}) < i + 1$, $l(A_{i+2}) < i + 2, \ldots, l(A_n) < n$, a contradiction. In the case $l(G) = \omega_0$ the assertion is an easy consequence of 3.15 and the case already proved.

4. BASIC PROPERTIES OF SUBDIRECTLY IRREDUCIBLE TORSION GROUPOIDS

4.1. Lemma. *Let G be a groupoid and r a congruence of G such that $r \cap t_G = \mathrm{id}_G$. Then $r \cap \bar{t}_G = \mathrm{id}_G$.*

Proof. It is easy to show by induction on i that $r \cap t_{G,i} = \mathrm{id}_G$ for any ordinal i.

4.2. Proposition. *Let G be a non-trivial torsion groupoid. Then G is subdirectly irreducible iff there exist elements $a, b \in G$ such that $a \neq b$ and $t_G = \{(a, b), (b, a)\} \cup \mathrm{id}_G$.*

Proof. Since G is a torsion groupoid, $t_G \neq \mathrm{id}_G$. Since every equivalence contained in t_G is a congruence, if G is subdirectly irreducible then t_G has only one block of cardinality $\geqq 2$ and this block contains exactly two elements. On the other hand, if $t_G = \{(a, b), (b, a)\} \cup \mathrm{id}_G$ where $a \neq b$, then for any congruence r such that $r \not\supseteq t_G$ we have $r \cap t_G = \mathrm{id}_G$ and so $r = \mathrm{id}_G$ by 4.1; consequently, G is subdirectly irreducible.

4.3. Proposition. *Let G be a subdirectly irreducible torsion groupoid and a, b the elements such that $a \neq b$ and $t_G = \{(a, b), (b, a)\} \cup \mathrm{id}_G$. Then either G is the two-element semigroup with zero multiplication or $a, b \in GG$.*

Proof. Suppose $a \notin G$. Then the congruence $r = (GG \times GG) \cup \mathrm{id}_G$ of G has the property $r \cap t_G = \mathrm{id}_G$. Hence $r = \mathrm{id}_G$ and $\mathrm{Card}\,(GG) = 1$. We see that G is a semigroup with zero multiplication and the rest is clear.

4.4. Proposition. *Let G be a regular subdirectly irreducible torsion groupoid; let a, b be the elements such that $a \neq b$ and $t_G = \{(a, b), (b, a)\} \cup \mathrm{id}_G$. Then:*

(1) *Every subgroupoid of G containing a, b is subdirectly irreducible.*

(2) *Either a or b is the idempotent of G.*

Proof. (1) is clear. Let us prove (2). By 3.8, G contains exactly one idempotent e. We shall proceed by induction on $l(G)$. The statement is clear for $l(G) \leqq 1$. Let $i = l(G) \geqq 2$ and assume first that i is not a limit ordinal. Then $GG \subseteq H$ for a block H of $t_{G,i-1}$. By 4.3, $a, b \in H$. On the other hand, $e \in H$ and H is a regular subdirectly irreducible torsion groupoid and $l(H) \leqq i - 1$. We get either $a = e$ or $b = e$ by the induction assumption. Now, let i be a limit ordinal. There is an ordinal $j < i$

with $(a, e) \in t_{G,j}$; we have $a, b, e \in K$ where K is the block of $t_{G,j}$ containing e. Evidently, K is a regular subdirectly irreducible torsion groupoid of length $\leqq j$; by the induction assumption we get either $a = e$ or $b = e$.

5. REGULAR SUBDIRECTLY IRREDUCIBLE GROUPOIDS OF LENGTH AT MOST TWO

Consider the groupoids $A(0), A(1), \ldots, A(7)$ defined by the following multiplication tables:

$A(0)$	a	b
a	a	a
b	a	a

$A(1)$	a	b	c
a	a	a	b
b	a	a	b
c	b	b	a

$A(2)$	a	b	c
a	a	a	a
b	a	a	a
c	b	b	b

$A(3)$	a	b	c
a	a	a	b
b	a	a	b
c	a	a	b

$A(4)$	a	b	c	d
a	a	a	a	b
b	a	a	a	b
c	b	b	b	a
d	b	b	b	a

$A(5)$	a	b	c	d
a	a	a	b	b
b	a	a	b	b
c	a	a	b	b
d	b	b	a	a

$A(6)$	a	b	c	d
a	a	a	a	b
b	a	a	a	b
c	b	b	b	a
d	a	a	a	b

$A(7)$	a	b	c	d	e
a	a	a	a	b	b
b	a	a	a	b	b
c	b	b	b	a	a
d	a	a	a	b	b
e	b	b	b	a	a

5.1. Proposition. *The groupoids $A(0), A(1), A(2), A(3), A(4), A(5), A(6), A(7)$ are pairwise non-isomorphic regular subdirectly irreducible torsion groupoids of length $\leqq 2$. Moreover, every regular subdirectly irreducible torsion groupoid of length $\leqq 2$ is isomorphic to one of these eight groupoids.*

Proof. The proof of the first assertion is an easy routine verification. Let G be a regular subdirectly irreducible torsion groupoid of length $\leqq 2$. Let a, b be the elements such that $t_G = \{(a, b), (b, a)\} \cup \mathrm{id}_G$. By 4.4, we can assume that a is the only idempotent of G. Let G be not isomorphic to $A(0)$. Then it follows from 4.3 that $GG = \{a, b\}$. By 1.3, $\mathrm{Card}\,(G/t_G) \leqq 4$ and so $\mathrm{Card}\,(G) \leqq 5$. We shall consider only the case $\mathrm{Card}\,(G) = 5$ (the other cases are similar). Let $G = \{a, b, c, d, e\}$. If $p_G \subseteq q_G$ then $p_G = t_G$ and p_G has four blocks, a contradiction with 1.3. Thus $p_G \nsubseteq q_G$; similarly $q_G \nsubseteq p_G$ and consequently both p_G and q_G have exactly two blocks. We have $\{a, b\} = A \cap C$ for a block A of p_G and a block C of q_G; put $B = G \setminus A$ and $D = G \setminus C$. Each of the sets $A \cap D$, $B \cap C$, $B \cap D$ contains at most one element. From this we get Card $(A) =$ Card $(C) = 3$. We can assume without loss of generality that $A = \{a, b, d\}$ and $C = \{a, b, c\}$. Now it is clear that G has the same multiplication table as $A(7)$.

5.2. Example. There exists a proper class of non-isomorphic subdirectly irreducible torsion groupoids of length 2. This follows from the fact that for every semigroup H with zero multiplication there exists a subdirectly irreducible torsion groupoid G with $G/t_G \simeq H$. Indeed, the groupoid G can be constructed in the following way. Denote by 0 the only idempotent of H and let a be an element not belonging to H. Put $G = H \cup \{a\}$; put $x \circ x = a$ for all $x \in G$ and $0 \circ a = a \circ 0 = a$; put $x \circ y = 0$ for all the remaining pairs x, y. Evidently, the groupoid $G(\circ)$ has the desired properties.

6. SUBDIRECTLY IRREDUCIBLE TORSION UNARS

6.1. Proposition. *Let G be either a left or a right unar. Put $f(x) = xx$ for all $x \in G$. Then G is a torsion groupoid iff G contains an idempotent 0 and for every $x \in G$ there exists a positive integer n such that $f^n(x) = 0$.*

Proof is easy.

6.2. Corollary. *Let G be a torsion groupoid which is either a left or a right unar. Then $l(G) \leqq \omega_0$.*

Define two infinite countable groupoids $B(\infty)$ and $C(\infty)$ as follows:

$$B(\infty) = \{a_1, a_2, \ldots\}\ ; \quad a_i a_j = a_{i-1} \quad \text{for all } i, j \text{ such that} \quad i \neq 1\ ;$$
$$a_1 a_j = a_1 \quad \text{for all } j\ .$$
$$C(\infty) = \{a_1, a_2, \ldots\}\ ; \quad a_i a_j = a_{j-1} \quad \text{for all } i, j \text{ such that} \quad j \neq 1\ ;$$
$$a_i a_1 = a_1 \quad \text{for all } i\ .$$

Moreover, for every integer $n \geqq 2$ denote by $B(n)$ the subgroupoid of $B(\infty)$ formed by the elements $a_1, \ldots, a_n$ and denote by $C(n)$ the subgroupoid of $C(\infty)$ formed by the elements $a_1, \ldots, a_n$.

6.3. Proposition. *The groupoids $B(\infty)$ and $B(n)$ (where $n \geqq 2$ is an integer) are subdirectly irreducible torsion left unars; every subdirectly irreducible torsion left unar is isomorphic either to $B(\infty)$ or to $B(n)$ for some integer $n \geqq 2$. The groupoids $C(\infty)$ and $C(n)$ (where $n \geqq 2$ is an integer) are subdirectly irreducible torsion right unars; every subdirectly irreducible torsion right unar is isomophic either to $C(\infty)$ or to $C(n)$ for some $n \geqq 2$. We have $B(2) \simeq C(2) \simeq A(0)$, $B(3) \simeq A(2)$ and $C(3) \simeq A(3)$.*

Proof is easy.

The aim of this section is to prove the following lemma.

7.1. Lemma. *Let G be a regular subdirectly irreducible torsion groupoid, let $n \geqq 2$ be an integer and H a subgroupoid of G such that $GG \subseteq H$ and $H \simeq B(n)$. Further, assume that* Card $(G/q_G) = 2$. *Then G/t_G is a semigroup with zero multiplication.*

In order to prove this lemma, it is enough to assume that $H = B(n)$. Since G is regular and H is a left unar, H is contained in a block A of q_G. Taking into account that G is regular and subdirectly irreducible, we see that $t_G = \{(a_1, a_2), (a_2, a_1)\} \cup \cup \mathrm{id}_G$ and $t_H = \{(a_1, a_2), (a_2, a_1)\} \cup \mathrm{id}_H$.

7.2. Lemma. *Let $x \in A \setminus H$. Then $xa = a_n$ for every $a \in A$.*

Proof. We have $xa_1 = xa$ for every $a \in A$. Suppose $xa_1 = a_i$ for some $i < n$. Then $xa_1 = a_i = a_{i+1}a_1$, $(x, a_{i+1}) \in p_G$, $(x, a_{i+1}) \in t_G$, $x \in H$, a contradiction. Hence $xa_1 \notin \{a_1, \ldots, a_{n-1}\}$ and so $xa_1 = a_n$.

7.3. Lemma. *Either $A = H$ or* Card $(A \setminus H) = 1$.

Proof. Let $x, y \in A \setminus H$. By 7.2, $(x, y) \in t_G$. Hence $x = y$.

According to 7.2 and 7.3, the subgroupoid A of G is a left unar and A is isomorphic either to $B(n)$ or to $B(n + 1)$. Hence there is no loss of generality in assuming $A = H$. In the following, q_G has exactly two blocks, namely H and $G \setminus H$. For every $b \in \in G \setminus H$ put $K_b = \{b\} \cup H$. Evidently, K_b is a subgroupoid of G. Evidently, it is enough to prove that for any $b \in G \setminus H$, the groupoid K_b/t_{K_b} is a semigroup with zero multiplication. On the other hand, K_b is a regular subdirectly irreducible torsion groupoid and Card $(K_b/q_{K_b}) = 2$. Hence it is enough to continue in the proof under the assumption Card $(G \setminus H) = 1$. Denote by b the only element of $G \setminus H$. Define a transformation f of $\{1, \ldots, n\}$ by $a_ib = a_{f(i)}$ for all $i \in \{1, \ldots, n\}$.

7.4. Lemma. *The following assertions are true*:

(1) $f(1) = f(2) \neq 1$.

(2) *If $f(i) = f(j)$ then either $i = j$ or $\{i, j\} = \{1, 2\}$.*

(3) $f(i) \neq 1$ *for all i.*

Proof. It follows from $t_G = \{(a_1, a_2), (a_2, a_1)\} \cup \mathrm{id}_G$ and $q_G = (H \times H) \cup \cup \{(b, b)\}$.

7.5. Lemma. *We have $f(1) = f(2) = 2$. Moreover, if $n \geqq 3$ then $f(3) = 1$.*

Proof. Since Card $(G/q_G) = 2$, G is not a semigroup with zero multiplication, $l(G) \geqq 2$ and there exists a pair $(c, d) \in t_{G,2} \setminus t_G$. Hence $(c, d) \notin t_G$ and $(ce, de) \in t_G$ and $(ec, ed) \in t_G$ for all $e \in G$. We shall distinguish the following two cases.

Case 1: $(c, d) \notin q_G$. Then either $c = b$ or $d = b$. It is enough to assume that $d = b$; then $c \in H$. We have $ec \neq eb$ for every $e \in G$, since $(c, b) \notin q_G$. But $(ec, eb) \in t_G$ and so $eb \in \{a_1, a_2\}$. This implies $Im(f) \subseteq \{1, 2\}$ and the assertion follows from 7.4.

Case 2: $(c, d) \in q_G$. Then $(c, d) \notin p_G$. We have $c, d \in H$; for every $e \in G$, $ce \neq de$ and so $ce, de \in \{a_1, a_2\}$. From this it follows that $c, d \in \{a_1, a_2, a_3\}$. Since $(c, d) \notin p_G$, we can assume that $d = a_3$. Then $c \in \{a_1, a_2\}$, $a_{f(1)} = cb \in \{a_1, a_2\}$ and $a_{f(3)} = db \in \{a_1, a_2\}$. According to 7.4, $f(1) = 2$ and $f(3) = 1$.

7.6. Lemma. $n \leqq 3$.

Proof. Suppose $n \geqq 4$. Using 7.5, it is easy to see that the equivalence $r = (\{a_1, a_2, a_3\} \times \{a_1, a_2, a_3\}) \cup \mathrm{id}_G$ is a congruence of G. The factor G/r is a non-trivial torsion groupoid and hence there are elements $c, d \in G$ with $(c, d) \notin r$ and $(ce, de) \in r$ and $(ec, ed) \in r$ for all $e \in G$. We shall distinguish the following cases:

Case 1: $c \in H$ and $d = b$. Then $ec \neq eb$ and $ec, eb \in \{a_1, a_2, a_3\}$ for all $e \in G$. In particular, $a_4 b \in \{a_1, a_2, a_3\}$, $f(4) \in \{1, 2, 3\}$, a contradiction with 7.4 and 7.5.

Case 2: $c \in H$, $d = a_i$, $i \geqq 5$. Then $(ca_1, a_i a_1) \in r$, $(ca_1, a_{i-1}) \in r$, $ca_1 = a_{i-1} = a_i a_1$, $(c, a_i) \in p_G$, $c = a_i = d$, a contradiction.

Case 3: $c \in \{a_1, a_2, a_3\}$, $d = a_4$. We have $(cb, a_4 b) \in r$, $(cb, a_{f(4)}) \in r$. But $cb \in \{a_1, a_2\}$; hence $f(4) \in \{1, 2, 3\}$, a contradiction with 7.4 and 7.5.

It is evident that at least one of these three or the three symmetric cases must take place. However, we got a contradiction in every one of these cases.

Denote by k, l the elements of $\{1, 2, 3\}$ such that $ba = a_k$ for every $a \in H$ and $bb = a_l$.

7.7. Lemma. *We have* $k, l \in \{1, 2\}$.

Proof. We can assume that $n = 3$. Since G is regular and $(a, b) \notin q_G$ for each $a \in H$, $k \neq l$ and we have either $k \in \{1, 2\}$ or $l \in \{1, 2\}$. First, let $k = 1$. Then $ba_1 = a_1 = a_1 a_1$, $(b, a_1) \in p_G$, $bb = a_1 b = a_2$, $l = 2$. Similarly, if $k = 2$, then $ba_3 = a_2 = a_3 a_3$, $(b, a_3) \in p_G$, $bb = a_3 b = a_1$, $l = 1$. Now, let $l = 1$. Then $bb = a_1 = a_3 b$, $(b, a_3) \in p_G$, $ba = a_3 a = a_2$ for all $a \in H$ and $k = 2$. Similarly, if $l = 2$, then $bb = a_2 = a_1 b$, $(b, a_1) \in p_G$, $ba = a_1 a = a_1$, $k = 1$.

This completes the proof of 7.1.

8. THE SECOND AUXILIARY RESULT

The aim of this section is to prove the following lemma.

8.1. Lemma. *Let G be a regular subdirectly irreducible torsion groupoid, let $n \geqq 2$ be an integer and H a subgroupoid of G such that $GG \subseteq H$ and $H \simeq B(n)$. Further, assume that* Card $(G/p_G) = 2$. *Then G/t_G is a semigroup with zero multiplication.*

In order to prove this lemma, it is enough to assume $H = B(n)$. Since Card $(G/p_G) = 2$ and G is regular, $p_H = p_G \mid H$, Card $(H/p_H) \leqq 2$ and $n \leqq 3$. On the other hand, if $n \leqq 2$, then G/t_G is obviously a semigroup with zero multiplication. Let $n = 3$. Denote by A the block of p_G with $a_1, a_2 \in A$; let B be the remaining block of p_G.

8.2. Lemma. *Let $a \in A$. Then $Ga \subseteq \{a_1, a_2\}$.*

Proof. We have $Aa = \{a_1 a\}$ and $Ba = \{a_3 a\}$. Hence it suffices to show that $a_1 a \in \{a_1, a_2\}$ and $a_3 a \in \{a_1, a_2\}$. Put $K = H \cup \{a\}$. Then K is a subgroupoid of G. It is enough to consider the case $a \notin H$. First, let $a_1 a = a_3$. If $a_3 a = a_2$ then $a_3 a = a_3 a_1$, $(a, a_1) \in q_G$, $(a, a_1) \in t_G$, $a \in H$, a contradiction. If $a_3 a = a_3$ then $a_3 a = a_2 a$, $(a_3, a_2) \in p_G$, a contradiction. Thus $a_3 a = a_1$ and K has the following multiplication table:

	a_1	a_2	a_3	a
a_1	a_1	a_1	a_1	a_3
a_2	a_1	a_1	a_1	a_3
a_3	a_2	a_2	a_2	a_1
a	a_1	a_1	a_1	a_3

However, this groupoid is not torsion, a contradiction. We have proved that $a_1 a \in \{a_1, a_2\}$. If $a_1 a = a_1$ then $a_1 a = a_1 a_1$, $(a, a_1) \in t_G$, $a \in H$, a contradiction. Therefore $a_1 a = a_2$. If $a_3 a = a_3$ then K has the following multiplication table:

	a_1	a_2	a_3	a
a_1	a_1	a_1	a_1	a_2
a_2	a_1	a_1	a_1	a_2
a_3	a_2	a_2	a_2	a_3
a	a_1	a_1	a_1	a_2

Again, this groupoid is not torsion, a contradiction. Thus $a_3 a \in \{a_1, a_2\}$. (In fact, we have $a_3 a = a_1$.)

8.3. Lemma. *Let $b \in B$. Then $Gb \subseteq \{a_1, a_2\}$.*

Proof is similar to that of 8.2.
It follows from 8.2 and 8.3 that $GG \subseteq \{a_1, a_2\}$. This completes the proof of 8.1.

9. THE THIRD AUXILIARY RESULT

The aim of this section is to prove the following lemma.

9.1. Lemma. *Let G be a regular subdirectly irreducible torsion groupoid such that $l(G) \leqq 3$; let H be a subgroupoid of G such that $GG \subseteq H$ and $H \simeq A(4)$. Further, assume that* Card $(G/p_G) = 2$. *Then G/t_G is a semigroup with zero multiplication.*

The proof of this lemma will be divided into the following four lemmas. Let $H = A(4) = \{a, b, c, d\}$.

9.2. Lemma. *Let $e \in G$. Then either $(a, e) \in p_G$ or $(c, e) \in p_G$.*

Proof. It follows from $(a, c) \notin p_G$ and Card $(G/p_G) = 2$.

9.3. Lemma. *Let $e \in G \setminus H$. Then $ee \in \{a, b\}$.*

Proof. Suppose, on the contrary, that either $ee = c$ or $ee = d$. Put $K = H \cup \{e\}$, so that K is a regular subdirectly irreducible torsion groupoid of length $\leqq 3$. Let us distinguish the following four cases.

Case 1: $(a, e) \in p_G$ and $ee = c$. Taking into account the regularity of K, we see that K has the following multiplication table:

	a	b	c	d	e
a	a	a	a	b	c
b	a	a	a	b	c
c	b	b	b	a	d
d	b	b	b	a	d
e	a	a	a	b	c

However, the length of this groupoid is equal to 4, a contradiction.

Case 2: $(a, e) \in p_G$ and $ee = d$. Then we can derive a contradiction similarly.

Case 3: $(c, e) \in p_G$ and $ee = c$. Then K has the following multiplication table:

	a	b	c	d	e
a	a	a	a	b	d
b	a	a	a	b	d
c	b	b	b	a	c
d	b	b	b	a	c
e	b	b	b	a	c

Again, $l(K) = 4$, a contradiction.

Case 4: $(c, e) \in p_G$ and $ee = d$. Then we can derive a contradiction similarly.

9.4. Lemma. *Let $e \in G \setminus H$. Then $ee = b$ and $(e, a) \in p_G$.*

Proof. By 9.2 and 9.3, it is enough to derive a contradiction in each of the following three cases:

Case 1: $ee = a$ and $(a, e) \in p_G$. Then $ee = a = aa = ea$, $(a, e) \in q_G$, $(a, e) \in t_G$, $e \in H$, a contradiction.

Case 2: $ee = a$ and $(c, e) \in p_G$. Then $(d, e) \in p_G$, $ee = a = cd = ed$, $(d, e) \in q_G$, $(d, e) \in t_G$, $d = e$, a contradiction.

Case 3: $ee = b$ and $(c, e) \in p_G$. Then $ee = b = cc = ec$, $(c, e) \in q_G$, $(c, e) \in t_G$, a contradiction.

9.5. Lemma. *G/t_G is a semigroup with zero multiplication.*

Proof. By 9.4, $(a, e) \in p_G$ and $ee = b$ for every element $e \in G \setminus H$. Hence $ee = b = ad = ed$, $(e, d) \in q_G$. We have $ae = be = ee = b$, $ce = de = dd = a$. If $e, f \in G \setminus H$ then $ef = af = ff = b$. We have proved that $GG \subseteq \{a, b\}$.

10. THE FOURTH AUXILIARY RESULT

The aim of this section is to prove the following lemma.

10.1. Lemma. *Let G be a regular subdirectly irreducible torsion groupoid such that $l(G) \leqq 3$; let H be a subgroupoid of G such that $GG \subseteq H$ and $H \simeq A(4)$. Further, assume that* Card $(G/q_G) = 2$. *Then G/t_G is a semigroup with zero multiplication.*

The proof of this lemma will be divided into the following four lemmas. Let $H = A(4) = \{a, b, c, d\}$.

10.2. Lemma. *Let $e \in G$. Then either $(a, e) \in q_G$ or $(d, e) \in q_G$.*

Proof. It follows from $(a, d) \notin q_G$ and Card $(G/q_G) = 2$.

10.3. Lemma. *Let $e \in G \setminus H$. Then $ee \in \{a, b\}$.*

Proof. Suppose, on the contrary, that either $ee = c$ or $ee = d$. Put $K = \{a, b, c, d, e\}$, so that K is a subgroupoid of G. Let us distinguish the following four cases.

Case 1: $(a, e) \in q_G$ and $ee = c$. Then K has the following multiplication table:

	a	b	c	d	e
a	a	a	a	b	a
b	a	a	a	b	a
c	b	b	b	a	b
d	b	b	b	a	b
e	c	c	c	d	c

However, this groupoid is not a torsion groupoid, a contradiction.

Case 2: $(a, e) \in q_G$ and $ee = d$. Then we can derive a contradiction similarly.

Case 3: $(d, e) \in q_G$ and $ee = c$. Then K has the following multiplication table:

	a	b	c	d	e
a	a	a	a	b	b
b	a	a	a	b	b
c	b	b	b	a	a
d	b	b	b	a	a
e	d	d	d	c	c

However, this groupoid is not a torsion groupoid, a contradiction.

Case 4: $(d, e) \in q_G$ and $ee = d$. Then we can derive a contradiction similarly.

10.4. Lemma. *Let $e \in G \setminus H$. Then $ee = b$ and either $(a, e) \in q_G$, $(c, e) \in p_G$ or $(d, e) \in q_G$, $(a, e) \in p_G$.*

Proof. By 10.2 and 10.3 it is enough to consider the following cases.

Case 1: $(a, e) \in q_G$ and $ee = a$. Then $ee = a = aa = ae$, $(a, e) \in p_G$, $(a, e) \in t_G$, $e \in H$, a contradiction.

Case 2: $(a, e) \in q_G$ and $ee = b$. Then $ee = b = ca = ce$, $(c, e) \in p_G$.

Case 3: $(d, e) \in q_G$ and $ee = a$. Then $ee = a = dd = de$, $(d, e) \in p_G$, $(d, e) \in t_G$, $e = d$, a contradiction.

Case 4: $(d, e) \in q_G$ and $ee = b$. Then $ee = b = ad = ae$, $(a, e) \in p_G$.

10.5. Lemma. *G/t_G is a semigroup with zero multiplication.*

Proof. Let $e \in G \setminus H$. By 10.4, $ee = b$. Further, either $(a, e) \in q_G$, $(c, e) \in p_G$ or $(d, e) \in q_G$, $(a, e) \in p_G$. Then either $ae = be = aa = a$, $ce = de = ca = b$ or $ae = be = ad = b$, $ce = de = cd = a$. Similarly, either $ea = eb = ec = cc = b$, $ed = cd = a$ or $ea = eb = ec = aa = a$, $ed = ad = b$. Finally, let $e, f \in G \setminus H$. Then either $ef = cf \in \{a, b\}$ or $ef = af \in \{a, b\}$. We have proved $GG \subseteq \{a, b\}$.

11. REGULAR SUBDIRECTLY IRREDUCIBLE TORSION GROUPOIDS

11.1. Lemma. *Let G be a regular subdirectly irreducible torsion groupoid. Then either* Card $(G/p_G) \leqq 2$ *or* Card $(G/q_G) \leqq 2$.

Proof. If G/t_G is trivial then G is a semigroup with zero multiplication and G contains only two elements. Let G/t_G be non-trivial. Let $a, b \in G$ be such that $a \neq b$ and $(a, b) \in t_G$. There are elements $c, d \in G$ with $(c, d) \notin t_G$ and $(ce, de) \in t_G$ and $(ec, ed) \in t_G$ for all $e \in G$. Assume $(c, d) \notin p_G$ (the other case is similar). Then $ce \neq de$ and $ce \in \{a, b\}$ for all $e \in G$. Since G is regular, Card $(G/q_G) \leqq 2$.

11.2. Proposition. *Let G be a regular subdirectly irreducible torsion groupoid of finite length. Then G is finite.*

Proof. We shall proceed by induction on $l(G)$. If $l(G) \leqq 1$ then the situation is clear. Let $l(G) \geqq 2$. Let $a, b \in G$ be such that $a \neq b$ and $a, b \in t_G$. By 4.3, $a, b \in GG$. Hence GG is a regular subdirectly irreducible torsion groupoid. However, $l(GG) < l(G)$ and so GG is finite by the induction assumption. Put $m =$ Card (GG). According to 1.3, Card $(G) \leqq m^2 + 1$.

11.3. Proposition. *Let G be a regular subdirectly irreducible torsion groupoid. Then every non-trivial subgroupoid of G is a regular subdirectly irreducible torsion groupoid.*

Proof is easy.

11.4. Proposition. *Let G be a regular subdirectly irreducible torsion groupoid such that GG is either a left or a right unar. Then G is isomorphic to one of the groupoids* $A(0), \ldots, A(7), B(4), B(5), \ldots, B(\infty), C(4), C(5), \ldots, C(\infty)$.

Proof. We shall assume that GG is a left unar and that G is not asemigroup with zero multiplication. Then GG is a subdirectly irreducible torsion groupoid. First, suppose that GG is finite. By 6.3, $GG \simeq B(n)$ for some $n \geqq 2$. If $\operatorname{Card}(G/q_G) = 1$ then G is a left unar and 6.3 can be applied. If $\operatorname{Card}(G/p_G) = 1$ then G is a right unar and again 6.3 can be applied. Hence we can assume that $\operatorname{Card}(G/p_G) \geqq 2$ and $\operatorname{Card}(G/q_G) \geqq 2$. By 11.1, either $\operatorname{Card}(G/p_G) = 2$ or $\operatorname{Card}(G/q_G) = 2$. By 7.1 and 8.1 we see that G/t_G is a semigroup with zero multiplication. Consequently, $l(G) = 2$ and 5.1 yields the result. Now, let GG be infinite. Then $GG \simeq B(\infty)$ by 6.3. Since G/p_G is infinite, $\operatorname{Card}(G/q_G) \leqq 2$. If $\operatorname{Card}(G/q_G) = 1$ then G is a left unar and 6.3 can be applied. If $\operatorname{Card}(G/q_G) = 2$ then, proceeding similarly as in the proof of 7.1, we obtain a contradiction.

11.5. Proposition. *The groupoids* $A(0), A(1), A(2), A(3), A(4), A(5), A(6), A(7), B(4), C(4)$ *are up to isomorphism the only regular subdirectly irreducible torsion groupoids of length* $\leqq 3$.

Proof. By 5.1 we can restrict ourselves to the case $l(G) = 3$. Then GG is a regular subdirectly irreducible torsion groupoid of length 2. If GG is either a left or a right unar, then 11.4 may be applied. Suppose that GG is neither a left nor a right unar. By 5.1 and 11.1, GG is isomorphic to one of the groupoids $A(1), A(4), A(5), A(6), A(7)$ and either $\operatorname{Card}(G/p_G) = 2$ or $\operatorname{Card}(G/q_G) = 2$. If GG is isomorphic to $A(4)$ then G/t_G is a semigroup with zero multiplication, as follows from 9.1 and 10.1, a contradiction. We can proceed similarly in the remaining cases.

11.6. Proposition. *The groupoids* $A(0), \ldots, A(7), B(4), B(5), \ldots, B(\infty), C(4), C(5), \ldots$ $\ldots, C(\infty)$ *are up to isomorphism the only strongly regular subdirectly irreducible torsion groupoids.*

Proof. Let G be a strongly regular subdirectly irreducible torsion groupoid. The case $l(G) \leqq 3$ is settled by 11.5. Let $l(G) \geqq 4$. For $i = 0, 1, \ldots$ let A_i denote the block of $t_{G,i}$ containing the unique idempotent 0 of G. By 3.17, G is the union of the chain $A_1, A_2, A_3, \ldots$ of regular subdirectly irreducible torsion groupoids and $l(A_3) = 3$. With respect to 11.5 we can assume that A_3 is a left unar (the other case is similar). Suppose that G is not a left unar. Then there is an $n \geqq 4$ which is the least positive integer such that A_n is not a left unar. However, $A_n A_n \subseteq A_{n-1}$ is a left unar, $l(A_n) \geqq 4$, $A_n \simeq B(n + 1)$ by 11.4, A_n is a left unar, a contradiction. We have proved that G is a left unar. The rest is clear.

11.7. Proposition. *Let G be a regular subdirectly irreducible torsion groupoid of length 4. Then* $5 \leqq \operatorname{Card}(G) \leqq 11$.

Proof. Put $H_1 = G/t_G$, $H_2 = G/t_{G,2}$, $H_3 = G/t_{G,3}$. Then $l(H_1) = 3$, $l(H_2) = 2$ and $l(H_3) = 1$. Hence $\mathrm{Card}(H_3) \geqq 2$, $\mathrm{Card}(H_2) \geqq 3$, $\mathrm{Card}(H_1) \geqq 4$ and $\mathrm{Card}(G) \geqq 5$. Denote by A the block of $t_{G,3}$ containing the unique idempotent of G. Then $l(A) \leqq 3$ and $\mathrm{Card}(A) \leqq 5$ by 11.5. Since $GG \subseteq A$, $\mathrm{Card}(GG) \leqq 5$ and $\mathrm{Card}(G/p_G) \leqq 5$, $\mathrm{Card}(G/q_G) \leqq 5$ by 1.3. On the other hand, either $\mathrm{Card}(G/p_G) \leqq 2$ or $\mathrm{Card}(G/q_G) \leqq 2$ by 11.1. Thus $\mathrm{Card}(G/t_G) \leqq 10$ and $\mathrm{Card}(G) \leqq 11$.

11.8. Example. Consider the groupoid $G = \{a, b, c, d, e\}$ with the following multiplication table:

	a	b	c	d	e
a	a	a	a	b	c
b	a	a	a	b	c
c	b	b	b	a	d
d	b	b	b	a	d
e	a	a	a	b	c

It is easy to check that G is a regular subdirectly irreducible torsion groupoid of length 4. Moreover, $l(GG) = 2$ and G is not strongly regular.

11.9. Proposition. *$A(0)$ and $A(1)$ are up to isomorphism the only commutative regular subdirectly irreducible torsion groupoids.*

Proof. Let G be a commutative regular subdirectly irreducible torsion groupoid. By 11.1, $\mathrm{Card}(G/t_G) \leqq 2$. Hence $l(G) \leqq 2$ and 5.1 can be applied.

Problem. Find all regular subdirectly irreducible torsion groupoids of length $\leqq 5$.

12. COMMUTATIVE TORSION GROUPOIDS WHOSE EVERY FACTOR IS REGULAR

Let A, B be two non-empty disjoint sets and a, b two different elements of A. Then we define a groupoid $U_{A,B,a,b}$ as follows: $U_{A,B,a,b} = A \cup B$; if $x, y \in A$ and $u, v \in B$ then $xy = uv = a$ and $xu = ux = b$.

12.1. Proposition. *Let A, B be two non-empty disjoint sets and $a, b \in A$, $a \neq b$. Then $U_{A,B,a,b}$ is a commutative torsion groupoid of length 2 and every factor of $U_{A,B,a,b}$ is regular.*

Proof. Put $G = U_{A,B,a,b}$. Evidently, $t_G = (A \times A) \cup (B \times B)$ and $G/t_G \simeq A(0)$. Hence G is a torsion groupoid of length 2; evidently, G is commutative. It remains to prove that G/r is a regular groupoid for any congruence r or G. Let r be a congruence of G. If $(a, b) \in r$ then G/r is a semigroup with zero multiplication, hence regular. Let $(a, b) \notin r$ and let $x, y, z \in G$ be such that $(xz, yz) \in r$. Since $xz, yz \in \{a, b\}$, either $xz = yz = a$ or $xz = yz = b$. In the first case, either $x, y, z \in A$ or $x, y, z \in B$. In the second case, either $x, y \in A$, $z \in B$ or $x, y \in B$, $z \in A$. In both cases, $(x, y) \in t_G$, so that $xu = yu$ and thus $(xu, yu) \in r$ for all $u \in G$.

12.2. Proposition. *The following two conditions are equivalent for any groupoid G:*

(1) *G is a commutative torsion groupoid and every factor of G is regular;*

(2) *either G is a semigroup with zero multiplication or there exist two non-empty disjoint sets A, B and elements $a, b \in A$ $(a \neq b)$ with $G = U_{A,B,a,b}$.*

Proof. By 12.1 it is enough to prove that (1) implies (2). Let G be a commutative torsion groupoid such that every factor of G is regular; assume that G is not a semigroup with zero multiplication. Then $\text{Card}(GG) \geqq 2$. By 11.9, every subdirectly irreducible factor of G is isomorphic to one of the groupoids $A(0)$, $A(1)$. Since every groupoid is isomorphic to a subdirect product of its subdirectly irreducible factors, we get $l(G) = 2$ and $xx = 0$ for all $x \in G$, where 0 is the unique idempotent of G. Denote by A the block of t_G containing 0, so that $GG \subseteq A$. Define a binary relation r on G as follows: $(x, y) \in r$ iff either $x = y$ or $x, y \in A \setminus \{0\}$ or $(x, y) \in t_G \setminus (A \times A)$. Evidently, r is a congruence of G and $r \subseteq t_G$. We are going to show that G/r is subdirectly irreducible. Let $(C, D) \in t_{G/r}$ and $C \neq D$. There are elements $c \in C$, $d \in D$; we have $(c, d) \notin r$ and $(cx, dx) \in r$ for all $x \in G$. Then $(cd, dd) \in r$, i.e. $(cd, 0) \in r$, $cd = 0$, $cc = cd$. Since G is regular, $(c, d) \in t_G$ and we get either $C = \{0\}$, $D = A \setminus \{0\}$ or $C = A \setminus \{0\}$, $D = \{0\}$. On the other hand, we have $(\{0\}, A \setminus \{0\}) \in t_{G/r}$ and G/r is subdirectly irreducible by 4.2. By 11.9, G/r contains at most three elements. From this it follows that G/t_G contains at most two elements. Since $l(G) = 2$, $\text{Card}(G/t_G) = 2$. Denote by B the block of t_G different from A. There are elements $a, b \in GG \subseteq A$ such that $xu = b$ and $uv = a$ for all $x \in A$ and $u, v \in B$. Then $a = uu = 0$ and $b \neq 0$, since G is regular. Finally, $xy = x0 = 00 = 0$ for all $x, y \in A$.

12.3. Corollary. *Let G be a commutative torsion groupoid such that every factor of G is regular. Then:*

(1) *Either $l(G) \leqq 1$ and GG, G/t_G are both trivial or $l(G) = 2$ and GG, G/t_G are isomorphic to $A(0)$.*

(2) *Every factor of any subgroupoid of G is regular.*

(3) *If $l(G) = 2$ then there exists a congruence r of $G \times G$ such that $(G \times G)/r$ is not regular.*

Problem. Describe all torsion groupoids G such that every factor of G is regular.

References

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