László Fuchs; John M. Irwin
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ON ELONGATIONS OF TOTALLY PROJECTIVE p-GROUPS BY $p^{\omega+n}$-PROJECTIVE p-GROUPS

LASZLO FUCHS, New Orleans and J. M. IRWIN, Detroit

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In this note, our goal is to classify a class of abelian p-groups that includes the totally projective p-groups and the separable $p^{\omega+n}$-projective p-groups. As is well known (see e.g. [5] or [2]), for the totally projective p-groups the Ulm invariants yield a complete system of invariants; and recently, it has been shown [3], [4] that the $p^{\omega+n}$-projective p-groups A are fully characterized by their p^n-socles $A[p^n] = \{a \in A \mid p^n a = 0\}$ as valuated abelian groups. The $p^{\omega+n}$-projective p-groups A can be defined by the property of containing a p^n-bounded (necessarily nice) subgroup P with A/P a direct sum of cyclic groups, so it is natural to investigate the class of those p-groups A that contain a p^n-bounded nice subgroup P such that A/P is totally projective. We could establish a structure theorem only on the subclass $\mathscr{G}_n$ consisting of those A for which P can be chosen so as to have no elements of infinite height, by showing that – like the $p^{\omega+n}$-projective p-groups – the groups in $\mathscr{G}_n$ can also be classified in terms of their p^n-socles, if viewed as valuated abelian groups.

Any group A in $\mathscr{G}_n$ is an extension of a totally projective p-group $p^\omega A$ by a separable $p^{\omega+n}$-projective p-group $A/p^\omega A$. Hence, in the sense of [6], A is an ω-elongation of a totally projective p-group by a separable $p^{\omega+n}$-projective p-group (but not every such elongation is a member of $\mathscr{G}_n$).

1. By a group we shall mean throughout an abelian p-group A where p is a fixed prime. For unexplained terminology and basic facts we refer to [2]. As usual, $p^\sigma A$ is defined for every ordinal σ by setting $p^{\sigma+1}A = p(p^\sigma A)$ and $p^\varrho A = \bigcap_{\sigma<\varrho} p^\sigma A$ if ϱ is a limit ordinal. We may and shall assume that A is reduced, i.e. $p^\tau A = 0$ for some ordinal τ. For $a \neq 0$ in A, the *height* $h(a)$ is σ if $a \in p^\sigma A \setminus p^{\sigma+1}A$, while $h(0) = \infty$. Then $p^\omega A$ is the set of all elements of infinite height in A; A is *separable* if $p^\omega A = 0$.

A subgroup P of A is *nice* if $p^\sigma(A/P) = (p^\sigma A + P)/P$ for every ordinal σ, i.e. if every coset of A mod P contains an element of the same height in A as the coset has in A/P. A subgroup P is necessarily nice if A/P is separable.

A *valuation* of A is a function $v : A \to \Gamma \cup \{\infty\}$ (where Γ stands for the class of ordinals and clearly, $\sigma < \infty$ for every $\sigma \in \Gamma$) such that

(i) $v(a) = \infty$ if and only if $a = 0$;

(ii) $v(ma) = v(a)$ or $> v(a)$ according as the integer m is or is not prime to p;

(iii) $v(a + b) \geqq \min(v(a), v(b))$, for all $a, b \in A$.

Two valuated groups are *isometric* if there is a value-preserving isomorphism between them.

2. We start our discussion with the following two simple lemmas.

Lemma 1. *If A is a p-group and P is a subgroup of A with $P \cap p^\omega A = 0$, then P is nice in A if and only if $G = P \oplus p^\omega A$ is nice in A.*

Suppose P is nice in A; to show G is nice in A, it suffices to show that A/G is separable. If $a + G$ $(a \in A)$ is of infinite height in A/G, then there exists a sequence $g_n \in G$ such that $h(a + g_n) \geqq n$ for every integer $n \geqq 1$. Write $g_n = x_n + b_n$ with $x_n \in P$, $b_n \in p^\omega A$; then $h(a + x_n) \geqq n$, so the coset $a + P$ has infinite height in A/P. By hypothesis, some $x \in P$ satisfies $h(a + x) \geqq \omega$ whence $a = -x + (a + x) \in P + + p^\omega A = G$, indeed. Conversely, if G is nice in A, then because of $p^\omega A \leqq G$, A/G has to be separable. It follows as before that a coset $a + P$ can have height ω in A/P only if it can be represented by an element of $p^\omega A$. This completes the proof.

Lemma 2. *If P is a p^n-bounded nice subgroup of A such that $P \cap p^\omega A = 0$ and A/P is totally projective, then*

(a) *$A/(P \oplus p^\omega A)$ is a direct sum of cyclics;*

(b) *$A/p^\omega A$ is a $p^{\omega+n}$-projective p-group.*

As P is nice in A, we have $p^\omega(A/P) = (p^\omega A + P)/P$. Hence $A/(P + p^\omega A) \cong \cong (A/P)/p^\omega(A/P)$ satisfies (a), as A/P is totally projective. Therefore $(P \oplus p^\omega A)/p^\omega A$ is a p^n-bounded subgroup of $A/p^\omega A$ modulo which the group is a direct sum of cyclics, so (b) follows.

The next result is a useful tool in recognizing the members of the class to be considered.

Lemma 3. *Let A be a p-group and P a p^n-bounded subgroup of A such that $P \cap p^\omega A = 0$. If $p^\omega A$ is totally projective and if $A/(P \oplus p^\omega A)$ is a direct sum of cyclic groups, then A/P is a totally projective p-group.*

Hypothesis implies that $P \oplus p^\omega A$ is nice in A, so by Lemma 1, P is a nice subgroup of A. Hence $p^\omega(A/P) = (P + p^\omega A)/P$ which is, because of $P \cap p^\omega A = 0$, isomorphic to $p^\omega A$. Furthermore, $(A/P)/p^\omega(A/P) = (A/P)/(P \oplus p^\omega A)/P \cong A/(P \oplus p^\omega A)$ is a direct sum of cyclics. It follows that A/P has to be totally projective.

Finally, we shall make use of two technical lemmas.

Lemma 4. *Let A be a p-group, P a p^n-bounded subgroup such that $P \cap p^\omega A = 0$. Then the relative invariants of $G = P \oplus p^\omega A$ can be computed by using $A[p^n]$ only.*

The σ-th relative invariant of G in A is the dimension of $p^\sigma A[p]/((p^{\sigma+1}A + G) \cap p^\sigma A[p])$ as a vector space over the prime field of characteristic p; cf. [2]. If $\sigma = m$ is an integer, then $p^{m+1}A + G = p^{m+1}A + P$. If $p(a + x) = 0$ where $a \in p^{m+1}A$, $x \in P$, then $p^n a = -p^n x = 0$, so $a \in p^{m+1} A[p^n]$ and $(p^{m+1}A + G)[p] = (p^{m+1} A[p^n] + P)[p]$ follows. If $\sigma \geqq \omega$, then $p^\sigma A \leqq G$ and the σ-th relative invariant is 0.

Recall that a p-group S with valuation is said to be *distinctive* (see [3]) if there is a monomorphism of S into a direct sum of cyclic p-groups that does not decrease valuation. We shall need the following result (see [3]) which is essentially a reformulation of a theorem by Dieudonné [1]:

Lemma 5. *Let G be a p-group and S a subgroup of G such that G/S is a direct sum of cyclic p-groups. If S is distinctive (equipped with the valuation given by the height function of G), then G is likewise a direct sum of cyclic groups.*

3. We now introduce the class of p-groups to be discussed.

Let $\mathcal{G}_n$ denote the class of p-groups A such that there is a p^n-bounded nice subgroup P of A, containing no elements of infinite height in A, with A/P totally projective.

It is evident that all totally projective p-groups and all separable $p^{\omega+n}$-projective p-groups as well as their direct sums belong to class $\mathcal{G}_n$. We have been unable to decide whether or not these are the only members of $\mathcal{G}_n$.

From the definition it is also clear that each class $\mathcal{G}_n$ is closed under arbitrary direct sums. We wish to show that the same holds under the formation of direct summands.

Theorem 1. *A direct summand of a group in $\mathcal{G}_n$ is again in $\mathcal{G}_n$.*

Suppose $A = B \oplus C$ belongs to $\mathcal{G}_n$, and P is a p^n-bounded nice subgroup of A such that $P \cap p^\omega A = 0$ and A/P is totally projective. Set $G_1 = B \cap (P + p^\omega A)$ and $G_2 = C \cap (P + p^\omega A)$. Then $B/G_1 \cong (B + P + p^\omega A)/(P + p^\omega A)$ is a direct sum of cyclic groups, since $A/(P + p^\omega A)$ is by Lemma 2 (a) a direct sum of cyclics. Now $G = G_1 \oplus G_2$ satisfies $p^\omega A \leqq G \leqq P + p^\omega A$, so $G = P' \oplus p^\omega A$ for $P' = P \cap G$. Furthermore, $p^\omega A = p^\omega B \oplus p^\omega C$, whence $G_1 = P_1 \oplus p^\omega B$ with $P_1 = G_1 \cap (P' \oplus p^\omega C)$. The subgroup P_1 of B is obviously p^n-bounded and has no elements of infinite height. As $p^\omega B$ is a summand of the totally projective group $p^\omega A$, we can apply Lemma 3 to B and P_1 to conclude that $B \in \mathcal{G}_n$, indeed.

4. Our main result states that the groups in class $\mathcal{G}_n$ are determined, up to isomorphisms, by their p^n-socles as valuated abelian groups.

Theorem 2. *Let $A, A' \in \mathcal{G}_n$. Then $A \cong A'$ if and only if there is a height-preserving isomorphism $\phi : A[p^n] \to A'[p^n]$.*

It is enough to establish the sufficiency of the condition. So, let ϕ be as stated. By hypothesis, there are p^n-bounded nice subgroups P and P' in A and A', respectively, such that $P \cap p^\omega A = 0$ and $P' \cap p^\omega A' = 0$, and A/P, A'/P' are totally projective. From Lemma 2 we know that A/G, A'/G' are direct sums of cyclic groups where $G = P \oplus p^\omega A$, $G' = P' \oplus p^\omega A'$.

We consider the exact sequence

$$0 \to G/H \to A/H \to A/G \to 0$$

where $H = (P \oplus p^\omega A) \cap (\phi^{-1}P' \oplus p^\omega A)$, and show that G/H is distinctive (valuation induced by the height function in A/H). Let $x + t + H = x + H$ be a coset ($x \in P$, $t \in p^\omega A$) of height $\geqq m$ in A/H, i.e. there is some $a \in A$ satisfying $p^m a - x \in H$. Thus $p^m a - x \in \phi^{-1}P' \oplus p^\omega A$, so $p^m a^* = x + \phi^{-1} y'$ for suitable $a^* \in A$, $y' \in P'$. We see that $\phi(x) + y'$ has height $\geqq m$ in A', and therefore the coset $\phi(x) + y' + G' = \phi(x) + G'$ has height $\geqq m$ in A'/G'. The map $x + H \mapsto \phi(x) + G'$ of G/H into A'/G' is easily seen to be monic, and since it does not decrease heights, G/H is distinctive, in fact. By Lemma 5, A/H is then a direct sum of cyclics.

Similarly, A'/H' with $H' = (\phi P \oplus p^\omega A') \cap (P' \oplus p^\omega A')$ is a direct sum of cyclic groups.

As ϕ preserves heights, it is clear that ϕ carries $H[p^n] = (P \oplus p^\omega A[p^n]) \cap (\phi^{-1}P' \oplus p^\omega A[p^n])$ onto $H'[p^n]$. If we set $Q = P \cap (\phi^{-1}P' \oplus p^\omega A[p^n])$, $Q' = P' \cap (\phi P \oplus p^\omega A'[p^n])$ then $H = Q \oplus p^\omega A$, $H' = Q' \oplus p^\omega A'$. From Lemma 3 we conclude that A/Q, A'/Q' are totally projective. Thus we see that P, P' can be replaced by Q, Q' which in addition satisfy: $\phi Q \oplus p^\omega A' = Q' \oplus p^\omega A'$. It follows that ϕ induces a height-preserving isomorphism $\phi_0 : Q \to Q'$.

The Ulm invariants of $p^\omega A$ and $p^\omega A'$ are the same, since these can be computed in their socles and ϕ guarantees that the results of computation are the same in $p^\omega A$ and $p^\omega A'$. These groups are totally projective, thus there is an isomorphism $\psi_0 : p^\omega A \to p^\omega A'$. Manifestly, ψ_0 has to preserve heights computed in A and A', respectively.

The isomorphisms ϕ_0 and ψ_0 give rise to an isomorphism $\psi : H \to H'$ where H and H' are nice in A and A', respectively. Since Q, Q' have elements of finite heights only and $p^\omega A$, $p^\omega A'$ have elements of heights $\geqq \omega$ only, ψ has to be height-preserving. By Lemma 4, the relative invariants of H in A can be computed in $A[p^n]$, and since ϕ carries $H[p^n]$ into $H'[p^n]$, the relative invariants of H in A are equal to those of H' in A'. It suffices to appeal to Hill's Theorem (see e.g. [2]) to conclude that $A \cong A'$, in fact.

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Authors' addresses: Laszlo Fuchs, Tulane University, New Orleans, Louisiana, U.S.A. and J. M. Irwin, Wayne State University, Detroit, Michigan, U.S.A.