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ON THE LIE ALGEBRA OF VERTICAL PROLONGATION OPERATORS

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We define a bracket of vertical prolongation operators on arbitrary fibered manifolds by means of a recently introduced operation of the so called strong difference, [2], [3], and we show that these operators constitute a Lie algebra. (Kosmann-Schwarzbach, [4], introduce such a bracket by means of the so called "linearization" of vertical differential section operators. However, one step of her construction cannot be applied to arbitrary fibered manifolds. Hence our construction is a purely geometrical treatment of the original idea by Y. Kosmann-Schwarzbach). Then we deduce some properties of a special class of vertical prolongation operators formed by generalized Lie derivatives of sections and morphisms. — All considerations are in the category C^∞.

1. Strong difference. We shall need the concept of strong difference introduced in [2] or [3]. Given the second tangent bundle $T(TM) = TTM$ of an arbitrary manifold M, we have besides the bundle projection $p_{TM}\colon TTM \to TM$ also the tangent map $Tp_M\colon TTM \to TM$ to the bundle projection of the (first) tangent bundle $p_M\colon TM \to M$. In [2] and [3] the author has proved that every two vectors $\boldsymbol{A}, \boldsymbol{B} \in TTM$ satisfying $p_{TM}(\boldsymbol{A}) = Tp_M(\boldsymbol{B})$ and $p_{TM}(\boldsymbol{B}) = Tp_M(\boldsymbol{A})$ determine a vector in TM denoted by $\boldsymbol{A} \dot{-} \boldsymbol{B}$ and called the strong difference of $\boldsymbol{A}$ and $\boldsymbol{B}$.

We apply the concept of the strong difference to the second vertical bundle VVY of an arbitrary fibered manifold Y over X. In this case we perform the construction described in [3] on each manifold Y_x (= the fiber over x). If $p\colon Y \to X$ is a fibered manifold and $q_Y\colon VY \to Y$ is its vertical bundle, we have besides the vector bundle $q_{VY}\colon VVY \to VY$ also a second vector bundle $Vq_Y\colon VVY \to VY$, where Vq_Y is the vertical tangent map to q_Y. Acording to [3], any two vectors $\boldsymbol{A}, \boldsymbol{B} \in VVY$ satisfy the conditions for the strong difference iff

$$(1) \qquad Vq_Y(\boldsymbol{A}) = q_{VY}(\boldsymbol{B})\,, \quad Vq_Y(\boldsymbol{B}) = q_{VY}(\boldsymbol{A})\,.$$

In natural local coordinates $(x^i, y^p, Y^p, \mathrm{d}y^p, \mathrm{d}Y^p)$ on VVY induced from some local coordinates (x^i, y^p) on Y, $\boldsymbol{A}, \boldsymbol{B} \in VVY$ satisfy the conditions for the strong difference iff $\boldsymbol{A} \equiv (x^i, y^p, a^p, b^p, A^p)$, $\boldsymbol{B} \equiv (x^i, y^p, b^p, a^p, B^p)$. Then $\boldsymbol{A} \dot{-} \boldsymbol{B} \equiv (x^i, y^p, A^p - B^p)$.

As a direct consequence of Theorem 1, [3], we obtain

Lemma 1. *Let $Y \to X$ and $Z \to X$ be two fibered manifolds over the same base and let $f: Y \to Z$ be a base-preserving morphism. If $A, B \in VVY$ satisfy the conditions for the strong difference, then $VVf(A)$, $VVf(B) \in VVZ$ also satisfy the conditions for the strong difference, and*

$$VVf(A) \dot{-} VVf(B) = Vf(A \dot{-} B)$$

holds.

2. Lie algebra of vertical prolongation operators. Let $Y \to X$ and $Z \to X$ be two fibered manifolds over the same base. A *differential operator A of Y into Z* is a rule transforming each section s of Y into a section As of Z. Operator A is said to be of the order r, if the value $As(x)$ depends only on the r-jet $j^r_x s$, $x \in X$. In this case we obtain an associated base-preserving morphism $\mathscr{A}: J^r Y \to Z$ which is assumed to be smooth. By definition, we have $\mathscr{A}(j^r_x s) = As(x)$.

For any r-th order differential operator A of Y into Z, we define its vertical prolongation VA which is an r-th order differential operator of $VY \to X$ into $VZ \to X$. Any section $\sigma: X \to VY$ can be expressed, at least locally, as

$$\sigma = \frac{\partial}{\partial t}\bigg|_0 s_t\,,$$

where s_t is a one-parameter family of sections of Y. Then we put

$$VA(\sigma) = VA\left(\frac{\partial}{\partial t}\bigg|_0 s_t\right) = \frac{\partial}{\partial t}\bigg|_0 As_t\,. \tag{2}$$

To demonstrate the global character of this definition, we use the associated morphism $\mathscr{A}: J^r Y \to Z$. The morphism $\mathscr{V}\mathscr{A}: J^r VY \to VZ$ associated with VA is of the form $\mathscr{V}\mathscr{A} = V\mathscr{A} \circ i^r_Y$, where $V\mathscr{A}: VJ^r Y \to VZ$ is the vertical tangent map to $\mathscr{A}$ and $i^r_Y: J^r VY \to VJ^r Y$ is the canonical identification defined by

$$i^r_Y\left(j^r_x \frac{\partial}{\partial t}\bigg|_0 s_t\right) = \frac{\partial}{\partial t}\bigg|_0 j^r_x s_t$$

for every one-parameter family of sections of Y and every $x \in X$, see [1]. Indeed,

$$VA(\sigma(x)) = VA\left(\frac{\partial}{\partial t}\bigg|_0 s_t(x)\right) = \frac{\partial}{\partial t}\bigg|_0 As_t(x) = \frac{\partial}{\partial t}\bigg|_0 (\mathscr{A}(j^r_x s_t)) = V\mathscr{A}\left(\frac{\partial}{\partial t}\bigg|_0 j^r_x s_t\right) =$$

$$= V\mathscr{A} \circ i^r_Y\left(j^r_x \frac{\partial}{\partial t}\bigg|_0 s_t\right).$$

If Z is a fibered manifold $q: Z \to Y$ over Y and A satisfies $qAs = s$ for all sections of Y, then A will be called a *prolongation differential operator*. In [4], prolongation differential operators are called differential section operators. A prolongation differential operator A is characterized by the property that $\mathscr{A}: J^rY \to Z$ is a morphism over id_Y. Then vertical prolongation VA is a prolongation differential operator with respect to projection Vq: $VZ \to VY$. Indeed,

$$Vq(VA(\sigma)) = Vq\left(VA\left(\frac{\partial}{\partial t}\bigg|_0 s_t\right)\right) = Vq\left(\frac{\partial}{\partial t}\bigg|_0 As_t\right) = \frac{\partial}{\partial t}\bigg|_0 (qAs_t) = \frac{\partial}{\partial t}\bigg|_0 s_t = \sigma$$

for every section $\sigma: X \to VY$.

Consider now the case $Z = VY$. A prolongation differential operator of Y into VY will be called a *vertical prolongation operator on* Y. Such operators form a real vector space, provided one defines

$$(k_1A + k_2B)\, s := k_1As + k_2Bs\,, \quad k_1, k_2 \in R$$

for every section s of Y. Each vertical prolongation operator A on Y can be prolonged in the above sense into an operator VA of VY into VVY. If A, B are two vertical prolongation operators on Y, we can construct $VA(Bs)$, $VB(As)$: $X \to VVY$ for every section s of Y. Both $VA \circ B$ and $VB \circ A$ are prolongation differential operators of Y into VVY. We have $Vq_Y(VA(Bs)) = Bs$ since VA is a prolongation differential operator with respect to the projection Vq_Y. Further, if Bs is tangent to s_t, $s_0 = s$, then

$$q_{VY}(VA(Bs)) = q_{VY}\left(\frac{\partial}{\partial t}\bigg|_0 As_t\right) = As\,.$$

Similarly we have $Vq_Y(VB(As)) = As$, $q_{VY}(VB(As)) = Bs$. Hence the conditions (1) for the strong difference are satisfied. If A and B are vertical prolongation operators of orders r and s, then $VA \circ B$ and $VB \circ A$ are prolongation differential operators of order $r + s$. The associated morphism to $VA \circ B$ is $V\mathscr{A} \circ i^r_Y \circ J^r\mathscr{B} : J^{r+s}Y \to VVY$ because of $VA(Bs(x)) = V\mathscr{A} \circ i^r_Y \circ j^r(\mathscr{B} \circ j^s_x s) = V\mathscr{A} \circ i^r_Y \circ J^r\mathscr{B}(j^{r+s}_x s)$, where $J^r\mathscr{B}$ is the restriction of the r-th jet prolongation of $\mathscr{B}$ to $J^{r+s}Y \subset J^r(J^sY)$. Similarly, the associated morphism to $VB \circ A$ is $V\mathscr{B} \circ i^s_Y \circ J^s\mathscr{A} : J^{r+s}Y \to VVY$.

We now define a vertical prolongation operator of order $r + s$ on Y, called the bracket of A, B, by

(3) $$[A, B]\,(s) := VA(Bs) \dot{-} VB(As) : X \to VY\,.$$

The associated morphism to $[A, B]$ is $[\mathscr{A}, \mathscr{B}] = V\mathscr{A} \circ i^r_Y \circ J^r\mathscr{B} \dot{-} V\mathscr{B} \circ i^s_Y \circ J^s\mathscr{A} :$ $: J^{r+s}Y \to VY$.

We are going to define the "value" of a vertical prolongation operator on any function f: $Y \to R$. Denote by δf: $VY \to R$ the fiber differential of f, [1]. For a vertical prolongation operator A with the associated morphism $\mathscr{A}$ we put

(4) $$Af := \delta f \circ \mathscr{A} : J^rY \to R\,.$$

Lemma 2. *If A, B are two vertical prolongation operators of order r such that $Af = Bf$ holds for every function f: $Y \to R$, then $A \equiv B$.*

Proof. In local coordinates (x^i, y^p) on Y and the induced coordinates (x^i, y^p, Y^p) on VY and $(x^i, y^p, y^p_i, \ldots, y^p_{i_1\ldots i_r})$ on J^rY, we have $f \equiv f(x^i, y^p)$, $\mathscr{A} \equiv x^i = x^i$, $y^p = y^p$, $Y^p = A^p(x^i, y^p, \ldots, y^p_{i_1\ldots i_r})$ and a similar expression for $\mathscr{B}$. Then

$$Af = \frac{\partial f}{\partial y^q} A^q , \quad Bf = \frac{\partial f}{\partial y^q} B^q .$$

Setting $f = y^p$, we obtain $A^p = B^p$ for every p, which implies $\mathscr{A} \equiv \mathscr{B}$. QED.

Further, if $f\colon J^sY \to R$ is a function, we have the fiber differential $\delta f : VJ^sY \to R$. Taking into account $J^s\mathscr{A} : J^{r+s}Y \to J^sVY$, we define

$$(5) \qquad Af := \delta f \circ i^s_Y \circ J^s\mathscr{A} : J^{r+s}Y \to R .$$

Using (4) and (5), we shall prove

Proposition 1. *The set of all vertical prolongation operators on an arbitrary fibered manifold forms a Lie algebra with respect to the bracket defined by* (3).

Proof. Any function $f\colon Y \to R$ can be considered as a base-preserving morphism $f\colon Y \to X \times R$, where $X \times R \to X$ is the product fibered manifold. Then δf is the second component of the vertical tangent map $Vf : VY \to V(X \times R) = X \times TR$ and Af is the second component of $Vf \circ \mathscr{A} : J^rY \to X \times TR$. Further, according to (5), $B(Af)$ is the fourth component of

$$VVf \circ V\mathscr{A} \circ i^r_Y \circ J^r\mathscr{B} : J^{r+s}Y \to X \times TTR .$$

Similarly, $A(Bf)$ is the fourth component of

$$VVf \circ V\mathscr{B} \circ i^s_Y \circ J^s\mathscr{A} : J^{r+s}Y \to X \times TTR .$$

But $V\mathscr{A} \circ i^r_Y \circ J^r\mathscr{B}$ and $V\mathscr{B} \circ i^s_Y \circ J^s\mathscr{A}$ satisfy the conditions for the strong difference and Lemma 1 implies

$$VVf \circ V\mathscr{A} \circ i^r_Y \circ J^r\mathscr{B} \dot{-} VVf \circ V\mathscr{B} \circ i^s_Y \circ J^s\mathscr{A} =$$
$$= Vf(V\mathscr{A} \circ i^r_Y \circ J^r\mathscr{B} \dot{-} V\mathscr{B} \circ i^s_Y \circ J^s\mathscr{A}) = Vf \circ [\mathscr{A}, \mathscr{B}] .$$

Hence the second component of the latter map is $[A, B]f$. On the other hand, the second component of $VVf \circ V\mathscr{A} \circ i^r_Y \circ J^r\mathscr{B} \dot{-} VVf \circ V\mathscr{B} \circ i^s_Y \circ J^s\mathscr{A}$ is $B(Af) - A(Bf)$, so that

$$(6) \qquad [A, B]f = B(Af) - A(Bf) .$$

Using (6), one finds easily

$$([[A, B], C] + [[B, C], A] + [[C, A], B])f = 0 .$$

By Lemma 2, we deduce the Jacobi identity. QED.

3. Generalized Lie derivatives of sections and morphisms. Let $Y \to X$ be a fibered manifold, η a projectable vector field on Y over a vector field ξ on X and s a section of Y. Then the Lie derivative (see [2]) of s with respect to η is

$$\mathscr{L}_\eta s = Ts \circ \xi - \eta \circ s : X \to VY .$$

Thus, every projectable vector field η transforms any section s of Y into a section $\mathscr{L}_\eta s : X \to VY$ and we can consider $\mathscr{L}_\eta$ as a vertical prolongation operator of order 1.

According to Proposition 1 the set of the Lie derivatives of sections with respect to the projectable vector fields forms a Lie algebra. The bracket is defined by the following formula

$$[\mathscr{L}_\eta, \mathscr{L}_{\bar{\eta}}]\, s = V\mathscr{L}_\eta \mathscr{L}_{\bar{\eta}} s \mathbin{\dot{-}} V\mathscr{L}_{\bar{\eta}} \mathscr{L}_\eta s . \tag{7}$$

Lemma 3. *For every projectable vector field η on Y and every section $\Phi : X \to VY$ we have*

$$(V\mathscr{L}_\eta)\, \Phi = i \circ \mathscr{L}_{V\eta} \Phi , \tag{8}$$

where i is the canonical involution of VVY and vector field $V\eta$ is the vertical prolongation of η defined by means of the vertical prolongation of the flow of η.

Proof. In local coordinates, $\Phi \equiv (x^i, \varphi^p(x), \Phi^p(x))$. Then $V\mathscr{L}_\eta$ transforms Φ into

$$\left(x^i, \varphi^p, (\mathscr{L}_\eta \varphi)^p, \Phi^p, \frac{\partial \Phi^p}{\partial x^i}\, \xi^i - \frac{\partial \eta^p}{\partial y^q}\, \Phi^q\right).$$

Further,

$$V_\eta \equiv \xi^i(x) \frac{\partial}{\partial x^i} + \eta^p(x, y) \frac{\partial}{\partial y^p} + \frac{\partial \eta^p}{\partial y^q}\, Y^q \frac{\partial}{\partial Y^p}$$

and

$$\mathscr{L}_{V_\eta} \Phi \equiv \left(\frac{\partial \varphi^p}{\partial x^i}\, \xi^i - \eta^p\right) \frac{\partial}{\partial y^p} + \left(\frac{\partial \Phi^p}{\partial x^i}\, \xi^i - \frac{\partial \eta^p}{\partial y^q}\, \Phi^q\right) \frac{\partial}{\partial Y^p} .$$

Thus $\mathscr{L}_{V_\eta}$ transforms Φ into

$$\left(x^i, \varphi^p, \Phi^p, (\mathscr{L}_\eta \varphi)^p, \frac{\partial \Phi^p}{\partial x^i}\, \xi^i - \frac{\partial \eta^p}{\partial y^q}\, \Phi^q\right).$$ QED.

Owing to Lemma 3, the bracket of Lie derivatives of sections can be expressed equivalently as

$$[\mathscr{L}_\eta, \mathscr{L}_{\bar{\eta}}]\, (s) = \mathscr{L}_{V\eta} \mathscr{L}_{\bar{\eta}} s \mathbin{\dot{-}} \mathscr{L}_{V\bar{\eta}} \mathscr{L}_\eta s . \tag{9}$$

In [2], Kolář has proved

$$\mathscr{L}_{V\eta} \mathscr{L}_{\bar{\eta}} s \mathbin{\dot{-}} \mathscr{L}_{V\bar{\eta}} \mathscr{L}_\eta s = \mathscr{L}_{[\eta, \bar{\eta}]} s . \tag{10}$$

From (9) and (10) it follows that $\mathscr{L}_{[\eta, \bar{\eta}]} s = [\mathscr{L}_\eta, \mathscr{L}_{\bar{\eta}}]\, (s)$ and we can consider $\mathscr{L}$ as a homomorphism of the Lie algebra of all projectable vector fields on Y into the Lie algebra of all vertical prolongation operators on Y.

Let $Y \to X$ and $Z \to X$ be two fibered manifolds over the same base. Let η be a projectable vector field on Y over a vector field ξ on X and ζ a projectable vector field on Z over the same vector field ξ. Then for every base-preserving morphism $f\colon Y \to Z$ we define its Lie derivative with respect to η and ζ by

$$\mathscr{L}_{(\eta,\zeta)}f := Tf \circ \eta - \zeta \circ f : Y \to VZ\,.$$

Remark. Let $G \to^q Z \to^p X$ be a double fibered manifold and $f\colon Y \to Z$ a base-preserving morphism. A *prolongation differential f-operator of Y into G* is a rule transforming each section s of Y into a section $As\colon X \to G$ such that $q(As) = f \circ s$. (Kosmann-Schwarzbach, [4], calls such an operator a differential section f-operator.) In the special case of $VZ \to^q Z \to^p X$, a prolongation differential f-operator will be called a vertical prolongation f-operator. Lie derivatives of morphisms can be considered as vertical prolongation f-operators transforming each section s of Y into a section $(\mathscr{L}_{(\eta,\zeta)}f) \circ s : X \to VZ$.

Let $\bar{\eta}$ and $\bar{\xi}$ be another pair of projectable vector fields on Y and Z over the same vector field $\bar{\zeta}$ on X. Then we define the iterated Lie derivative by

$$\mathscr{L}_{(\bar{\eta},V\bar{\xi})}\mathscr{L}_{(\eta,\zeta)}f : Y \to VVZ. \tag{11}$$

Proposition 2. *$\mathscr{L}_{(\eta,V\zeta)}\mathscr{L}_{(\bar{\eta},\bar{\zeta})}f$ and $\mathscr{L}_{(\bar{\eta},V\bar{\xi})}\mathscr{L}_{(\eta,\zeta)}f$ satisfy the conditions for the strong difference and*

$$\mathscr{L}_{(\eta,V\zeta)}\mathscr{L}_{(\bar{\eta},\bar{\xi})}f \dot{-} \mathscr{L}_{(\bar{\eta},V\bar{\xi})}\mathscr{L}_{(\eta,\zeta)}f = \mathscr{L}_{([\eta,\bar{\eta}],[\zeta,\bar{\zeta}])}f\,. \tag{12}$$

Proof. The generalized Lie derivative of a map $f\colon M \to N$ with respect to a pair of vector fields ξ and η on M and N is defined by the formula $\mathscr{L}_{(\xi,\eta)}f := Tf \circ \xi - \eta \circ f : M \to TN$, [3]. In general, it is easy to see that if the values of f lie in a submanifold $Q \subset N$ and the vector field η is tangent to Q, then $\mathscr{L}_{(\xi,\eta)}f = \mathscr{L}_{(\xi,\tilde{\eta})}f$, where $\tilde{\eta}$ is the restriction of η to Q. Since ζ is a projectable vector field on Z and the values of $\mathscr{L}_{(\bar{\eta},\bar{\zeta})}f$ lie in VZ, we have, in our case, $\mathscr{L}_{(\eta,T\zeta)}\mathscr{L}_{(\bar{\eta},\bar{\zeta})}f = \mathscr{L}_{(\eta,V\zeta)}\mathscr{L}_{(\bar{\eta},\bar{\zeta})}f$, where $T\zeta$ is the prolongation of ζ with respect to the tangent functor constructed by means of flows, [3]. Our proposition is then a special case of Theorem 2 of [3]. QED.

Owing to Proposition 2, the set of Lie derivatives of morphisms forms the Lie algebra with a bracket defined by

$$[\mathscr{L}_{(\eta,\zeta)}, \mathscr{L}_{(\bar{\eta},\bar{\zeta})}]f = \mathscr{L}_{(\eta,V\zeta)}\mathscr{L}_{(\bar{\eta},\bar{\zeta})}f \dot{-} \mathscr{L}_{(\bar{\eta},V\bar{\zeta})}\mathscr{L}_{(\eta,\zeta)}f\,. \tag{13}$$

A simpler situation occurs, if $Z = E$ is a vector bundle. Then $\mathscr{L}_{(\eta,\zeta)}f$ can be considered as a base-preserving morphism of Y into E as well. Hence we can construct the iterated Lie derivative $\mathscr{L}_{(\eta,\zeta)}\mathscr{L}_{(\bar{\eta},\bar{\zeta})}f : Y \to E$. A projectable vector field ζ on E is called linear, [2], if its flow is formed by linear fiber isomorphisms.

Proposition 3. *Let $Y \to X$ be a fibered manifold, $E \to X$ a vector bundle over the same base, and $f: Y \to E$ a base-preserving morphism. Let $\eta, \bar{\eta}$ be two projectable vector fields on Y over vector fields $\xi, \bar{\xi}$ on X, and $\zeta, \bar{\zeta}$ two linear vector fields on E over the same vector fields $\xi, \bar{\xi}$, respectively. Then*

$$\mathscr{L}_{(\bar{\eta},\bar{\zeta})}\mathscr{L}_{(\eta,\zeta)}f - \mathscr{L}_{(\eta,\zeta)}\mathscr{L}_{(\bar{\eta},\bar{\zeta})}f = \mathscr{L}_{([\bar{\eta},\eta],[\bar{\zeta},\zeta])}f \,. \tag{14}$$

Proof represents an easy direct calculation in local coordinates. We remark that in the vector bundle case we do not need the strong difference. However, the condition of linearity of ζ and $\bar{\zeta}$ is essential, i.e. (14) does not hold for general projectable vector fields ζ and $\bar{\zeta}$.

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