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DISTRIBUTIVITY OF INTERVALS OF TORSION RADICALS

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Let $\mathcal{R}$ be the class of all torsion radicals of lattice ordered groups [4] (for the definitions cf. § 1 below) and let $\mathcal{G}$ be the class of all lattice ordered groups. The class $\mathcal{R}$ is partially ordered as follows: for $\sigma_1, \sigma_2 \in \mathcal{R}$ we put $\sigma_1 \leqq \sigma_2$ if $\sigma_1(G) \subseteq \sigma_2(G)$ is valid for each $G \in \mathcal{G}$. Then $\mathcal{R}$ is a complete lattice (in the sense that for each subclass X of $\mathcal{R}$ there exist the least upper bound of X and the greatest lower bound of X in $\mathcal{R}$); moreover, the distributive law $\sigma \wedge (\sigma_1 \vee \sigma_2) = (\sigma \wedge \sigma_1) \vee (\sigma \wedge \sigma_2)$ is fulfilled in $\mathcal{R}$ (cf. [4]).

In [5], Theorem 1.5 it was asserted that $\mathcal{R}$ is completely distributive. But in [6] it was remarked that the proof of Thm. 1.5 in [5] was not complete. Hence the question whether $\mathcal{R}$ is completely distributive remained open.

For $\sigma_1, \sigma_2 \in \mathcal{R}$ with $\sigma_1 \leqq \sigma_2$ we denote by $[\sigma_1, \sigma_2]$ the class of all $\sigma \in \mathcal{R}$ such that $\sigma_1 \leqq \sigma \leqq \sigma_2$. Let $\bar{0}$ be the zero torsion radical (i.e., $\bar{0}(G) = \{0\}$ for each $G \in \mathcal{G}$). In this paper the following results will be established:

There exists $\sigma \in \mathcal{R}$ such that the interval $[\bar{0}, \sigma]$ is not infinitely distributive; thus $\mathcal{R}$ fails to be completely distributive. Let $\mathcal{R}_d$ be the class of all $\sigma \in \mathcal{R}$ having the property that $[\bar{0}, \sigma]$ is completely distributive. Then $\mathcal{R}_d$ possesses the greatest element and the class $\mathcal{R}_d$ is a proper class. (In fact, a slightly more general result will be obtained.) *Let $\sigma \in \mathcal{R}$ and suppose that the torsion class corresponding to σ is generated by linearly ordered groups. Then $[\bar{0}, \sigma]$ is completely distributive.*

1. PRELIMINARIES

For the basic notions and notations. cf. Conrad [1] and Fuchs [2]. We recall some definitions concerning torsion radicals that will be needed in the sequel.

For $G \in \mathcal{G}$ let $c(G)$ be the system of all convex l-subgroups of G; $c(G)$ is partially ordered by inclusion. Then $c(G)$ is a complete lattice. The lattice operations in $c(G)$ are denoted by $\bigwedge, \bigvee$.

Let σ be a mapping of $\mathcal{G}$ into $\mathcal{G}$ such that the following conditions are fulfilled for each $G \in \mathcal{G}$:

(i) $\sigma(G) \in c(G)$;
(ii) if $G_1 \in c(G)$, then $\sigma(G_1) = \sigma(G) \cap G_1$;
(iii) if φ is a homomorphism of G into a lattice ordered group G_1, then $\varphi(\sigma(G)) = \sigma(\varphi(G))$.

Under these assumptions σ is said to be a *torsion radical.*

A nonempty class C of lattice ordered groups is called a *torsion class* if it has the following properties:

(a) $G \in C$ and $G_1 \in c(G)$ implies that $G_1 \in C$;
(b) if $G \in \mathcal{G}$ and if $\{G_i\}_{i\in I} \subseteq c(G) \cap C$, then $\bigvee_{i\in I} G_i \in C$;
(c) the class C is closed with respect to homomoiphisms.

Let $\sigma \in \mathcal{R}$. We denote by $C^0(\sigma)$ the class of all $G \in \mathcal{G}$ with $\sigma(G) = G$. Then C^0 is a one-to-one mapping of the class $\mathcal{R}$ onto class Rad consisting of all radical classes; moreover, for each pair $\sigma_1, \sigma_2 \in \mathcal{R}$ we have

(1) $$\sigma_1 \leqq \sigma_2 \Leftrightarrow C^0(\sigma_1) \subseteq C^0(\sigma_2)\,.$$

Hence if we consider Rad as a partially ordered class (with respect to inclusion), then Rad is isomorphic to $\mathcal{R}$; thus, from the fact that R is a complete lattice [4] it follows that Rad is a complete lattice as well.

Let $\mathcal{R}_1 = \{\sigma_i\}_{i\in I}$ be a nonempty subclass of $\mathcal{R}$. For each $G \in \mathcal{G}$ we put

$$\sigma_1(G) = \bigvee_{i\in I} \sigma_i(G)\,, \quad \sigma_2(G) = \bigwedge_{i\in I} \sigma_i(G)\,.$$

Then in the lattice $\mathcal{R}$ we have $\sigma_1 = \bigvee_{i\in I} \sigma_i$, $\sigma_2 = \bigwedge_{i\in I} \sigma_i(G)$. For each $\tau \in R$ the relation

(2) $$\tau \wedge \left(\bigvee_{i\in I} \sigma_i\right) = \bigvee_{i\in I} (\tau \wedge \sigma_i)$$

is valid. (Cf. [4].) From (2) it follows that $\mathcal{R}$ is distributive.

In view of (1), the identity analogous to (2) is valid for the lattice Rad. It is easy to verify that if $\{C_i\}_{i\in I}$ is a nonempty subclass of Rad, then $\bigwedge_{i\in I} C_i = \bigcap_{i\in I} C_i$.

Let A be a subclass of Rad. The intersection of all torsion classes B with $A \subseteq B$ will be said to be the torsion class generated by A and it will be denoted by $T(A)$. For each $G \in \mathcal{G}$ we denote $T(\{G\}) = G^\wedge$.

2. THE LATTICE ORDERED GROUP H

In this section it will be shown that the relation analogous to the dual of (2) does not hold in general in the lattice Rad.

Let A be a nonempty class of lattice ordered groups. Suppose that A is closed with respect to isomorphisms. Let us denote by

$S_c(A)$ – the class of all lattice ordered groups H' such that H' is a convex l-subgroup of some lattice ordered group belonging to A;

$H(A)$ – the class of all homomorphic images of lattice ordered groups belonging to A;

$l(A)$ – the class of all lattice ordered groups H' that can be expressed as $H' = \bigcup_{i\in I} H_i$, where $\{H_i\}_{i\in I} \subseteq c(H)$ and the system $\{H_i\}_{i\in I}$ (partially ordered by inclusion) is a chain;

$u(A)$ – the class of all lattice ordered groups H' that can be written as $H' = \bigvee_{i\in I} H_i$, where $\{H_i\}_{i\in I} \subseteq c(H) \cap A$.

2.1. Proposition. (Cf. [3], 2.4.) *Let $A \neq \emptyset$ be a class of lattice ordered groups. Then $T(A) = u(H(S_c(A)))$.*

2.2. Proposition. (Cf. [3], Thm. 2.6.) *Let $A \neq \emptyset$ be a class of linearly ordered groups. Let $G \in \mathcal{G}$. Then the following conditions are equivalent:*

(α) $G \in T(A)$,

(β) G can be expressed as $G = \sum_{j\in J} G_j$, where each G_j belongs to $l(H(S_c(A)))$.

Let N^0 and Q be the additive group of all integers or of all rationals, respectively (under the natural linear order). Let N be the set of all positive integers and for each $n \in N$ let G_n be an l-subgroup of Q such that (i) G_n fails to be isomorphic to G_m whenever n and m are distinct positive integers, (ii) $1 \in G_n$ for each $n \in N$, and (iii) $G_1 = N^0$. (Such a system $\{G_n\}_{n\in N}$ obviously does exist.) Put $G_0 = \prod_{n\in N} G_n$. Let $g_0 \in G_0$ with $g_0(n) = 1$ for each $n \in N$. Further, let H be the subgroup of the group G_0 generated by the set $\{g_0\} \cup (\sum_{n\in N} G_n)$. Under the induced partial order, H is an l-subgroup of G_0.

For $n \in N$ let $B_n = \{g \in H : g(m) = 0 \text{ for each } m \leqq n\}$. Put $A_0 = \sum_{n\in N} G_n$. Then A_0 and B_n $(n = 1, 2, \ldots)$ are convex l-subgroups of H. For each $n \in N$ we have $A_0 \vee B_n = H$. This implies

$$A_0^\wedge \vee B_n^\wedge = H^\wedge \quad \text{for each} \quad n \in N\,,$$

hence

$$\bigwedge_{n\in N} (A_0^\wedge \vee B_n^\wedge) = H^\wedge\,. \tag{3}$$

Let $n \in N$ be fixed. Put $N_n = \{m \in N : m > n\}$, $\{B_n\} = A$. The lattice ordered group B_n is a direct factor of H; let $g_n = g_n[B_n]$ be the component of g_0 in B_n (thus $g_n(k) = 1$ for $k > n$ and $g_n(k) = 0$ otherwise). Then g_n is a strong unit of B_n. For a convex l-subgroup X of B_n we put

$$N(X) = \{k \in N : \text{there is } x \in X \text{ with } x(k) \neq 0\}\,.$$

From the definition of B_n we immediately infer:

2.3. Lemma. *Let $X \in c(B_n)$. If $g_n \in X$, then $X = B_n$. If $g_n \notin X$, then $X = \sum_{m\in N(X)} G_m$.* (If $M \neq \emptyset$, then $\sum_{m\in M} G_m$ is understood to be the zero group $\{0\}$.)

As a consequence of 2.3 we obtain:

2.4. Lemma. *Let $X \in c(B_n)$ and let K be an l-ideal in X with $X/K \neq \{0\}$. Then one of the following possibilities holds*: (ii) *X/K is isomorphic with the subgroup of H generated by the set $\{g_n\} \cup (\sum_{m \in N(X) \setminus N(K)} G_m)$ (under the induced partial order), or* (ii) *X/K is isomorphic with $\sum_{m \in N(X) \setminus N(K)} G_m$.*

2.4.1. Remark. If we put $X = B_n$, $K = \sum_{m \in N_n} G_m$, then X/K is isomorphic to G_1.

2.5. Corollary. *Let $\{0\} \neq Y \in H(S_c(A))$. Assume that Y fails to be isomorphic with G_1. Then* (i) *there exist $Y' \in c(Y)$ and $m \in N_n$ such that Y' is isomorphic with G_m, and* (ii) *if $k \in N \setminus N_n$, $k > 1$, then no convex l-subgroup of Y is isomorphic with G_k.*

2.6. Lemma. *Let $Z \in u(H(S_c(A)))$. Assume that Z is not isomorphic with G_1. There exists $Z' \in c(Z)$ and $m \in N_n$ such that Z' is isomorphic to G_m. If $k \in N$, $1 < k \leqq n$, then no convex l-subgroup of Z is isomorphic with G_k.*

Proof. The first assertion immediately follows from 2.5. Let $1 < k \in N$, $k \leqq n$ and suppose that Z' is a convex l-subgroup of Z isomorphic with G_k. There exists a set $\{Y_i\}_{i \in I} \subseteq c(Z) \cap H(S_c(A))$ with $Z = \bigvee_{i \in I} Y_i$. Hence

$$Z' = Z' \wedge Z = \bigvee_{i \in I} (Z' \wedge Y_i) .$$

Thus there is $i \in I$ such that $Z' = Z' \wedge Y_i$ and therefore $Z' \in c(Y_i)$, which contradicts 2.5 (ii).

2.7. Lemma. $\bigwedge_{k \in N} B_k^{\wedge} = \{\{0\}\} \cup G_1^{\wedge}$.

Proof. Clearly $\{0\} \in \bigwedge_{k \in N} B_k^{\wedge}$. From 2.4.1 it follows that $G_1 \in B_k^{\wedge}$ for each $k \in N$. Assume that there exists $Z \in \bigwedge_{k \in N} B_k^{\wedge} = \bigcap_{k \in N} B_n^{\wedge}$ such that $Z \neq \{0\}$ and $Z \notin G_1^{\wedge}$. Then Z is not isomorphic with G_1. Choose $n \in N$. In view of 2.1 and 2.6 (i) there is $m \in N$ with $m > n$ such that G_m is isomorphic with some $Z' \in c(Z)$. Because of $Z \in B_m^{\wedge}$ we have a contradiction with 2.6 (ii).

2.8. Lemma. *Let $\{0\} \neq G \in \mathscr{G}$. Then the following conditions are equivalent*: (i) *$G \in A_0^{\wedge}$*; (ii) *G can be expressed as a direct sum of lattice ordered groups G^j ($j \in J$) such that for each $j \in J$, G^j is isomorphic to some G_n with $n \in N$.*

This follows from 2.2 and from the fact that for each $n \in N$, G_n has only trivial convex l-subgroups.

2.9. Corollary. $\bigwedge_{n \in N} (A_0^{\wedge} \vee B_n^{\wedge}) \neq A_0^{\wedge} \vee (\bigwedge_{n \in N} B_n^{\wedge})$.

Proof. In view of 2.7 and 2.8 we have $A_0^{\wedge} \vee (\bigwedge_{n \in N} B_n^{\wedge}) = A_0^{\wedge}$; moreover, 2.8 implies that H does not belong to $A_0^{\wedge}$. Now it suffices to apply (3).

As a consequence of 2.9 we infer:

2.10. Proposition. *The lattice* Rad *(and hence also the lattice $\mathscr{R}$) fails to be infinitely distributive.*

Since complete distributivity (cf. § 3 below) implies infinite distributivity, we have

2.11. Corollary. *The lattices* Rad *and* $\mathscr{R}$ *fail to be completely distributive.*

3. HIGHER DEGREES OF DISTRIBUTIVITY

Let $\sigma_1, \sigma_2 \in \mathscr{R}$, $\sigma_1 < \sigma_2$. We can consider the following conditions for the interval $[\sigma_1, \sigma_2]$ of $\mathscr{R}$:

(2′) If $\tau \in [\sigma_1, \sigma_2]$ and if $\{\sigma_i\}_{i\in I}$ is a subclass of $[\sigma_1, \sigma_2]$, then
(*) $\tau \vee (\bigwedge_{i\in I} \sigma_i) = \bigwedge_{i\in I} (\tau \vee \sigma_i)$.

(2″) If $\tau \in [\sigma_1, \sigma_2]$ and if $\{\sigma_i\}_{i\in I}$ is a set, $\{\sigma_i\}_{i\in I} \subseteq [\sigma_1, \sigma_2]$, then the relation (*) is valid.

If σ_1 covers σ_2, then obviously (2′) holds. (The set of prime intervals in $\mathscr{R}$ is infinite; cf. e.g. [3], Propos .4.4.)

3.1. Lemma. *Let* $\sigma_1, \sigma_2 \in \mathscr{R}$, $\sigma_1 < \sigma_2$. *The conditions* (2′) *and* (2″) *are equivalent.*

Proof. We obviously have (2′) ⇒ (2″). Assume that (2′) fails to hold. Hence there is a subclass $\{\tau\} \cup \{\sigma_i\}_{i\in I}$ of $[\sigma_1, \sigma_2]$ such that (*) does not hold. Thus there exists $G \in \mathscr{G}$ with

$$(\tau \vee (\bigwedge_{i\in I} \sigma_i))(G) \subset (\bigwedge_{i\in I} (\tau \vee \sigma_i))(G) .$$

There is a set $J \subseteq I$ such that

$$(\tau \vee (\bigwedge_{i\in I} \sigma_i))(G) = (\tau \vee (\bigwedge_{i\in J} \sigma_i))(G) ,$$
$$(\bigwedge_{i\in I} (\tau \vee \sigma_i))(G) = (\bigwedge_{i\in J} (\tau \vee \sigma_i))(G) .$$

Therefore

$$\tau \vee (\bigwedge_{i\in J} \sigma_i) \neq \bigwedge_{i\in J} (\tau \vee \sigma_i) ,$$

which implies that (2″) does not hold.

The interval $[\sigma_1, \sigma_2]$ is called completely distributive, if, whenever $\{o_{s,t}\}_{s\in S, t\in T}$ is a subclass of $[\sigma_1, \sigma_2]$, then

$$\bigwedge_{s\in S} \bigvee_{t\in T} \sigma_{st} = \bigvee_{\varphi\in T^S} \bigwedge_{s\in S} \sigma_{s,\varphi(s)} \tag{4}$$

holds and also the relation dual to (4) is valid.

By analogous reasoning as in the proof of 3.1 we can verify that $[\sigma_1, \sigma_2]$ is completely distributive if and only if the above condition is valid for the case when S and T are sets.

Let α be an finite cardinal. If the above condition is fulfilled whenever S and T are sets with card $S \leqq \alpha$, card $T \leqq \alpha$, then $[\sigma_1, \sigma_2]$ is called *α-distributive.*

Let $\mathscr{R}_d$, $\mathscr{R}_D$ and $\mathscr{R}_\alpha$ be the class of all torsion radicals σ such that the interval $[\bar{0}, \sigma]$ is completely distributive, infinitely distributive or α-distributive, respectively.

3.2. Proposition. *Let $\beta \in \{d, D, \alpha\}$. Then $\mathscr{R}_\beta$ possesses the greatest element.*

Proof. Let $\beta = d$, $\mathscr{R}_d = \{\sigma_i\}_{i \in I}$, $\sigma_d = \bigvee_{i \in I} \sigma_i$. We have to verify that σ_d belongs to $\mathscr{R}_d$. By way of contradiction, assume that $[\bar{0}, \sigma_d]$ fails to be completely distributive. Hence there are sets S, T and torsion radicals $\sigma_{s,t}(s \in S,\ t \in T)$ belonging to $[\bar{0}, \sigma_d]$ such that either the relation (4) or the relation dual to (4) fails to hold. Assume that (4) does not hold (the dual case is analogous). Therefore,

$$(5) \qquad \sigma_u = \bigvee_{\varphi \in T^S} \bigwedge_{s \in S} \sigma_{s,\varphi(s)} < \bigwedge_{s \in S} \bigvee_{t \in T} \sigma_{st} = \sigma_v \,.$$

Clearly $\sigma_u, \sigma_v \in [\bar{0}, \sigma_d]$, hence according to (2) we have

$$(6) \qquad \sigma_u = \sigma_u \wedge \sigma_d = \bigvee_{i \in I} (\sigma_u \wedge \sigma_i)\,, \quad \sigma_v = \sigma_v \wedge \sigma_d = \bigvee_{i \in I} (\sigma_v \wedge \sigma_i)\,.$$

Next we infer from (5) (in view of (2)) that

$$\sigma_u \wedge \sigma_i = \bigvee_{\varphi \in T^S} \bigwedge_{s \in S} (\sigma_{s,\varphi(s)} \wedge \sigma_i)\,,$$

$$\sigma_v \wedge \sigma_i = \bigwedge_{s \in S} \bigvee_{t \in T} (\sigma_{st} \wedge \sigma_i)\,.$$

Because $[\bar{0}, \sigma_i]$ is completely distributive, we have $\sigma_u \wedge \sigma_i = \sigma_v \wedge \sigma_i$ for each $i \in I$, whence with respect to (6) we infer that $\sigma_u = \sigma_v$, which contradicts (5). The proofs for $\beta = D$ and $\beta = \alpha$ are analogous.

Clearly $\mathscr{R}_d \subseteq \mathscr{R}_\alpha \subseteq \mathscr{R}_d$ is valid for each infinite cardinal α. Next, each two-element lattice is completely distributive. According to Proposition 4.4 in [3] the class of all torsion radicals covering $\bar{0}$ is infinite. Hence $\mathscr{R}_d$, $\mathscr{R}_\alpha$ and $\mathscr{R}_D$ are infinite classes.

4. TORSION CLASSES GENERATED BY LINEARLY ORDERED GROUPS

A torsion class A is said to be *generated by linearly ordered groups* if there exists a class $X \subset \mathscr{G}$ such that $A = T(X)$ and each lattice ordered group belonging to X is linearly ordered. We also say that the torsion radical $C^0(A)$ is *generated by linearly ordered groups.*

In this section it will be shown that if $\{\sigma_{ts}\}_{t \in T, s \in S}$ are torsion radicals generated by linearly ordered groups, then the relation (4) and the relation dual to (4) are valid. In other words this can be expressed as follows: Let X_0 be the class of all linearly ordered groups; then $C^0(T(X_0)) \in \mathscr{R}_d$.

For each class $B \subseteq \mathscr{G}$ we denote $L(B) = B \cap X_0$.

Let C_{ts} $(t \in T,\ s \in S)$ be torsion classes.

4.1. Lemma. $\bigcap_{t \in T} l(\bigcup_{s \in S} L(C_{ts})) \subseteq l(\bigcup_{\varphi \in S^T} (\bigcap_{t \in T} L(C_{t,\varphi(t)})))$.

Proof. Denote

$$A = \bigcap_{t \in T} l(\bigcup_{s \in S} L(C_{ts}))\,, \quad B = l(\bigcup_{\varphi \in S^T} (\bigcap_{t \in T} L(C_{t,\varphi(t)})))\,.$$

Then

$$(7) \qquad B = l(\bigcap_{t\in T} \bigcup_{s\in S} L(C_{ts}))\,.$$

Let $R \in A$. For each $t \in T$ there exist linearly ordered groups R_{tj} $(j \in J_t)$ belonging to $\bigcup_{s\in S} L(C_{ts})$ such that

$$(8) \qquad R = \bigcup R_{tj}\,(j \in J_t)\,.$$

Now we distinguish two cases.

a) Assume that there exists $t \in T$ such that $R_{tj} \neq R$ for each $j \in J_t$. Let $t' \in T$, $j \in J_t$. From

$$R = \bigcup R_{t'j'}\,(j' \in J_{t'})$$

it follows that there exists $j' \in J_{t'}$ with $R_{tj} \subseteq R_{t'j'}$, whence $R_{tj} \in \bigcup_{s\in S} L(C_{t's}))$. Therefore in view of (7) and (8) we infer that R belongs to B.

b) Assume that there exists no $t \in T$ fulfilling $R_{tj} \neq R$ for each $j \in J_t$. Hence $R \in \bigcup_{s\in S} L(C_{ts}))$ holds for each $t \in T$. This together with (7) implies $R \in B$.

4.2. Lemma. *Let C_t $(t \in T)$ be torsion classes generated by linearly ordered groups. Then $L(\bigwedge_{t\in T} C_t) = \bigcap_{t\in T} L(C_t)$ and $L(\bigvee_{t\in T} C_t) = l(\bigcup_{t\in T} L(C_t))$.*

Proof. The first assertion is a consequence of $\bigwedge_{t\in T} C_t = \bigcap_{t\in T} C_t$. The second assertion follows from 2.2 and from the fact that each linearly ordered group is directly indecomposable.

4.3. Lemma. *Let C_{ts} $(t \in T, s \in S)$ be torsion classes generated by linearly ordered groups. Then $\bigwedge_{t\in T} \bigvee_{s\in S} C_{ts} = \bigvee_{\varphi\in S^T} \bigwedge_{t\in T} C_{t,\varphi(t)}$.*

Proof. Denote $A_1 = \bigwedge_{t\in T} \bigvee_{s\in S} C_{ts}$, $B_1 = \bigvee_{\varphi\in S^T} \bigwedge_{t\in T} C_{t,\varphi(t)}$. We obviously have $B_1 \subseteq A_1$. Further, both A_1 and B_1 are generated by linearly ordered groups. Hence for proving that $A_1 \subseteq B_1$ holds it suffices to verify that $L(A_1) \subseteq L(B_1)$ is valid. Let A and B be as above. According to 4.2, $L(A_1) = A$ and $L(B_1) = B$. Thus in view of 4.1 we infer that $L(A_1) \subseteq L(B_1)$ is valid.

From 4.1, 4.3 and (7) we immediately obtain:

4.4. Lemma. *Let C_{ts} $(t \in T, s \in S)$ be torsion classes generated by linearly ordered groups. Then $\bigcap_{t\in T} l(\bigcup_{s\in S} L(C_{ts})) = l(\bigcap_{t\in T} \bigcup_{s\in S} L(C_{ts}))$.*

4.5. Lemma. *Let C_{ts} $(t \in T, s \in S)$ be torsion classes generated by linearly ordered groups. Then $\bigvee_{t\in T} \bigwedge_{s\in S} C_{ts} = \bigwedge_{\varphi\in S^T} \bigvee_{t\in T} C_{t,\varphi(t)}$.*

Proof. Denote $A_1 = \bigvee_{t\in T} \bigwedge_{s\in S} C_{ts}$, $B_1 = \bigwedge_{\varphi\in S^T} \bigvee_{t\in T} C_{t,\varphi(t)}$. Since A_1 and B_1 are generated by linearly ordered groups, it suffices to verify that $L(B_1) = L(A_1)$

holds. According to 4.2,

$$L(A_1) = l(\bigcup_{t\in T} \bigcap_{s\in S} L(C_{ts})) = l(\bigcap_{\varphi\in S^T} \bigcup_{t\in T} L(C_{t,\varphi(t)}))\,,$$

$$L(B_1) = \bigcap_{\varphi\in S^T} l(\bigcup_{t\in T} L(C_{t,\varphi(t)}))\,.$$

Therefore in view of 4.4 we obtain $L(A_1) = L(B_1)$.

Because of the isomorphism between Rad and $\mathscr{R}$ we infer from 4.3 and 4.5:

4.6. Theorem. *Let X_0 be the class of all linearly ordered groups, $\sigma_c = C^0(T(X_0))$. Then the interval $[\bar{0}, \sigma_c]$ of R is completely distributive.*

From 2.2 it follows that $[\bar{0}, \sigma_c]$ is a proper class. Therefore we have:

4.7. Corollary. *Let $\beta \in \{d, \alpha, D\}$. Then $\mathscr{R}_\beta$ is a proper class.*

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