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# SIMULTANEOUS INTEGRABILITY OF AN ALMOST COMPLEX AND AN ALMOST TANGENT STRUCTURE

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**0. Introduction.** In this paper we study the integrability of a $G$-structure defined on a manifold $M$ by a couple $(J, T)$ consisting of an almost complex structure $J$ and an almost tangent structure $T$. In other words we consider on $M$ two tensor fields $J$ and $T$ of type $(1,1)$ satisfying $J^2 = -I$, $T^2 = 0$ and $\ker T = \operatorname{im} T$. We distinguish three cases described by the relations

(1) $JT = TJ$,

(2) $JT = -TJ$,

(3) $JT + TJ = I$.

These possiblities arise quite naturally as the only ones when we require that the subalgebra of the associative algebra of tensor fields of type $(1,1)$ generated by $J$ and $T$ has dimension $\leqq 4$. The main results are formulated as theorems 1,2 and 3.

For the sake of simplicity we assume all structures to be of class $C^\omega$.

**1. Algebraic preliminaries.** Let us consider a real associative algebra $A$ with the unit element (which we denote by $e$). We are going to describe all such algebras with two generators $j$, $t \neq 0$ satisfying

$$j^2 = -e\,, \quad t^2 = 0 \tag{1}$$

under the restriction $\dim A \leqq 4$.

It can be easily checked that $e, j, t$ are linearly independent. As a consequence we have $\dim A = 3$ or $\dim A = 4$. Of course, the five elements $jt$, $tj, j, t, e$ must be dependent. We shall need the following

**Lemma 1.** *If there is*

$$\alpha jt + \beta tj + \gamma j + \delta t + \varepsilon e = 0 \tag{2}$$

*then the combination is either trivial or* $\alpha\beta \neq 0$.

Proof. Let us suppose the above combination to be non-trivial. Then obviously

$\alpha^2 + \beta^2 > 0$. Let us suppose that $\alpha \neq 0$, $\beta = 0$. Then

$$jt = \gamma_1 j + \delta_1 t + \varepsilon_1 e \tag{3}$$

and multiplying this equality by $t$ from the right we get

$$\gamma_1 jt + \varepsilon_1 t = 0 .$$

Substituting (3) into this last formula we have

$$\gamma_1^2 j + (\gamma_1 \delta_1 + \varepsilon_1) t + \gamma_1 \varepsilon_1 e = 0$$

which implies $\gamma_1 = \varepsilon_1 = 0$. Thus (3) has the form

$$jt = \delta_1 t .$$

Multiplication of this identity by $j$ from the left gives

$$-t = \delta_1 jt , \quad (1 + \delta_1^2) t = 0$$

which implies $t = 0$ and this is a contradiction. Similarly we proceed in the case $\alpha = 0$, $\beta \neq 0$. Our lemma is proved.

Now it is obvious that there exist $a$, $b$, $c$, $d$ such that

$$jt = at_j + bj + ct + de . \tag{4}$$

Multiplying (4) by $t$ from the left and from the right respectively we obtain

$$tjt = btj + dt , \quad 0 = atjt + bjt + dt .$$

These two equations combined with (4) give

$$2abtj + b^2 j + (ad + bc + d) r + bde = 0$$

which by virtue of Lemma 1 shows that there are two possibilities:

(i) $b = 0$, $d = \quad 0$,

(ii) $b = 0$, $a = -1$.

In the case (i) the equation (4) multiplied by $j$ from the left and from the right respectively gives

$$-t = ajtj + cjt , \quad jtj = -at + ctj .$$

From these two equations using (4) we get

$$2actj + (c^2 - a^2 + 1) t = 0 .$$

This proves that either $c = 0$ and $a = 1$ or $c = 0$ and $a = -1$. Thus in the case (i) the generators $j$, $t$ satisfy either $jt = tj$ or $jt = -tj$.

In the case (ii) we multiply again the equation (4) by $j$ from the left and from the right thus obtaining respectively

$$-t + jtj = cjt + dj , \quad jtj - t = ctj + dj .$$

Hence we have

$$c(jt - tj) = 0\,.$$

Because $jt = tj$ in this case yields a contradiction (the proof of which we leave to the reader), we have $c = 0$, and thus $jt + tj = de$. If $d \neq 0$ it is obvious that there is no substantial difference between the case $jt + tj = de$ and the case $jt + tj = e$. In the sense of this remark we may state the following

**Proposition 1.** *There are exactly three real associative algebras with the unit element having two generators $j, t \neq 0$ such that $j^2 = -e$, $t^2 = 0$ and having dimension $\leqq 4$. These three algebras are completely described by the following identites:*

(i) $jt = tj$,

(ii) $jt = -tj$,

(iii) $jt + tj = e$.

*The corresponding algebras we denote by the symbols $\mathscr{A}_1, \mathscr{A}_2, \mathscr{A}_3$, respectively.*

**2. The commutative case.** Here we shall study the structure which corresponds to the algebra $\mathscr{A}_1$ from Proposition 1. Let $M$ be a manifold provided with a couple $J, T$ of tensor fields of type $(1, 1)$ satisfying $J^2 = -I$, $T^2 = 0$, $JT = TJ$, and $\ker T = \operatorname{im} T$. This structure will be called a *$(J, T)_1$-structure*. We write $m = \dim M$. We shall denote by $D$ the distribution $\ker T = \operatorname{im} T$. It can be immediately seen that there is $JD \subseteq D$, $TD \subseteq D$. This shows that $m \equiv 0 \bmod 4$, so that we can write $m = 4n$.

**Proposition 2.** *There is a 1-1 correspondence between the set of $(J, T)_1$-structures on $M$ and the set of $G_1$-structures on $M$, where $G_1$ is the Lie group consisting of all regular matrices of the form*

$$\begin{pmatrix} A_1^1 & A_1^2 & 0 & 0 \\ -A_1^2 & A_1^1 & 0 & 0 \\ A_3^1 & A_3^2 & A_1^1 & A_1^2 \\ -A_3^2 & A_3^1 & -A_1^2 & A_1^1 \end{pmatrix}$$

*with $A_1^1, A_1^2, A_3^1, A_3^2$ being $(n \times n)$-matrices.*

Proof. Let us consider a manifold $M$ provided with a $(J, T)_1$-structure, and let us take any point $x \in M$. The tensor $J$ defines a complex vector space structure on the tangent space $T_x(M)$. Obviously $D_x$ is a complex subspace of $T_x(M)$. Let us take any complex basis $v_1, \ldots, v_n$ of $T_x(M)$ over $D_x$. It is easy to see that $v_1, \ldots, v_n, Jv_1, \ldots \ldots, Jv_n, Tv_1, \ldots, Tv_n, TJv_1, \ldots, TJv_n$ is a real basis of $T_x(M)$. The reader can easily verify that the set of all bases of this type all over $M$ is a $G_1$-structure over $M$.

Conversely, let a $G_1$-structure on $M$ be given. We take any basis $v_1, \ldots, v_{4n}$ of $T_x(M)$ belonging to this $G_1$-structure and define

$$J_x v_i = \sum_{k=1}^{4n} J_i^k v_k, \quad T_x v_i = \sum_{k=1}^{4n} T_i^k v_k,$$

where $(J_i^k)$ and $(T_i^k)$ are the matrices

$$\begin{pmatrix} 0 & -I & 0 & 0 \\ I & 0 & 0 & 0 \\ 0 & 0 & 0 & -I \\ 0 & 0 & I & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \end{pmatrix},$$

respectively. We denote here by $I$ the unit $(n \times n)$-matrix. It is not difficult to show that the definition of $J$ and $T$ does not depend on the choice of a basis from the $G_1$-structure. All the other details of the proof we leave to the reader.

For the formulation of the main theorem of this section we shall need a tensor field $\{F, G\}$ of type $(1, 2)$ associated with any couple $F, G$ of tensor fields of type $(1, 1)$ satisfying $FG = GF$. This tensor field is defined by the formula

$$\{F, G\}(X, Y) = [FX, GY] + FG[X, Y] - F[X, GY] - G[FX, Y].$$

We shall prove

**Theorem 1.** *A $(J, T)_1$-structure is integrable if and only if*

$$\{J, J\} = \{J, T\} = \{T, T\} = 0.$$

Proof. $J$ is an almost complex structure on the manifold $M$. But because $\{J, J\} = 0$ the almost complex structure $J$ is integrable, i.e. $M$ is a complex manifold and we can use complex charts. Now it is not difficult to see that the $(J, T)_1$-structure is integrable if and only if to any point $a \in M$ there exists its open neighborhood with a complex chart $(z^1, \ldots, z^{2n})$ on it such that

$$T^C \frac{\partial}{\partial z^i} = \frac{\partial}{\partial z^{n+i}}, \quad T^C \frac{\partial}{\partial z^{n+i}} = 0, \quad i = 1, 2, \ldots, n,$$

where $T^C$ denotes the complexification of $T$.

We shall show first that $T^C$ is a holomorphic tensor field. For this purpose it suffices to prove that for any holomorphic vector field $Z$ the vector field $T^C Z$ is again holomorphic. Taking a holomorphic vector field $Z$ we can write it in the form $Z = \frac{1}{2}(X - iJX)$, where $X$ is a real vector field satisfying $\mathscr{L}_X J = 0$. The symbol $\mathscr{L}_X$ denotes the Lie derivative with respect to $X$. We have $T^C Z = \frac{1}{2}(TX - iJTX)$. By virtue of the assumption $\{J, T\} = 0$ we get

$$(\mathscr{L}_{TX} J)(Y) = (\mathscr{L}_{TX} J)(Y) - T(\mathscr{L}_X J)(Y) =$$
$$= [TX, JY] - J[TX, Y] - T[X, JY] + TJ[X, Y] = -\{J, T\}(Y, X) = 0$$

which shows that $T^C Z$ is a holomorphic vector field.

The distribution $D$ is integrable. This can be easily seen because taking two vector fields $X$, $Y \in D = \operatorname{im} T$ we can write $X = TX$, $Y = TY_1$ and by virtue of $\{T, T\} = 0$ we get

$$[X, Y] = [TX_1, TY_1] = T([TX_1, Y_1] + [X_1, TY_1])$$

which belongs to $D = \operatorname{im} T$. Because $D$ is invariant under $J$, it is possible for any point $a \in M$ to find its open neighborhood $\mathscr{U}_1$ with a complex chart $(z^1, \ldots, z^{2n})$ on it such that the equations

$$dz^1 = \ldots = dz^n = 0$$

define on $\mathscr{U}_1$ the complexification $D^C$ of $D$.

Obviously there are holomorphic functions $A_i^j$; $i, j = 1, \ldots, n$ such that

$$T^C\left(A_i^j \frac{\partial}{\partial z^j}\right) = \frac{\partial}{\partial z^{n+i}}, \quad i = 1, \ldots, n .$$

We shall consider the differential system

$$\frac{\partial H^{n+j}}{\partial z^{n+i}} = A_i^j, \quad i, j = 1, \ldots, n .$$

There are holomorphic functions $H^{n+j}(z^1, \ldots, z^{2n})$, $j = 1, \ldots, n$ defined on a neighborhood $\mathscr{U}_2 \subseteq \mathscr{U}_1$ of $a$ which form a solution of this system if and only if

$$\frac{\partial A_i^j}{\partial z^{n+k}} = \frac{\partial A_k^j}{\partial z^{n+i}}, \quad i, j, k = 1, \ldots, n . \tag{5}$$

But this last condition is a consequence of the assumption $\{T, T\} = 0$. We have namely

$$0 = \{T^C, T^C\}\left(A_i^j \frac{\partial}{\partial z^j}, \ A_k^j \frac{\partial}{\partial z^j}\right) = \left[\frac{\partial}{\partial z^{n+i}}, \ \frac{\partial}{\partial z^{n+k}}\right] - T^C\left[A_i^j \frac{\partial}{\partial z^j}, \ \frac{\partial}{\partial z^{n+k}}\right] -$$

$$- T^C\left[\frac{\partial}{\partial z^{n+j}}, \ A_k^j \frac{\partial}{\partial z^j}\right] = \left(\frac{\partial A_i^j}{\partial z^{n+k}} - \frac{\partial A_k^j}{\partial z^{n+i}}\right) T^C \frac{\partial}{\partial z^\gamma},$$

which implies (5). Thus on an open neighborhood $\mathscr{U}_3 \subseteq \mathscr{U}_2$ of $a$ we can introduce a complex chart $('z^1, \ldots, 'z^{2n})$ by

$$'z^i = z^i, \quad 'z^{n+i} = H^{n+i}(z^1, \ldots, z^{2n}), \quad i = 1, \ldots, n .$$

With respect to this new chart we have by virtue of (5)

$$A_i^j T^C \frac{\partial}{\partial' z^j} = A_i^j T^C \frac{\partial}{\partial z^j} = \frac{\partial}{\partial z^{n+i}} = \frac{\partial H^{n+j}}{\partial z^{n+i}} \frac{\partial}{\partial' z^{n+j}} = A_i^j \frac{\partial}{\partial' z^{n+j}}$$

which implies

$$T^C \frac{\partial}{\partial' z^i} = \frac{\partial}{\partial' z^{n+i}} .$$

The relation

$$T^C \frac{\partial}{\partial' z^{n+i}} = 0$$

is obvious. The theorem is proved.

**3. The anticommutative case.** In this section we consider a manifold $M$ provided with a couple $J$, $T$ of tensor fields of type $(1, 1)$ satisfying $J^2 = -I$, $T^2 = 0$, $JT = -JT$ and $\ker T = \operatorname{im} T$. We shall call such a structure a $(J, T)_2$*-structure*. We denote $m = \dim M$. It is easy to see that again $m \equiv 0 \bmod 4$ so that we can write $m = 4n$. As in the preceding section we denote by $D$ the distribution $\ker T = \operatorname{im} T$. Obviously $JD \subseteq D$, $TD \subseteq D$ holds again.

**Proposition 3.** *There is a* 1-1 *correspondence between the set of* $(J, T)_2$*-structures on* $M$ *and the set of* $G_2$*-structures on* $M$, *where* $G_2$ *is the Lie group consisting of all regular matrices of the form*

$$\begin{pmatrix} A_1^1 & A_1^2 & 0 & 0 \\ A_1^2 & A_1^1 & 0 & 0 \\ A_3^1 & A_3^2 & A_1^1 & -A_1^2 \\ -A_3^2 & A_3^1 & A_1^2 & A_1^1 \end{pmatrix}$$

*with* $A_1^1, A_3^1, A_1^2, A_3^2$ *being* $(n \times m)$*-matrices.*

Proof. Let $M$ be a manifold provided with a $(J, T)_2$-structure. We construct a $G_2$-structure on $M$ in the following way. On any tangent space $T_x(M)$ the tensor $J$ induces a complex vector space structure. $D_x$ is obviously a subspace of this complex vector space. Let $v_1, \ldots, v_n$ be any basis of the complex space $T_x(M)$ over the subspace $D_x$. Then the vectors $v_1, \ldots, v_n, Jv_1, \ldots, Jv_n, Tv_1, \ldots, Tv_n, JTv_1, \ldots, JTv_n$ form a real basis of $T_x(M)$ considered as a real vector space. It is easy to check that all bases of this form at all points of $M$ constitute a $G_2$-structure on $M$.

Conversely, let a $G_2$-structure be given on a manifold $M$. Let us take any basis $v_1, \ldots, v_{4n}$ of $T_x(M)$ belonging to the $G_2$-structure. We define $J_x$ and $T_x$ by the formulas

$$J_x v_i = \sum_{k=1}^{4n} J_i^k v_k\,, \quad T_x v_i = \sum_{k=1}^{4n} T_i^k v_k\,, \quad i = 1, \ldots, 4n\,,$$

where $(J_i^k)$ and $(T_i^k)$ are $(4n \times 4n)$-matrices

(6) $$\begin{pmatrix} 0 & -I & 0 & 0 \\ I & 0 & 0 & 0 \\ 0 & 0 & 0 & -I \\ 0 & 0 & I & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ I & 0 & 0 & 0 \\ 0 & -I & 0 & 0 \end{pmatrix},$$

respectively. By $I$ we denote here the unit $(n \times n)$-matrix. It is easy to see that the definition of $J_x$ and $T_x$ does not depend on the choice of a basis from the $G_2$-structure. All the other details of the proof we leave again to the reader.

Let us recall now that for any two tensor fields $F, G$ of type $(1, 1)$ we can define the Nijenhuis tensor field $[F, G]$ (which is a tensor field of type $(1, 2)$) by the formula

$$[F, G](X, Y) = [FX, GY] + FG[X, Y] - F[X, GY] - G[FX, Y] + \\ + [GX, FY] + GF[X, Y] - G[X, FY] - F[GX, Y].$$

These Nijenhuis tensor fields, or briefly Nijenhuis tensors, will appear in the following proposition. Let us remark that $[F, F] = 2\{F, F\}$. For our special tensors $J$ and $T$ we introduce one more tensor of type $(1, 2)$ by the formula

$$(J, T)(X, Y) = T(J[JX, JTY] + [X, JTY] + J[X, TY] - [JX, TY]).$$

Now we may state the following proposition, which provides us with necessary tools for the proof of the main theorem of this section.

**Proposition 4.** *Let $M$ be a manifold with a $(J, T)_4$-structure such that*

$$[J, J] = [J, T] = [T, T] = (J, T) = 0.$$

*Then there exists on $M$ a torsionless connection $\nabla$ such that $\nabla J = \nabla T = 0$.*

Proof. We recall first a well known result from the theory of almost complex manifolds (see [1], p. 143). Let $M$ be a manifold endowed with an almost complex structure $J$. Taking any torsionless connection $\nabla$ on $M$ we can define a connection $\tilde{\nabla}$ by $\tilde{\nabla}_X Y = \nabla_X Y - Q(X, Y)$ with

$$Q(X, Y) = \tfrac{1}{4}(\nabla_{JY} J)(X) + \tfrac{1}{4}J(\nabla_Y J)(X) + \tfrac{1}{2}J(\nabla_X J)(Y).$$

With respect to this new connection we have $\tilde{\nabla} J = 0$. Moreover, the torsion tensor of $\tilde{\nabla}$ is equal to $\frac{1}{8}[J, J]$.

Now we start to consider a manifold $M$ with a $(J, T)_2$-structure. Our plan is the following one. $T$ is an almost tangent structure on $M$. Because $[T, T] = 0$ this structure is integrable and therefore there exists a torsionless connection $\nabla$ on $M$ such that $\nabla T = 0$ (see [2], pp. 238 and 240). Applying the above described $\sim$-procedure to $\nabla$ and $J$ we get a connection with respect to which $J$ is parallel and which is torsionless because $[J, J] = 0$. Unfortunately $T$ need not be parallel with respect to this new connection. In order to improve this we shall alter the connection $\nabla$. More precisely, instead of $\nabla$ we shall consider the connection $\dot{\nabla}$ defined by $\dot{\nabla}_X Y = \nabla_X Y - P(X, Y)$ with a suitably chosen $P(X, Y)$. The connection $\dot{\nabla}$ should fulfil the following requirements. It should be torsionless, satisfy $\dot{\nabla} T = 0$, and the $\sim$-procedure applied to $\dot{\nabla}$ and $J$ should provide us with a connection, which we denote again by $\tilde{\nabla}$, satisfying $\tilde{\nabla} T = 0$. We remark that the second requirement is not necessary, but we shall work with it because it simplifies the proof. In other words our plan is to find $P(X, Y)$ with the following three properties

(i) $P(X, Y) = P(Y, X)$,

(ii) $P(X, TY) = TP(X, Y)$,

(iii) $\tilde{\nabla} T = 0$,

where $\dot{Q}$ is defined by the same formula as $Q$ with $\dot{\nabla}$ instead of $\nabla$.

A calculation shows us that

$$(7)\qquad (\tilde{\nabla}_X T)(Y) = -\tfrac{1}{2}TP(JX, JY) + \tfrac{1}{2}JTP(JX, Y) + \tfrac{1}{4}T(\nabla_{JY}J)(X) + \\ + \tfrac{1}{4}TJ(\nabla_Y J)(X) - \tfrac{1}{4}(\nabla_{JTY}J)(X) - \tfrac{1}{4}J(\nabla_{TY}J)(X).$$

We introduce the following notation:

$$V(X, Y) = \tfrac{1}{4}T(\nabla_{JY}J)(X) + \tfrac{1}{4}TJ(\nabla_Y J)(X) + \\ + \tfrac{1}{4}(\nabla_{JTY}J)(X) + \tfrac{1}{4}J(\nabla_{TY}J)(X),$$

$$W(X, Y) = TP(X, Y).$$

With this notation the condition (iii) can be expressed in the form

$$(8)\qquad \tfrac{1}{2}JW(X, JY) + \tfrac{1}{2}W(X, Y) = V(X, Y).$$

Let us denote by $\mathscr{S}$ the vector space of tensor fields of type $(1, 2)$ on $M$. On $\mathscr{S}$ we define an endomorphism $\mathscr{a} : \mathscr{S} \to \mathscr{S}$ by

$$(\mathscr{a}S)(X, Y) = \tfrac{1}{2}JS(X, JY) + \tfrac{1}{2}S(X, Y), \quad S \in \mathscr{S}.$$

We easily find $\mathscr{a}^2 = \mathscr{a}$, or equivalently $\mathscr{a}(\mathscr{a} - \mathscr{I}) = 0$, where $\mathscr{I}$ is the identity map. Thus we obtain a decomposition $\mathscr{S} = \mathscr{S}_0 \oplus \mathscr{S}_1$ with $\mathscr{S}_0 = \ker \mathscr{a}$ and $\mathscr{S}_1 = \ker(\mathscr{a} - \mathscr{I})$. The corresponding projectors are $\mathscr{I} - \mathscr{a}$ and $\mathscr{a}$. Using the endomorphism $\mathscr{A}$ we can rewrite (8) into the form

$$(9)\qquad \mathscr{a}W = V.$$

Now it is obvious that we can find $W$ satisfying (9) if and only if $\mathscr{a}V = V$. We have

$$(\mathscr{a}V)(X, Y) = -\tfrac{1}{8}JT(\nabla_Y J)(X) + \tfrac{1}{8}T(\nabla_{JY}J)(X) + \tfrac{1}{8}J(\nabla_{TY}J)(X) - \\ -\tfrac{1}{8}(\nabla_{TJY}J)(X) + \tfrac{1}{8}T(\nabla_{JY}J)(X) + \tfrac{1}{8}TJ(\nabla_Y J)(X) - \\ -\tfrac{1}{8}(\nabla_{JTY}J)(X) + \tfrac{1}{8}J(\nabla_{TY}J)(X) = V(X, Y).$$

This shows that (9) has a solution. This solution is obviously $W = V$, and any other solution has the form $V - N$, where $N$ is a tensor field of type $(1, 2)$ satisfying $\mathscr{a}N = 0$, or equivalently $N(X, JY) = JN(X, Y)$. Let us remark that we are not interested in any solution $W$ of (9). Our final goal is to find $P$ which by (i) is to be symmetric.

Then, of course, $W = TP$ must be also symmetric. Therefore we shall now try to find $N$ such that

(i)′ $N(X, JY) = JN(X, Y)$,

(ii)′ $V(X, Y) - N(X, Y) = V(Y, X) - N(Y, X)$.

A calculation shows that

$$V(X, Y) - V(Y, X) = -\tfrac{1}{2}T(\nabla_{JX}J)(Y) + \tfrac{1}{2}T(\nabla_{JY}J)(X) - \\ - \tfrac{1}{4}[J, JT](X, Y) - \tfrac{1}{4}J[J, T](X, Y). \tag{10}$$

We shall need the formula (see [3], $p$. 94)

$$[H, KL] + [K, HL] = K[H, L] + H[K, L] + [H, K]L + [H, K] . L$$

which is valid for any three tensor fields of type (1, 1). In this formula $[H, K]L$ and $[H, K] . L$ are tensor fields of type (1, 2) defined by $([H, K]L)(X, Y) = [H, K](LX, Y)$ and $([H, K] . L)(X, Y) = [H, K](X, LY)$, respectively. Using this formula we easily find

$$[J, JT] = -T[J, J] - J[T, J] - [J, T]J - [J, T] . J .$$

Now by virtue of the assumptions we get from (10)

$$V(X, Y) - V(Y, X) = -\tfrac{1}{2}T(\nabla_{JX}J)(Y) + \tfrac{1}{2}T(\nabla_{JY}J)(X) .$$

This result shows that if we set

$$N(X, Y) = -\tfrac{1}{2}T(\nabla_{JX}J)(Y)$$

then the condition (ii)′ is obviously satisfied. But it is easy to see that this $N$ satisfies also (i)′. We get

$$N(X, JY) = -\tfrac{1}{2}T(\nabla_{JX}J)(JY) = -\tfrac{1}{2}T(\nabla_{JX}(-Y)) + \tfrac{1}{2}TJ(\nabla_{JX}JY) = \\ = -\tfrac{1}{2}JT(\nabla_{JX}JY) + \tfrac{1}{2}T\nabla_{JX}Y = JN(X, Y) .$$

Thus we have found

$$W(X, Y) = V(X, Y) - N(X, Y) = \tfrac{1}{4}T(\nabla_{JY}J)(X) + \tfrac{1}{4}TJ(\nabla_{Y}J)(X) + \\ + \tfrac{1}{4}(\nabla_{JTY}J)(X) + \tfrac{1}{4}J(\nabla_{TY}J)(X) + \tfrac{1}{2}T(\nabla_{JX}J)(Y)$$

which is a solution of (9), and which is moreover symmetric, i.e. $W(X, Y) = W(Y, X)$.

In order to find $P$ such that $TP(X, Y) = W(X, Y)$ we must prove first that $TW(X, Y) = 0$. We obtain

$$TW(X, Y) = \tfrac{1}{4}TJ(J[JX, JTY] + [X, JTY] + J[X, TY] - [JX, TY]) = \\ = -\tfrac{1}{4}J(J, T)(X, Y) = 0 .$$

Further calculation shows that

$$W(X, TY) = \tfrac{1}{4}TJ(J[JX, JTY] + [X, JTY] + J[X, TY] - [JX, TY]) =$$
$$= -\tfrac{1}{4}J(J, T)(X, Y) = 0 .$$

These two results enable us to define $P$ as follows. First we choose a distribution $\tilde{D}$ on $M$ such that $D \oplus \tilde{D} = T(M)$. We need the following notation: if $X \in D$ we denote by $\tilde{X}$ the unique element from $\tilde{D}$ satisfying $T\tilde{X} = X$. Now we define

$$\begin{aligned} P(X, Y) &= 0 && \text{for } X, Y \in D , \\ P(X, Y) &= W(\tilde{X}, Y) && \text{for } X \in D,\ Y \in \tilde{D}, \\ P(X, Y) &= W(X, \tilde{Y}) && \text{for } X \in \tilde{D},\ Y \in D, \\ P(X, Y) &= T^{-1}W(X, Y) && \text{for } X, Y \in \tilde{D} . \end{aligned}$$

$T^{-1}W(X, Y)$ denotes here the unique element from $\tilde{D}$ the image of which under $T$ is $W(X, Y)$. $P(X, Y)$ satisfies (i) because $W(X, Y)$ is symmetric. (ii) follows from the very definition of $P(X, Y)$. Finally, $P(X, Y)$ satisfies (iii) because $\mathscr{a}W = V$. Our proposition is proved.

Now let us consider again a manifold $M$ with a $(J, T)_2$-structure satisfying

$$[J, J] = [J, T] = [T, T] = (J, T) = 0 .$$

By the preceding proposition we can find on $M$ a torsionless connection with respect to which $J$ and $T$ are parallel. Let $\nabla$ and $\nabla'$ be two connections with these properties. We shall write $\nabla'_X Y = \nabla_X Y - S(X, Y)$. Obviously we have $S(X, Y) = S(Y, X)$, $S(X, JY) = JS(X, Y)$ and $S(X, TY) = TS(X, Y)$. If we denote by $R$ and $R'$ the curvature tensors of $\nabla$ and $\nabla'$, respectively, we have

$$\text{(11)} \qquad R'(X, Y) Z = R(X, Y) Z - (\nabla_X(S(Y, Z)) - \nabla_Y(S(X, Z))) -$$
$$- (S(X, \nabla_Y Z) - S(Y, \nabla_X Z)) + (S(X, S(Y, Z) - S(Y, S(X, Z)) +$$
$$+ S([X, Y], Z) .$$

If $Z \in D$ we can find $\hat{Z}$ such that $T\hat{Z} = Z$. Then for any $X$ we have

$$\begin{aligned} S(X, Z) &= -J^2S(X, T\hat{Z}) = -J^2S(T\hat{Z}, X) = -JS(T\hat{Z}, JX) = \\ &= -JS(JX, T\hat{Z}) = -JTS(JX, \hat{Z}) = TJS(\hat{Z}, JX) = \\ &= TJ^2S(\hat{Z}, X) = -TS(X, \hat{Z}) = -S(X, T\hat{Z}) = -S(X, Z) \end{aligned}$$

which implies $S(X, Z) = 0$. Using this result and (11) we easily find that for any $Z \in D$ we have

$$R'(X, Y) Z = R(X, Y) Z .$$

Moreover, because $\nabla T = 0$ we have $R(X, Y)Z \in D$. This leads us to the following

**Definition 1.** Let $M$ be a manifold with a $(J, T)_2$-structure satisfying

$$[J, J] = [J, T] = [T, T] = (J, T) = 0\,.$$

Then we define a 3-linear mapping

$$B : T(M) \times T(M) + D \to D$$

by the formula $B(X, Y, Z) = R(X, Y)Z$, where $R$ is the curvature tensor of any torsionless connection $\nabla$ satisfying $\nabla T = \nabla J = 0$.

Now we have prepared all necessary tools for the proof of the main theorem of this section.

**Theorem 2.** *A $(J, T)_2$-structure is integrable if and only if the following conditions are satisfied:*

(i) $[J,J] = [J, T] = [T, T] = (J, T) = 0$,

(ii) $B = 0$.

Proof. Let a $(J, T)_2$-structure on a manifold $M$ be integrable. That means that for any point of $M$ there exists its open neighborhood with a chart $(x^1, \ldots, x^{4n})$ on it such that with respect to the basis $\partial/\partial x^1, \ldots, \partial/\partial x^{4n}$ the tensors $J$ and $T$ have the matrix expression (6). This immediately implies (i). But if this is the case $B$ can be defined. Taking any connection $\nabla$ with $\nabla J = \nabla T = 0$ we have

$$B\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^{2n+K}}\right) =$$

$$= \left(\frac{\partial \Gamma^r_{j,2n+K}}{\partial x^i} - \frac{\partial \Gamma^r_{i,2n+K}}{\partial x^j} + \Gamma^s_{j,2n+K}\Gamma^r_{is} - \Gamma^s_{i,2n+K}\Gamma^r_{js}\right)\frac{\partial}{\partial x^r}$$

for $i, j, r = 1, \ldots, 4n$, $K = 1, \ldots, 2n$. The $\Gamma$'s denote here as usual the Christoffel coefficients of $\nabla$. We shall prove that $\Gamma^i_{j,2n+K} = 0$ for $i, j = 1, \ldots, 4n$ and $K = 1, \ldots, 2n$ which will imply $B = 0$. We denote by lower-case Greek letters the integers $1, \ldots, n$ and we write $\nabla_i$ instead of $\nabla_{\partial/\partial x^i}$. We have

$$\Gamma^i_{j,2n+\alpha}\frac{\partial}{\partial x_i} = \nabla_j \frac{\partial}{\partial x^{2n+\alpha}} = \nabla_j\left(T\frac{\partial}{\partial x^\alpha}\right) = T\nabla_j\frac{\partial}{\partial x^\alpha} = T\left(\Gamma^i_{j\alpha}\frac{\partial}{\partial x^i}\right) =$$

$$= \Gamma^\beta_{j\alpha}\frac{\partial}{\partial x^{2n+\beta}} - \Gamma^{n+\beta}_{j\alpha}\frac{\partial}{\partial x^{n+\beta}}$$

which implies $\Gamma^i_{j,2n+\alpha} = 0$ for $1 \leqq i \leqq 2n$. Along the same lines we obtain

$\Gamma^i_{j,2n+n+\alpha} = 0$. Furthermore,

$$\Gamma^I_{\alpha,2n+\beta} \frac{\partial}{\partial x^I} = \nabla_\alpha \frac{\partial}{\partial x^{2n+\beta}} = \nabla_\alpha \left(T \frac{\partial}{\partial x^\beta}\right) = T\left(\nabla_\alpha \frac{\partial}{\partial x^\beta}\right) = T\left(\nabla_\beta \frac{\partial}{\partial x^\alpha}\right) =$$

$$= T\nabla_\beta\left(-J \frac{\partial}{\partial x^{n+\alpha}}\right) = -TJ\nabla_\beta \frac{\partial}{\partial x^{n+\alpha}} = JT\nabla_\beta \frac{\partial}{\partial x^{n+\alpha}} = JT\nabla_{n+\alpha} \frac{\partial}{\partial x^\beta} =$$

$$= J\nabla_{n+\alpha}\left(T \frac{\partial}{\partial x^\beta}\right) = J\nabla_{n+\alpha} \frac{\partial}{\partial x^{2n+\beta}} = J\nabla_{2n+\beta} \frac{\partial}{\partial x^{2n+\alpha}} = \nabla_{2n+\beta}\left(J \frac{\partial}{\partial x^{n+\alpha}}\right) =$$

$$= -\nabla_{2n+\beta} \frac{\partial}{\partial x^\alpha} = -\nabla_\alpha \frac{\partial}{\partial x^{2n+\beta}} = -\Gamma^I_{\alpha,2n+\beta} \frac{\partial}{\partial x^I}$$

from which we get $\Gamma^I_{\alpha,2n+\beta} = 0$. Similarly we proceed in the remaining cases.

Conversely, let us suppose that a $(J, T)_2$-structure on a manifold $M$ satisfies (i) and (ii), and let us take a point $a \in M$. The condition $[T, T] = 0$ implies the integrability of the distribution $D = \operatorname{im} T$. Thus we can find an open neighborhood $\mathcal{U}_1$ of $a$ and a chart $(x_1^1, \ldots, x_1^{4n})$ on it such that $D$ is spanned by the vectors $\partial/\partial x_1^{2n+1}, \ldots, \partial/\partial x_1^{4n}$. Generally $[J, J] = 0$ implies the integrability of the almost complex structure $J$. At this place we shall use this fact only partially. Namely, inspecting the proof of integrability of an almost complex structure (see e.g. [1], Appendix 8), we obtain easily its modification depending on an arbitrary number of parameters. More precisely this means that on a smaller open neighborhood $\mathcal{U}_2 \subseteq \mathcal{U}_1$ of $a$ we can find a chart $(x_2^1, \ldots, x_2^{4n})$ with

$$x_2^I = x_1^I\,, \quad x_2^{2n+I} = f^{2n+I}(x_1^1, \ldots, x_1^{4n})\,, \quad I = 1, \ldots, 2n$$

such that $(x_2^{2n+1} + ix_2^{2n+n+1}, \ldots, x_2^{2n+n} + ix_2^{2n+n+n})$ is a complex chart on any leaf of the distribution $D$ contained in $\mathcal{U}_2$.

Let us write now

$$J \frac{\partial}{\partial x_2^I} = c_I^K \frac{\partial}{\partial x_2^K} + d_I^K \frac{\partial}{\partial x_2^{2n+K}}\,, \quad I = 1, \ldots, 2n\,.$$

We get easily $c_I^K c_K^L = -\delta_I^L$. This enables us to introduce on each leaf of the complementary distribution $D' = [\partial/\partial x_2^1, \ldots, \partial/\partial x_2^{2n}]$ an auxiliary almost complex structure $J'$ by

$$J' \frac{\partial}{\partial x_2^I} = c_I^K \frac{\partial}{\partial x_2^K}\,.$$

We want to show now that this almost complex structure has relatively nice properties. For this purpose we rewrite the tensor $(J, T)$ into the form

$$(J, T)(X, Y) = TJ[J, J](X, TY) + 2T(J[X, TY] - [JX, TY]) =$$
$$= 2T(J[X, TY] - [JX, TY])$$

and apply the assumption $(J, T) = 0$. Taking $X = \partial/\partial x_2^I$ and $Y$ such that $TY = \partial/\partial x_2^{2n+L}$ we get

$$0 = \tfrac{1}{2}(J, T)(X, Y) = T\left(J\left[\frac{\partial}{\partial x_2^I}, \frac{\partial}{\partial x_2^{2n+L}}\right] - \left[c_I^K \frac{\partial}{\partial x_2^K} + d_I^K \frac{\partial}{\partial x_2^{2n+K}}, \frac{\partial}{\partial x_2^{2n+L}}\right]\right) = $$

$$= \frac{\partial c_I^K}{\partial x_2^{2n+L}} T \frac{\partial}{\partial x_2^K}$$

which implies $\partial c_I^K / \partial x_2^{2n+L} = 0$. Another important property of $J'$ follows from the assumption $[J, J] = 0$. With $X = \partial/\partial x_2^I$, $Y = \partial/\partial x_2^K$ we obtain

$$0 = [J, J]\left(\frac{\partial}{\partial x_2^I}, \frac{\partial}{\partial x_2^K}\right) = \left[c_I^L \frac{\partial}{\partial x_2^L}, c_K^M \frac{\partial}{\partial x_L^M}\right] - J\left[c_I^L \frac{\partial}{\partial x_2^L}, \frac{\partial}{\partial x_2^K}\right] -$$

$$- J\left[\frac{\partial}{\partial x_2^I}, c_K^M \frac{\partial}{\partial x_2^M}\right] + \text{element of } D .$$

This shows that $[J', J'] = 0$ on any leaf of $D$. This fact together with the independence of $c_I^K$ on $x_2^{2n+1}, \ldots, x_2^{4n}$ enables us to find a chart $(x_3^1, \ldots, x_3^{4n})$ on an open neighborhood $\mathscr{U}_3 \subseteq \mathscr{U}_2$ of $a$ with

$$x_3^I = g^I({}_2^1x, \ldots, x_2^{2n}), \quad x_3^{2n+I} = x_2^{2n+I}, \quad I = 1, \ldots, 2n$$

and such that

$$(12) \qquad J \frac{\partial}{\partial x_3^\alpha} = \frac{\partial}{\partial x_3^{n+\alpha}} + d_\alpha^K \frac{\partial}{\partial x_3^{2n+K}},$$

$$J \frac{\partial}{\partial x_3^{n+\alpha}} = -\frac{\partial}{\partial x_3^\alpha} + d_{n+\alpha}^K \frac{\partial}{\partial x_3^{2n+K}}, \quad \alpha = 1, \ldots, n .$$

(The coefficients $d$ differ of course from those above denoted by the same letter.) It is evident that $(x_3^{2n+1} + ix_3^{2n+n+1}, \ldots, x_3^{2n+n} + ix_3^{2n+n+n})$ is again a complex chart on any leaf of $D$ contained in $\mathscr{U}_3$.

We now introduce an auxiliary complex structure on $\mathscr{U}_3$ using the complex chart $(z^1, \ldots, z^{2n})$ defined by

$$z^\alpha = x_3^\alpha + ix_3^{n+\alpha}, \quad z^{n+\alpha} = x_3^{2n+\alpha} + ix_3^{2n+n+\alpha}, \quad \alpha = 1, \ldots, n .$$

Of course, this complex structure on $\mathscr{U}_3$ need not coincide with the complex structure induced by $J$. Nevertheless the both structures coincide on any leaf of $D$. We shall use this complex chart in order to rewrite the equations (12) into a complex form. Let us mention first that by virtue of $J^2 = -I$ we obtain from (12) the equalities

$$d_{n+\alpha}^\beta = d_\alpha^{n+\beta}, \quad d_{n+\alpha}^{n+\beta} = -d_\alpha^\beta; \quad \alpha, \beta = 1, \ldots, n .$$

Using this we find easily the complex form of (12), namely

(13) $$J \frac{\partial}{\partial z^\alpha} = \mathrm{i} \frac{\partial}{\partial z^\alpha} + (d_\alpha^\beta - \mathrm{i}\, d_\alpha^{n+\beta}) \frac{\partial}{\partial \bar{z}^{n+\beta}} \,.$$

Moreover, if we introduce the notation $D_\alpha^\beta = d_\alpha^\beta + \mathrm{i} d_\alpha^{n+\beta}$ we can write (13) in an equivalent form

(14) $$J \frac{\partial}{\partial z^\alpha} = \mathrm{i} \frac{\partial}{\partial z^\alpha} + \bar{D}_\alpha^\beta \frac{\partial}{\partial \bar{z}^{n+\beta}} \,.$$

The assumption $[J, J] = 0$ provides us with an important information concerning the $D$'s. Calculating $[J, J] (\partial/\partial z^\alpha, \partial/\partial z^{n+\beta})$ we obtain

(15) $$\frac{\partial D_\alpha^\beta}{\partial \bar{z}^{n+\gamma}} = 0 \,; \quad \alpha, \beta, \gamma = 1, \ldots, n \,.$$

Let us consider now the differential system

(16) $$\frac{\partial H^\beta}{\partial \bar{z}^\alpha} = -\tfrac{1}{2}\mathrm{i} D_\alpha^\delta \frac{\partial H^\beta}{\partial z^{n+\delta}} \,; \quad \alpha, \beta = 1, \ldots, n$$

for the functions $H_\beta(z^1, \ldots, z^{2n}, \bar{z}_1, \ldots, \bar{z}^{2n})$. Considering the equation

$$\frac{\partial}{\partial \bar{z}^\gamma} \left( D_\alpha^\delta \frac{\partial H^\beta}{\partial z^{n+\delta}} \right) = \frac{\partial}{\partial \bar{z}^\alpha} \left( D_\gamma^\delta \frac{\partial H^\beta}{\partial z^{n+\delta}} \right)$$

we come to the conclusion that the system (16) is involutive (see [4]) if and only if the following conditions are satisfied:

$$\frac{\partial D_\alpha^\varepsilon}{\partial \bar{z}^\gamma} - \frac{\partial D_\gamma^\varepsilon}{\partial \bar{z}^\alpha} - \tfrac{1}{2}\mathrm{i} \left( D_\alpha^\delta \frac{\partial D_\gamma^\varepsilon}{\partial z^{n+\delta}} - D_\gamma^\delta \frac{\partial D_\alpha^\varepsilon}{\partial z^{n+\delta}} \right) = 0 \,, \quad \alpha, \beta, \varepsilon = 1, \ldots, n \,.$$

Bur fortunately this is the case in our situation. It suffices to use again$[J, J] = 0$ and to calculate $[J, J] (\partial/\partial \bar{z}^\alpha, \partial/\partial \bar{z}^\gamma)$. Thus we have proved that the system (16) is involutive. Taking the corresponding Cauchy data in the form

$$H^\beta(z^1, \ldots, z^{2n}, 0, \ldots, 0, \bar{z}^{n+1}, \ldots, \bar{z}^{2n}) = \varphi^\beta(z^1, \ldots, z^{2n}) \,,$$

where $\varphi^\beta$ are holomorphic with respect to the variables $z^{n+1}, \ldots, z^{2n}$ and such that $\det (\partial \varphi^\beta / \partial z^{n+\alpha}) \neq 0$ we can see that by virtue of (15) there is a solution

$$H^\beta(z^1, \ldots, z^{2n}, \bar{z}^1, \ldots, \bar{z}^n) \,, \quad \beta = 1, \ldots, n$$

of (16) which is holomorphic with respect to the variables $z^{n+1}, \ldots, z^{2n}$. We denote

$$h^{2n+\beta}(x_3^1, \ldots, x_3^{4n}) \quad = \mathrm{Re}\, H^\beta(z^1, \ldots, z^{2n}, \bar{z}^1, \ldots, \bar{z}^n) \,,$$
$$h^{2n+n+\beta}(x_3^1, \ldots, x_3^{4n}) = \mathrm{Im}\, H^\beta(z^1, \ldots, z^{2n}, \bar{z}^1, \ldots, \bar{z}^n) \,, \quad \beta = 1, \ldots, n$$

and perform the coordinate change

$$x_4^I = x_3^I \,, \quad x_4^{2n+I} = h^{2n+I}(x_3^1, \ldots, x_3^{4n}) \,, \quad I = 1, \ldots, 2n \,.$$

This new chart is defined on an open neighborhood $\mathscr{U}_4 \subseteq \mathscr{U}_3$ of $a$. Obviously $[\partial/\partial x_4^{2n+1}, \ldots, \partial/\partial x_4^{4n}]$ is again a basis of $D$, and $(x^{2n+1} + \mathrm{i}x^{2n+n+1}, \ldots, x^{2n+n} + \mathrm{i}x^{2n+n+n})$ is a complex chart on any leaf of $D$ contained in $\mathscr{U}_4$. Moreover we have

$$(17) \qquad J\frac{\partial}{\partial x_3^\alpha} = J\left(\frac{\partial}{\partial x_4^\alpha} + \frac{\partial h^{2n+\beta}}{\partial x_3^\alpha}\cdot\frac{\partial}{\partial x_4^{2n+\beta}} + \frac{\partial h^{2n+n+\beta}}{\partial x_3^\alpha}\frac{\partial}{\partial x_4^{2n+n+\beta}}\right) = $$

$$= J\frac{\partial}{\partial x_4^\alpha} + \frac{\partial h^{2n+\beta}}{\partial x_3^\alpha}\frac{\partial}{\partial x_4^{2n+n+\beta}} - \frac{\partial h^{2n+n+\beta}}{\partial x_3^\alpha}\frac{\partial}{\partial x_3^{2n+\beta}}$$

and by virtue of (12) also

$$(18) \qquad J\frac{\partial}{\partial x_3^\alpha} = \frac{\partial}{\partial x_3^{n+\alpha}} + d_\alpha^K\frac{\partial}{\partial x_3^{2n+K}} = \frac{\partial}{\partial x_4^{n+\alpha}} + \frac{\partial h^{2n+\beta}}{\partial x_3^{n+\alpha}}\frac{\partial}{\partial x_4^{2n+\beta}} + $$

$$+ \frac{\partial h^{2n+n+\beta}}{\partial x_3^{n+\alpha}}\cdot\frac{\partial}{\partial x_4^{2n+n+\beta}} + d_\alpha^K\frac{\partial h^{2n+L}}{\partial x_3^{2n+K}}\frac{\partial}{\partial x_4^{2n+L}}\,.$$

Taking the real and the imaginary part of (16) we obtain the equations

$$\frac{\partial h^{2n+\beta}}{\partial x_3^\alpha} - \frac{\partial h^{2n+n+\beta}}{\partial x_3^{n+\alpha}} = d_\alpha^\gamma\frac{\partial h^{2n+n+\beta}}{\partial x_3^{2n+\gamma}} + d_\alpha^{n+\gamma}\frac{\partial h^{2n+n+\beta}}{\partial x_3^{2n+n+\gamma}}\,,$$

$$\frac{\partial h^{2n+\beta}}{\partial x_3^{n+\alpha}} + \frac{\partial h^{2n+n+\beta}}{\partial x_3^\alpha} = -d_\alpha^\gamma\frac{\partial h^{2n+\beta}}{\partial x_3^{2n+\gamma}} - d_\alpha^{n+\gamma}\frac{\partial h^{2n+\beta}}{\partial x_3^{2n+n+\gamma}}\,.$$

Using these equations we get easily from (17) and (18) the result $J(\partial/\partial x_4^\alpha) = \partial/\partial x_4^{n+\alpha}$. Similarly we get $J(\partial/\partial x^{n+\alpha}) = -\partial/\partial x^\alpha$.

Let us introduce now a complex chart $(v^1, \ldots, v^{2n})$ on $\mathscr{U}_4$ by

$$v^\alpha = x_4^\alpha + \mathrm{i}x_4^{n+\alpha}\,, \quad v^{n+\alpha} = x_4^{2n+\alpha} + \mathrm{i}x_4^{2n+n+\alpha}\,, \quad \alpha = 1, 2, \ldots, n\,.$$

This time it is easy to see that the complex structure defined on $\mathscr{U}_4$ by this coincides with the structure induced by the almost complex structure $J$. We notice that the relation $JT = -TJ$ implies immediately that the image under $T$ of any complex vector field of type $(1, 0)$ is a vector field of type $(0, 1)$. This enables us to write

$$T\frac{\partial}{\partial v^\alpha} = a_\alpha^\beta\frac{\partial}{\partial \bar{v}^{n+\beta}}\,; \quad \alpha = 1, \ldots, n$$

where the coefficients $a_\alpha^\beta$ are complex functions on $\mathscr{U}_4$. We denote by $b_\alpha^\beta$ the elements of the inverse matrix to $(a_\alpha^\beta)$.

Now we are going to perform the last step of the proof. First we must prove that the functions

$$(19) \qquad b_\beta^\gamma\,\overline{\frac{\partial a_\gamma^\alpha}{\partial \bar{v}^I}}\,; \quad I = 1, \ldots, 2n\,; \quad \alpha, \beta = 1, \ldots, n$$

are holomorphic. Let us choose on $M$ any symmetric connection $\nabla$ with $\nabla J = \nabla T =$

$= 0$. We need to notice that $\nabla J = 0$ implies that the covariant derivative with respect to any field of a complex vector field of type $(1, 0)$ is again a complex vector field of type $(1, 0)$. The same holds for vector fields of type $(0, 1)$. By virtue of this and of the symmetry of $\nabla$ we find that $\nabla_{\partial/\partial v^I} \partial/\partial \bar{v}^K = \nabla_{\partial/\partial \bar{v}^K} \partial/\partial v^I$ is a vector field both of type $(1, 0)$ and $(0, 1)$ which means

$$\nabla_{\partial/\partial \bar{v}^I} \frac{\partial}{\partial \bar{v}^K} = \nabla_{\partial/\partial v^K} \frac{\partial}{\partial v^I} = 0\,, \quad I, K = 1, \dots, 2n\,.$$

The assumption (ii) implies

$$0 = B\left(\frac{\partial}{\partial \bar{v}^I}, \frac{\partial}{\partial v^K}, \frac{\partial}{\partial v^{n+\alpha}}\right) = R\left(\frac{\partial}{\partial \bar{v}^I}, \frac{\partial}{\partial v^K}\right)\frac{\partial}{\partial v^{n+\alpha}} =$$

$$= \nabla_{\partial/\partial \bar{v}^I} \nabla_{\partial/\partial v^K} \frac{\partial}{\partial v^{n+\alpha}} - \nabla_{\partial/\partial v^K} \nabla_{\partial/\partial \bar{v}^I} \frac{\partial}{\partial v^{n+\alpha}} = \nabla_{\partial/\partial \bar{v}^I} \nabla_{\partial/\partial v^K}\left(T\left(\bar{b}_\alpha^\beta \frac{\partial}{\partial \bar{v}^\beta}\right)\right) =$$

$$= T\nabla_{\partial/\partial \bar{v}^I} \nabla_{\partial/\partial v^K}\left(\bar{b}_\alpha^\beta \frac{\partial}{\partial \bar{v}^\beta}\right) = T\nabla_{\partial/\partial \bar{v}^I}\left(\frac{\partial \bar{b}_\alpha^\beta}{\partial v^K} \frac{\partial}{\partial \bar{v}^\beta}\right) =$$

$$= \nabla_{\partial/\partial \bar{v}^I}\left(\frac{\partial \bar{b}_\alpha^\beta}{\partial v^K} T \frac{\partial}{\partial \bar{v}^\beta}\right) = \nabla_{\partial/\partial \bar{v}^I}\left(\frac{\partial \bar{b}_\alpha^\beta}{\partial v^K} \bar{a}_\beta^\gamma \frac{\partial}{\partial v^{n+\gamma}}\right) = \frac{\partial}{\partial \bar{v}^I}\left(\frac{\partial \bar{b}_\alpha^\beta}{\partial v^K} \bar{a}_\beta^\gamma\right)\frac{\partial}{\partial v^{n+\gamma}} =$$

$$= -\frac{\partial}{\partial \bar{v}^I}\left(b_\alpha^\beta \frac{\partial a_\beta^\gamma}{\partial \bar{v}^K}\right)\frac{\partial}{\partial v^{n+\gamma}}$$

which shows that the functions (19) are holomorphic. (By $R$ we have denoted the curvature tensor of $\nabla$.) This enables us to study the question of existence of holomorphic functions $\Phi_\alpha^\beta(v^1, \dots, v^{2n})$; $\alpha, \beta = 1, \dots, n$ which are solutions of the system

$$\frac{\partial \Phi_\alpha^\beta}{\partial v^I} = -\overline{b_\alpha^\gamma \frac{\partial a_\gamma^\varepsilon}{\partial \bar{v}^I}}\, \Phi_\varepsilon^\beta\,; \quad I = 1, \dots, 2n\,; \quad \alpha, \beta = 1, \dots, n\,. \tag{20}$$

The standard procedure shows that this system is involutive, and this implies the existence of holomorphic solutions $\Phi_\alpha^\beta$. It is easy to see that we can find $\Phi_\alpha^\beta$ such that $\Phi_\alpha^\beta(a) = \delta_\alpha^\beta$. Let us use now the assumption $[T, T] = 0$. We have

$$0 = [T, T]\left(\frac{\partial}{\partial v^\alpha}, \frac{\partial}{\partial v^\beta}\right) = \left(a_\alpha^\gamma \frac{\partial a_\beta^\varepsilon}{\partial \bar{v}^{n+\gamma}} - a_\beta^\gamma \frac{\partial a_\alpha^\varepsilon}{\partial \bar{v}^{n+\gamma}}\right)\frac{\partial}{\partial \bar{v}^{n+\varepsilon}}$$

or equivalently

$$b_\alpha^\varepsilon \frac{\partial a_\varepsilon^\eta}{\partial \bar{v}^{n+\gamma}} = b_\gamma^\varepsilon \frac{\partial a_\varepsilon^\eta}{\partial \bar{v}^{n+\alpha}}\,; \quad \alpha, \gamma, \eta = 1, \dots, n\,.$$

Using this and (20) we find easily $\partial \Phi_\alpha^\beta / \partial v^{n+\gamma} = \partial \Phi_\gamma^\beta / \partial v^{n+\alpha}$. Therefore we can find holomorphic functions $K^{n+\alpha}(v^1, \dots, v^{2n})$; $\alpha = 1, \dots, n$ such that

$$\frac{\partial K^{n+\alpha}}{\partial v^{n+\beta}} = \Phi_\beta^\alpha\,. \tag{21}$$

The functions

$$a_\alpha^\beta \frac{\partial \bar{K}^\gamma}{\partial \bar{v}^{n+\beta}}\,; \quad \alpha, \gamma = 1, \ldots, n \tag{22}$$

are holomorphic because by virtue of (20) we have

$$\frac{\partial}{\partial \bar{v}^I}\left(a_\alpha^\beta \frac{\partial \bar{K}^\gamma}{\partial \bar{v}^{n+\beta}}\right) = \frac{\partial}{\partial v^I}\left(\bar{a}_\alpha^\beta \frac{\partial K^\gamma}{\partial v^{n+\beta}}\right) = \frac{\partial}{\partial v^I}\left(\bar{a}_\alpha^\beta \Phi_\beta^\gamma\right) =$$

$$= \frac{\partial \bar{a}_\alpha^\beta}{\partial v^I}\Phi_\beta^\gamma + \bar{a}_\alpha^\beta \frac{\partial \Phi_\beta^\gamma}{\partial v^I} = \frac{\partial \bar{a}_\alpha^\beta}{\partial v^I}\Phi_\beta^\gamma - \bar{a}_\alpha^\beta b_\beta^\varepsilon \overline{\frac{\partial a_\varepsilon^\eta}{\partial \bar{v}^I}}\Phi_\eta^\gamma =$$

$$= \frac{\partial \bar{a}_\alpha^\beta}{\partial v^I}\Phi_\beta^\gamma - \frac{\partial \bar{a}_\alpha^\eta}{\partial v^I}\Phi_\eta^\gamma = 0\,.$$

Moreover we have

$$\frac{\partial}{\partial v^{n+\varepsilon}}\left(a_\alpha^\beta \frac{\partial \bar{K}^\gamma}{\partial \bar{v}^{n+\beta}}\right) = \frac{\partial a_\alpha^\beta}{\partial v^{n+\varepsilon}} \frac{\partial \bar{K}^\gamma}{\partial \bar{v}^{n+\beta}} = 0\,.$$

This follows immediately from the assumption $[J, T] = 0$. Namely,

$$0 = [J, T]\left(\frac{\partial}{\partial v^{n+\varepsilon}}, \frac{\partial}{\partial v^\alpha}\right) = 2\mathrm{i}\,\frac{\partial a_\alpha^\beta}{\partial v^{n+\varepsilon}} \frac{\partial}{\partial \bar{v}^{n+\beta}}$$

which implies $\partial a_\alpha^\beta / \partial v^{n+\varepsilon} = 0$. Thus we have proved that the functions (20) are holomorphic and depend on the variables $v^1, \ldots, v^n$ only. We can therefore consider the system

$$\frac{\partial K^\gamma}{\partial v^\alpha} = a_\alpha^\varepsilon \frac{\partial \bar{K}^{n+\gamma}}{\partial \bar{v}^{n+\varepsilon}}\,; \quad \alpha, \beta = 1, \ldots, n\,. \tag{23}$$

The equality

$$\frac{\partial}{\partial v^\beta}\left(a_\alpha^\varepsilon \frac{\partial \bar{K}^{n+\gamma}}{\partial \bar{v}^{n+\varepsilon}}\right) = \frac{\partial}{\partial v^\alpha}\left(a_\beta^\varepsilon \frac{\partial \bar{K}^{n+\gamma}}{\partial \bar{v}^{n+\varepsilon}}\right)$$

is obviously equivalent to

$$\frac{\partial a_\alpha^\gamma}{\partial v^\beta} = \frac{\partial a_\beta^\gamma}{\partial v^\alpha}\,.$$

But even this last equality follows from the assumption $[J, T] = 0$. We get namely

$$0 = [J, T]\left(\frac{\partial}{\partial v^\alpha}, \frac{\partial}{\partial v^\beta}\right) = 2\mathrm{i}\left(\frac{\partial a_\beta^\gamma}{\partial v^\alpha} - \frac{\partial a_\alpha^\gamma}{\partial v^\beta}\right)\frac{\partial}{\partial \bar{v}^{n+\gamma}}\,.$$

Now it is easy to see that the system (23) is involutive. Then there exist holomorphic functions $K^\gamma(v^1, \ldots, v^n)$; $\gamma = 1, \ldots, n$ which satisfy (23).

It remains to perform the coordinate change

$$w^\alpha = K^\alpha(v^1, \dots, v^n)\,, \quad w^{n+\alpha} = K^{n+\alpha}(v^1, \dots, v^n, v^{n+1}, \dots, v^{2n})\,, \quad \alpha = 1, \dots, n$$

introducing in this way a complex chart on an open neighborhood $\mathcal{U}_5 \subseteq \mathcal{U}_4$ of $a$. We shall write

$$w^\alpha = x_5^\alpha + ix_5^{n+\alpha}\,, \quad w^{n+\alpha} = x_5^{2n+\alpha} + ix_5^{2n+n+\alpha}\,, \quad \alpha = 1, \dots, n\,.$$

Obviously the chart $(w^1, \dots, w^{4n})$ induces again the same complex structure on $\mathcal{U}_5$ as the almost complex structure $J$. This implies that the matrix expression of $J$ with respect to the basis $\partial/\partial x_5^1, \dots, \partial/\partial x_5^{4n}$ is exactly the left matrix in (6). Concerning $T$ we obtain

$$T\frac{\partial}{\partial v^\alpha} = T\left(\frac{\partial K^\beta}{\partial v^\alpha}\frac{\partial}{\partial w^\beta} + \frac{\partial K^{n+\beta}}{\partial v^\alpha}\frac{\partial}{\partial w^{n+\beta}}\right) = \frac{\partial K^\beta}{\partial v^\alpha}T\frac{\partial}{\partial w^\beta}\,,$$

$$T\frac{\partial}{\partial v^\alpha} = a_\gamma^\alpha\frac{\partial}{\partial \bar{v}^{n+\gamma}} = a_\alpha^\gamma\frac{\partial \bar{K}^{n+\beta}}{\partial \bar{v}^{n+\gamma}}\cdot\frac{\partial}{\partial \bar{w}^{n+\beta}}\,.$$

From these two equalities by virtue of (23) we get $T(\partial/\partial w^\alpha) = \partial/\partial \bar{w}^{n+\alpha}$. Now it is easy to see that the matrix expression of $T$ with respect to the basis $\partial/\partial x_5^1, \dots, \partial/\partial x_5^{4n}$ is exactly the right hand matrix in (6). The proof is complete.

**Corollary.** *A $(J, T)_2$-structure is integrable if and only if there exists on $M$ a torsionless connection $\nabla$ such that*

(i) $\nabla J = \nabla T = 0$,

(ii) $R(X, Y)\,Z = 0$, $X, Y \in T(M)$, $Z \in D$

*where $R$ denotes the curvature tensor of $\nabla$.*

**4. The case** $JT + TJ = I$. In this section we consider the last case, namely a manifold $M$ provided with a couple $J, T$ of tensor fields of type $(1, 1)$ satisfying $J^2 = -I$, $T^2 = 0$ and $JT + TJ = I$. We shall call this structure a *$(J, T)_3$-structure.* It is not necessary here to suppose that $\ker T = \operatorname{im} T$ because, as we shall see in the next lemma, this appears as a consequence of our assumptions. We denote again $m = \dim M$. Obviously we can write $m = 2n$.

**Lemma 2.** *We have* $\ker T = \operatorname{im} T$.

Proof. Obviously $\operatorname{im} T \subseteq \ker T$. If $X \in \ker T$ we can write $X = JTX + TJX = TJX$ which shows that $X \in \operatorname{im} T$.

We introduce on $M$ two tensor fields $P_1, P_2$ of type $(1, 1)$ by

$$P_1 = JT\,, \quad P_2 = TJ\,.$$

**Lemma 3.** *$P_1$ and $P_2$ are complementary projectors.*

Proof. We see immediately that $P_1^2 = P_1$, $P_2^2 = P_2$, $P_1 + P_2 = I$.

This result enables us to introduce two complementary distributions $D_1$, $D_4$ on $M$. We set

$$D_1 = \operatorname{im} P_1\,, \quad D_2 = \operatorname{im} P_2\,.$$

**Lemma 4.** *We have $D_1 = \ker(J + T)$, $D_2 = \ker T$. Moreover, $J$ and $T$ map $D_1$ isomorphically onto $D_2$ and thus $\dim D_1 = \dim D_2 = n$.*

Proof. Let $X \in D_1$. Then there exists $Y \in T(M)$ such that $X = JTY$. We get $(J + T)(X) = (J + T)\, JTY = -TY + TY = 0$, which implies $D_1 \subseteq \ker(J + T)$. Conversely, let $(J + T)X = 0$. We have $JX = -TX$, and hence $X = JTX$ which shows $\ker(J + T) \subseteq D_1$.

Let $X \in D_2$. This yields the existence of $Y \in T(M)$ such that $X = TJY$. Obviously $TX = T^2JY$ so that $D_4 \subseteq \ker T$. For $X \in \ker T$ we can write $X = JTX + TJX = TJX$ which gives $\ker T \subseteq D_2$. The proof of the last assertion we leave to the reader.

**Proposition 5.** *There is a 1-1 correspondence between the set of $(J, T)_3$-structures on $M$ and the set of $G_3$-structures on $M$, where $G_3$ is the Lie group consisting of all matrices of the form*

$$\begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}$$

*with $A$ being a regular $(n \times n)$-matrix.*

Proof. Let us consider a $(J, T)_3$-structure on $M$. We define the corresponding $G_3$-structure in the following way. Let us take a point $x \in M$ and a basis $v_{n+1}, \ldots, v_{2n}$ of $D_{2x}$. Furthermore let us set $v_i = J_x v_{n+i}$; $i = 1, \ldots, n$. Now it is easy to check that all the bases of the form $J_x v_{n+1}, \ldots, J_x v_{2n}, v_{n+1}, \ldots, v_{2n}$ at all points of $M$ constitute a $G_3$-structure on $M$.

Conversely, let us have a $G_3$-structure on $M$. We choose any basis $v_1, \ldots, v_{2n}$ of $T_x(M)$ belonging to the $G_3$-structure. We define $J_x$ and $T_x$ by the formulas

$$J_x v_i = \sum_{k=1}^{2n} J_i^k v_k\,, \quad T_x v_i = \sum_{k=1}^{2n} T_i^k v_k$$

where $(J_i^k)$ and $(T_i^k)$ are $(n \times n)$-matrices

$$(24) \qquad \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 0 \\ I & 0 \end{pmatrix},$$

respectively. We have denoted here by $I$ the unit $(n \times n)$-matrix. An easy computation shows that this definition does not depend on the choice of a basis from the $G_3$-structure. The reader can verify that in this way we get a $(J, T)_3$-structure on $M$.

Similarly as in the preceding section our main tool will be

**Proposition 6.** *Let $M$ be a manifold with a $(J, T)_3$-structure such that*

$$[J, J] = [J, T] = [T, T] = 0\,.$$

*Then there exists on $M$ a unique torsionless connection $\nabla$ such that $\nabla J = \nabla T = 0$.*

Proof. We shall proceed along the same lines as in Proposition 4. We refer the reader to the proof of this proposition, where he can find the details which we omit here.

We introduce first an auxiliary tensor $E = JT - TJ$. One finds easily

$$E^2 = I\,, \quad JE = -EJ\,, \quad TE = -ET\,, \quad T = -J(E + I)\,.$$

The tensor $E$ is nothing else than the tensor associated with the almost product structure $(D_1, D_2)$ consisting of the two distributions $D_1$ and $D_2$. Using the formula

$$[H, KL] + [K, HL] + K[H, L] + H[K, L] + [H, K]\,L + [H, K]\,.\,L$$

mentioned in the previous section we can calculate

$$[P_1, P_2] = -[T, T] + 2JT[J, T] + 2J[J, T]\,T +$$
$$+ 2J[J, T]\,.\,T + [J, J]\,T\,.\,T = 0$$

where $([J, J]\,T\,.\,T)(X, Y) = [J, J]\,(TX, TY)$. This shows that the almost product structure $(D_1, D_2)$ is integrable, and therefore (see e.g. [3], p. 97) there exists on $M$ a torsionless connection $\nabla$ such that $\nabla E = 0$. We introduce a new connection $\mathring{\nabla}$ by $\mathring{\nabla}_X Y = \nabla_X Y - P(X, Y)$, where $P$ is a tensor field which is to be determined. It must satisfy

(i) $P(X, Y) = P(Y, X)$,

(ii) $P(X, EY) = EP(X, Y)$.

Furthermore, we define on $M$ a connection $\tilde{\nabla}$ by $\tilde{\nabla}_X Y = \mathring{\nabla}_X Y - \dot{Q}(X, Y)$ where

$$\dot{Q}(X, Y) = \tfrac{1}{4}(\mathring{\nabla}_{JY} J)(X) + \tfrac{1}{4}J(\mathring{\nabla}_Y J)(X) + \tfrac{1}{2}(\mathring{\nabla}_X J)(Y)\,.$$

The last requirement on $P$ is the following one:

(iii) $\tilde{\nabla} E = 0$.

The calculation of $(\tilde{\nabla}_X E)(Y)$ gives us the following result:

$$(\tilde{\nabla}_X E)(Y) = -\tfrac{1}{2}EP(JX, JY) - \tfrac{1}{2}EJP(JX, Y) + \tfrac{1}{4}E(\nabla_{JY} J)(X) +$$
$$+ \tfrac{1}{4}EJ(\nabla_Y J)(X) - \tfrac{1}{4}(\nabla_{JEY} J)(X) - \tfrac{1}{4}J(\nabla_{EY} J)(X)\,.$$

Similarly as in the preceding section we introduce the notation

$$V(X, Y) = \tfrac{1}{4}J(\nabla_Y J)(X) + \tfrac{1}{4}(\nabla_{JY} J)(X) + \tfrac{1}{4}EJ(\nabla_{EY} J)(X) - \tfrac{1}{4}E(\nabla_{EJY} J)(X)\,,$$
$$W(X, Y) = P(X, Y)\,.$$

The equation (iii) can be rewritten with the use of this notation in the form

$$\frac{1}{2}W(X, Y) - \frac{1}{2}JW(X, JY) = V(X, Y). \tag{25}$$

Now we use again the space $\mathscr{S}$ of tensor fields of type (1, 2) on $M$. In this space we define an endomorphism $a$, this time by

$$(aS)(X, Y) = \frac{1}{2}S(X, Y) - \frac{1}{2}S(X, JY)\,; \quad S \in \mathscr{S}$$

and find $a^2 = a$ or equivalently $a(a - \mathscr{I}) = 0$. Thus we can write $\mathscr{S} = \mathscr{S}_0 \oplus \mathscr{S}_1$ with $\mathscr{S}_0 = \ker a$ and $\mathscr{S}_1 = \ker(a - \mathscr{I})$. Instead of (25) we can write now simply

$$aW = V. \tag{26}$$

We know that this equation has a solution if and only if $aV = V$. We calculate

$$\begin{aligned}(aV)(X, Y) = {} & \tfrac{1}{8}J(\nabla_Y J)(X) + \tfrac{1}{8}(\nabla_{JY}J)(X) + \tfrac{1}{8}EJ(\nabla_{EY}J)(X) - \\ & - \tfrac{1}{8}E(\nabla_{EJY}J)(X) + \tfrac{1}{8}(\nabla_{JY}J)(X) + \tfrac{1}{8}J(\nabla_Y J)(X) - \\ & - \tfrac{1}{8}E(\nabla_{EJY}J)(X) + \tfrac{1}{8}EJ(\nabla_{EY}J)(X) = V(X, Y).\end{aligned}$$

Thus (26) has a solution, and each of its solutions has the form $V - N$ where $N$ is a tensor field of type (1, 2) satisfying $aN = 0$, i.e. $N(X, JY) = -JN(X, Y)$. Because $P$ has to be symmetric we shall be looking for $N$ satisfying

(i)′ $N(X, JY) = -JN(X, Y)$,
(ii)′ $V(X, Y) - N(X, Y) = V(Y, X) - N(Y, X)$.

For this purpose we calculate

$$\begin{aligned}V(X, Y) - V(Y, X) = {} & -\tfrac{1}{2}J(\nabla_X J)(Y) + \tfrac{1}{2}J(\nabla_Y J)(X) + \tfrac{1}{4}[EJ, EJ](X, Y) + \\ & + \tfrac{1}{4}[E, E](X, Y) - \tfrac{1}{4}[E, E](JX, JY) - \\ & - \tfrac{1}{4}[EJ, EJ](JX, JY) = -\tfrac{1}{2}(\nabla_X J)(Y) + \tfrac{1}{2}J(\nabla_Y J)(X).\end{aligned}$$

Here we have used the fact that $[E, E] = 0$, $[EJ, EJ] = 0$. The proof of this we leave to the reader.

The last result suggests to take

$$N(X, Y) = -\tfrac{1}{2}J(\nabla_X J)(Y).$$

This $N$ obviously satisfies (ii)′. But it is not difficult to see that it satisfies also (i)′. We have namely

$$\begin{aligned}N(X, JY) &= -\tfrac{1}{2}J(\nabla_X J)(JY) = \tfrac{1}{2}J^2(\nabla_X J)(Y) = \\ &= -J(-\tfrac{1}{2}J(\nabla_X J)(Y)) = -JN(X, Y).\end{aligned}$$

Summarizing we can say now that we have found

$$\begin{aligned}P(X, Y) = {} & \tfrac{1}{4}J(\nabla_Y J)(X) + \tfrac{1}{4}(\nabla_{JY}J)(X) + \tfrac{1}{4}EJ(\nabla_{EY}J)(X) - \\ & - \tfrac{1}{4}E(\nabla_{EJY}J)(X) + \tfrac{1}{2}J(\nabla_X J)(Y)\end{aligned}$$

which satisfies (i) and (iii). It remains to prove that it satisfies also (ii).

$$P(X, EY) - EP(X, Y) = \tfrac{1}{4}J(\nabla_{EY}J)(X) + \tfrac{1}{4}(\nabla_{JEY}J)(X) + \tfrac{1}{4}EJ(\nabla_Y J)(X) +$$
$$+ \tfrac{1}{4}E(\nabla_{JY}J)(X) + \tfrac{1}{2}J(\nabla_X J)(EY) - \tfrac{1}{4}EJ(\nabla_Y J)(X) - \tfrac{1}{4}E(\nabla_{JY}J)(X) -$$
$$- \tfrac{1}{4}J(\nabla_{EY}J)(X) + \tfrac{1}{4}(\nabla_{EJY}J)(X) - \tfrac{1}{2}EJ(\nabla_X J)(Y) =$$
$$= \tfrac{1}{2}J\nabla_X(JEY) + \tfrac{1}{2}\nabla_X(EY) - \tfrac{1}{2}EJ(\nabla_X J)(Y) =$$
$$= -\tfrac{1}{2}JE\,\nabla_X(JY) + \tfrac{1}{2}E\nabla_X Y - \tfrac{1}{2}EJ(\nabla_X J)(Y) =$$
$$= -\tfrac{1}{2}JE(\nabla_X J)(Y) - \tfrac{1}{2}E\,\nabla_X Y + \tfrac{1}{2}E\,\nabla_X Y - \tfrac{1}{2}EJ(\nabla_X J)(Y) = 0\,.$$

Thus we have proved the existence of a torsionless connection with respect to which the tensor fields $J$ and $E$ are parallel. But because $T = -J(E + I)$, the tensor field $T$ is parallel with respect to this connection, too. This proves the existence assertion of the proposition.

In order to prove the uniqueness assertion let us consider two torsionless connections $\nabla$ and $\nabla'$ such that $J$ and $T$ are parallel with respect to both of them. We can write $\nabla'_X Y = \nabla_X Y - U(X, Y)$. The tensor field $U$ obviously satisfies

$$U(X, Y) = U(Y, X)\,, \quad U(X, JY) = JU(X, Y)\,, \quad U(X, TY) = TU(X, Y)\,.$$

We notice moreover that

$$JTU(X, Y) = JU(X, TY) = JU(TY, X) = U(TY, JX) = U(JX, TY) =$$
$$= TU(JX, Y) = TU(Y, JX) = TJU(Y, X) = TJU(X, Y)\,.$$

Using this and Lemma 3 we find

$$U = JTU + TJU = JTJTU + TJTJU = (JTTJ + TJJT)\,U = 0$$

which completes the proof of the proposition.

For the formulation of the main theorem of this section we shall need a certain 3-linear mapping, which is very similar to the mapping $B$ from the previous section. Let us suppose that a $(J, T)_3$-structure satisfies $[J, J] = [J, T] = [T, T] = 0$. Then according to Proposition 6 there exists a unique torsionless connection $\nabla$ with $\nabla T = \nabla J = 0$. Let us denote by $R$ its curvature tensor. We introduce a 3-linear mapping

$$C : D_2 \times D_1 \times D_1 \to D_1$$

by $C(X, Y, Z) = R(X, Y)\,Z$. Now we can already formulate the main

**Theorem 3.** *A $(J, T)_3$-structure is integrable if and only if the following conditions are satisfied*:

(i) $[J, J] = [J, T] = [T, T] = 0$,
(ii) $C = 0$.

Proof: If a $(J, T)_3$-structure on $M$ is integrable then for any point $a \in M$ there exists its open neighborhood with a chart $(x^1, \ldots, x^{2n})$ on it such that with respect to the basis $\partial/\partial x^1, \ldots, \partial/\partial x^{2n}$ the tensors $J$ and $T$ have the matrix expressions (24). Using this chart it can be very easily checked that (i) is satisfied. In order to prove (ii) let us notice first that $\nabla J = \nabla T = 0$ implies $\nabla P_1 = \nabla P_2 = 0$. Consequently We have $\nabla_X Y \in D_i$ for any $X$ and for $Y \in D_i$, $i = 1, 2$. Because $\nabla$ is torsionless we have

$$\nabla_{\partial/\partial x^i} \frac{\partial}{\partial x^{n+j}} = \nabla_{\partial/\partial x^{n+j}} \frac{\partial}{\partial x^i} = 0 .$$

Having this in mind we get

$$C\left(\frac{\partial}{\partial x^{n+i}}, \frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k}\right) = R\left(\frac{\partial}{\partial x^{n+i}}, \frac{\partial}{\partial x^j}\right)\frac{\partial}{\partial x^k} = R\left(\frac{\partial}{\partial x^{n+i}}, \frac{\partial}{\partial x^j}\right)\left(J\frac{\partial}{\partial x^{n+k}}\right) =$$

$$= \nabla_{\partial/\partial x^{n+i}} \nabla_{\partial/\partial x^j}\left(J\frac{\partial}{\partial x^{n+k}}\right) - \nabla_{\partial/\partial x^j} \nabla_{\partial/\partial x^{n+i}}\left(J\frac{\partial}{\partial x^{n+k}}\right) =$$

$$= J\left(\nabla_{\partial/\partial x^{n+i}} \nabla_{\partial/\partial x^j} \frac{\partial}{\partial x^{n+k}} - \nabla_{\partial/\partial x^j} \nabla_{\partial/\partial x^{n+i}} \frac{\partial}{\partial x^{n+k}}\right) = -J\,\nabla_{\partial/\partial x^j} \nabla_{\partial/\partial x^{n+i}} \frac{\partial}{\partial x^{n+k}} =$$

$$= -J\,\nabla_{\partial/\partial x^j} \nabla_{\partial/\partial x^{n+i}}\left(T\frac{\partial}{\partial x^k}\right) = -JT\nabla_{\partial/\partial x^j} \nabla_{\partial/\partial x^{n+i}} \frac{\partial}{\partial x^k} = 0$$

which proves (ii).

Conversely, let us suppose that the conditions (i) and (ii) are satisfied. We know from the proof of Proposition 6 that (i) implies the integrability of the almost product structure $(D_1, D_2)$. Thus for any point $a$ of $M$ we can find its open neighborhood $U_1$ with a chart $(x^1, \ldots, x^{2n})$ on it such that $\partial/\partial x^1, \ldots, \partial/\partial x^n$ and $\partial/\partial x^{n+1}, \ldots, \partial/\partial x^{2n}$ are local bases of $D_1$ and $D_2$ on $\mathcal{U}_1$, respectively. There are uniquely determined functions $a_i^j$, $i, j = 1, \ldots, n$ such that

$$T\left(a_i^j \frac{\partial}{\partial x^j}\right) = \frac{\partial}{\partial x^{n+i}}\,; \quad i = 1, \ldots, n .$$

We denote by $(b_i^j)$ the inverse matrix to the matrix $(a_i^j)$.

First we shall consider the differential system

$$\frac{\partial \varphi_i^j}{\partial x^k} = -b_i^l \frac{\partial a_l^p}{\partial x^k} \varphi_p^j\,; \quad i, j, k = 1, \ldots, n . \tag{27}$$

The coefficients $b_i(\partial a_l^p/\partial x^k)$ do not depend on the variables $x^{n+1}, \ldots, x^{2n}$. In order to prove this we can write

$$0 = \nabla_{\partial/\partial x^k}\left(\frac{\partial}{\partial x^{n+l}}\right) = \nabla_{\partial/\partial x^k} T\left(a_l^p \frac{\partial}{\partial x^p}\right) = T\nabla_{\partial/\partial x^k}\left(a_l^p \frac{\partial}{\partial x^p}\right) =$$

$$= T\left(\frac{\partial a_l^p}{\partial x^k}\frac{\partial}{\partial x^p} + a_l^p \Gamma_{kp}^j \frac{\partial}{\partial x^j}\right) = \left(\frac{\partial a_l^p}{\partial x^k} + a_l^j \Gamma_j^p\right) T\frac{\partial}{\partial x^p}$$

which implies

$$\Gamma^p_{ki} = -b^l_i \frac{\partial a^p_l}{\partial x^k} .$$

By virtue of (ii) we get

$$0 = C\left(\frac{\partial}{\partial x^{n+j}}, \frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^i}\right) = \nabla_{\partial/\partial x^{n+j}} \nabla_{\partial/\partial x^k} \frac{\partial}{\partial x^i} = \nabla_{\partial/\partial x^{n+j}} \left(\Gamma^p_{ki} \frac{\partial}{\partial x^p}\right) =$$

$$= \left(\frac{\partial}{\partial x^{n+j}} \Gamma^p_{ki}\right) \frac{\partial}{\partial x^p}$$

and therefore

$$\frac{\partial}{\partial x^{n+j}} \left(b^l_i \frac{\partial a^p_l}{\partial x^k}\right) = 0 .$$

This enables us to consider the system (27) as a system in variables $x^1, \ldots, x^n$ only. Considering the equation

$$\frac{\partial}{\partial x^h} \left(b^l_i \frac{\partial a^p_l}{\partial x^k} \varphi^j_p\right) = \frac{\partial}{\partial x^k} \left(b^l_i \frac{\partial a^p_l}{\partial x^h} \varphi^j_p\right)$$

we find easily that the system (27) is involutive. Therefore we can find solutions $\varphi^j_i(x^1, \ldots, x^n)$, $i, j = 1, \ldots, n$ of (27) defined on an open neighborhood $\mathcal{U}_2 \subseteq \mathcal{U}_1$ of $a$ and satisfying $\det(\varphi^j_i) \neq 0$ on $\mathcal{U}_2$. We shall investigate now the difference

$$\frac{\partial \varphi^i_j}{\partial x^k} - \frac{\partial \varphi^i_k}{\partial x^j} = \left(b^l_j \frac{\partial a^p_l}{\partial x^k} - b^l_k \frac{\partial a^p_l}{\partial x^j}\right) \varphi^i_p .$$

One can easily see that this difference vanishes if and only if

$$a^l_k \frac{\partial a^j_i}{\partial x^l} - a^l_i \frac{\partial a^j_k}{\partial x^l} = 0 .$$

We shall show now that the above identity is a consequence of the assumption $[J, J] = 0$. For

$$X = a^l_k \frac{\partial}{\partial x^l}, \quad Y = a^l_i \frac{\partial}{\partial x^l}$$

we get

$$0 = [J, J](X, Y) = [JX, JY] - [X, Y] - J[JX, Y] - J[X, JY] =$$

$$= [TX, TY] - [X, Y] - T[TX, Y] = T[X, TY] =$$

$$= \left[\frac{\partial}{\partial x^{n+k}}, \frac{\partial}{\partial x^{n+i}}\right] - \left[a^l_k \frac{\partial}{\partial x^l}, a^l_i \frac{\partial}{\partial x^l}\right] - T\left[\frac{\partial}{\partial x^{n+k}}, a^l_i \frac{\partial}{\partial x^l}\right] -$$

$$- T\left[a^l_k \frac{\partial}{\partial x^l}, \frac{\partial}{\partial x^{n+i}}\right] = -\left(a^l_k \frac{\partial a^j_i}{\partial x^l} - a^l_i \frac{\partial a^j_k}{\partial x^l}\right) \frac{\partial}{\partial x^j} - \left(\frac{\partial a^j_i}{\partial x^{n+k}} - \frac{\partial a^j_k}{\partial x^{n+k}}\right) T \frac{\partial}{\partial x^j}$$

which implies

(28) $$a_k^l \frac{\partial a_i^j}{\partial x^l} - a_i^l \frac{\partial a_k^j}{\partial x^l} = 0 \,,$$

(29) $$\frac{\partial a_i^j}{\partial x^{n+k}} - \frac{\partial a_k^j}{\partial x^{n+i}} = 0 \,.$$

Thus we have proved that $\partial\varphi_j^i/\partial x^k - \partial\varphi_k^i/\partial x^j = 0$ and therefore on an open neighborhood $\mathcal{U}_3 \subseteq \mathcal{U}_2$ of $a$ there exist functions $h^i(x^1, \ldots, x^n)$, $i = 1, \ldots, n$ such that

$$\frac{\partial h^i}{\partial x^j} = \varphi_j^i \,; \quad i, j = 1, \ldots, n \,.$$

Our next task is to investigate the functions $a_i^k(\partial h^j/\partial x^h)$; $i, j = 1, \ldots, n$. By virtue of (27) we get

$$\frac{\partial}{\partial x^l}\left(a_i^k \frac{\partial h^j}{\partial x^k}\right) = \frac{\partial}{\partial x^l}\left(a_i^k \varphi_k^j\right) = a_i^k \frac{\partial \varphi_k^j}{\partial x^l} + \frac{\partial a_i^k}{\partial x^l}\varphi_k^j = - a_i^k b_k^p \frac{\partial a_p^q}{\partial x^l}\varphi_q^j + \frac{\partial a_i^k}{\partial x^l}\varphi_k^j = 0$$

which shows that our functions do not depend on the variables $x^1, \ldots, x^n$. Moreover, using (29) we get

$$\frac{\partial}{\partial x^{n+l}}\left(a_i^k \frac{\partial h^j}{\partial x^k}\right) - \frac{\partial}{\partial x^{n+i}}\left(a_l^k \frac{\partial h^j}{\partial x^k}\right) = \left(\frac{\partial a_i^k}{\partial x^{n+l}} - \frac{\partial a_l^k}{\partial x^{n+i}}\right)\frac{\partial h^j}{\partial x^k} = 0$$

which implies the existence of functions $h^{n+i}(x^{n+1}, \ldots, x^{2n})$, $i = 1, \ldots, n$ defined on an open neighborhood $\mathcal{U}_4 \subseteq \mathcal{U}_3$ of $a$ satisfying

(30) $$\frac{\partial h^{n+j}}{\partial x^{n+i}} = a_i^k \frac{\partial h^j}{\partial x^k} \,.$$

On an open neighborhood $\mathcal{U}_5 \subseteq \mathcal{U}_4$ of $a$ we introduce a new chart $('x', \ldots, 'x^{2n})$ by

$$'x^i = h^i(x^1, \ldots, x^n) \,, \quad 'x^{n+i} = h^{n+i}(x^{n+1}, \ldots, x^{2n}) \,, \quad i = 1, \ldots, n \,.$$

With respect to this new chart we get by virtue of (30)

$$a_i^k \frac{\partial h^j}{\partial x^k} T \frac{\partial}{\partial' x^j} = T\left(a_i^k \frac{\partial h^j}{\partial x^k}\frac{\partial}{\partial' x^j}\right) = T\left(a_i^k \frac{\partial}{\partial x^k}\right) = \frac{\partial}{\partial x^{n+i}} =$$

$$= \frac{\partial h^{n+j}}{\partial x^{n+i}}\frac{\partial}{\partial' x^{n+j}} = a_i^k \frac{\partial h^j}{\partial x^k}\frac{\partial}{\partial' x^{n+j}}$$

which implies

$$T\frac{\partial}{\partial' x^i} = \frac{\partial}{\partial' x^{n+i}} \,.$$

This together with the obvious identity $T(\partial/\partial' x^{n+i}) = 0$ shows that the matrix expression of $T$ with respect to the basis $\partial/\partial' x^1, \ldots, \partial/\partial' x^{2n}$ is exactly the second matrix from (24). Finally, using Lemma 4 we get

$$J \frac{\partial}{\partial' x^i} = -T \frac{\partial}{\partial' x^i} = - \frac{\partial}{\partial' x^{n+i}}$$

and consequently $J(\partial/\partial' x^{n+i}) = \partial/\partial' x^i$ thus showing that the matrix expression of $J$ with respect to the basis $\partial/\partial' x^1, \ldots, \partial/\partial' x^{2n}$ is the first matrix from (24). This completes the proof of the theorem.

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