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ON LIMITS OF L_p-NORMS OF LINEAR OPERATORS

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1. INTRODUCTION

This paper is devoted to a non-commutative extension of an assertion from the classical theory of integration. It is well known that for a non-negative measure μ and a measurable function f one has

$$\lim_{p\to\infty}\left(\int |f|^p \,\mathrm{d}\mu\right)^{1/p} = \|f\|_\infty$$

on the assumption that for some $q \in \langle 1, \infty)$ $\int |f|^q \,\mathrm{d}\mu$ is finite. In the non-commutative case the role of the function f is taken over by a linear operator and the role of the essential supremum of a function is played by the spectral norm of the operator. The integral of a linear operator will be assumed in the sense [11].

In the commutative case the above mentioned assertion has little practical importance since to find an approximate essential supremum for a given function by means of evaluating the integral $(\int |f|^p \,\mathrm{d}\mu)^{1/p}$ for a sufficiently large p is usually a far more difficult problem. On the other hand, in the non-commutative case this assertion acquires practical significance since for a given operator the calculation of its spectral norm (and, therefore, of its spectral radius for a normal operator) is a well known problem which is also approximately solvable by calculating the above mentioned integral foı p sufficiently laıge (i.e. the L_p-norm of an operator).

The paper is organized as follows. In Section 2 the necessary basic notations, concepts and results from the integration theory of linear operators are briefly summarized. Section 3 contains the main results of this paper. We prove that, under similar assumptions as in the case of functions, the L_p-norms of a linear operator converge to its spectral norm. The rate of this convergence is estimated.

2. NON-COMMUTATIVE INTEGRATION

In this section only the basic facts, definitions and notations needed for the subsequent treatment are briefly summarized. For a comprehensive exposition of the integration theory of linear operators the reader is refered to the fundamental paper by Segal [11] and also to papers [4] and [13]. Further development of the theory will be found e.g. in [1–3], [5–7], [9], [10], [14–16]. A presentation of the results and of the directions of research in the field of abstract integration can be found in the expository paper [12].

We start with the definition of a gage space that plays a central role in the developed theory. A *gage space* $\Gamma = (H, \mathscr{A}, m)$ is a system composed of a Hilbert space H, a von Neumann algebra $\mathscr{A}$ of operators on H and a non-negative valued function m (called a *gage*) on the projections in $\mathscr{A}$, where m is completely additive, unitarily invariant, and such that every projection in $\mathscr{A}$ is the l.u.b. of projections on which m is finite. A gage space is called *regular* when the only projection of gage zero is the zero projection.

An operator T in H is called measurable with respect to $\mathscr{A}$ if $T \eta \mathscr{A}$, if T is closed, and if there exists an increasing sequence of closed linear subspaces K_n in the domain of T such that the corresponding projections P_n belong to $\mathscr{A}$ and have finite orthocomplements $I - P_n \to 0$. A sequence $\{T_n\}$ of measurable operators with respect to $\mathscr{A}$ is said to *converge almost everywhere* to a measurable operator T provided for every $\varepsilon > 0$ there exists a sequence of projections P_n in $\mathscr{A}$ such that $\|(T_n - T) P_n\| < \varepsilon$, $I - P_n$ is finite and $\downarrow 0$ as $n \uparrow \infty$.

Further, the concept of the ring E of elementary operators in $\mathscr{A}$ is introduced, consisting of those operators in $\mathscr{A}$ whose ranges are contained in the range of a projection of a finite gage. There exists a unique linear extension of m on the ring E. Finally, in a regular gage space Γ an operator T is called *integrable* (symbolically $T \in L_1(\Gamma)$) if there exists a sequence $\{T_n\}$ of elementary operators converging to T almost everywhere which satisfies the additional condition that $m(|T_n - T_k|) \to 0$ as $n, k \to \infty$. Its integral (trace) $m(T)$ is defined as $\lim_{n \to \infty} m(T_n)$, it exists and is unique.

For a measurable operator T let $U|T|$ be the canonical polar decomposition of T and let $\int \lambda \, dE_\lambda$ be the spectral resolution of $|T|$. Then U belongs to $\mathscr{A}$ and $|T|$ is measurable and the identity

$$m(|T|) = \int_0^\infty \lambda \, dm(E_\lambda)$$

holds. For $p \in \langle 1, \infty)$ we introduce the spaces $L_p(\Gamma)$ of measurable operators T by the condition

$$\|T\|_p = (m(|T|^p))^{1/p} = \left(\int_0^\infty \lambda^p \, dm(E_\lambda)\right)^{1/p} < \infty .$$

Finally, for an operator T we set $\|T\|_\infty$ equal to the bound of T ($\sup_{\|x\|=1} \|Tx\|$) if T is bounded and otherwise we put $\|T\|_\infty = \infty$.

Let $M = [R, \mathscr{R}, r]$ be a measure space which is σ-finite (weaker assumption on M is sufficient, see [11: Example 1.1]). Then M can be identified with the gage space $(L_2(M), L_\infty(M), r)$, an element of $L_\infty(M)$ being identified with the corresponding multiplicative operator. Conversely, a regular commutative gage space is algebraically equivalent to a gage space built on a measure space. Here, then, the notions of measurable operator and convergence a.e. of measurable operators correspond exactly to the classical definitions of measurable function and convergence a.e. of functions (for details see [11: Th. 2 and Th. 7]).

As a non-commutative example take H to be finite dimensional and let $\mathscr{A}$ be the set of all linear operators on H. Define $m(P)$ = dimension (range P) for any projection P in $\mathscr{A}$. It can be verified readily that $(H, \mathscr{A}, m)$ is a gage space and for any A in $\mathscr{A}$ $m(A)$ is then the ordinary trace of A. Further examples can be found in [8].

3. CONVERGENCE OF L_p-NORMS OF OPERATORS

First we shall show that, under assumptions analogous to those in the commutative case, we have $\|A\|_p \to \|A\|_\infty$ for $p \to \infty$ with A being an operator.

Theorem 3.1. *Let $\Gamma = (H, \mathscr{A}, m)$ be a regular gage space and let there exists $q \in \langle 1, \infty)$ such that $A \in L_q(\Gamma)$. Then* $\lim\limits_{p\to\infty} \|A\|_p = \|A\|_\infty$.

Proof. Let $A = WB$ be a canonical polar decomposition of an operator A and $B = \int \lambda \, \mathrm{d}E_\lambda$ a spectral resolution of the operator B. The proof will be divided into two parts.

Part 1. $\|A\|_\infty = \infty$. Since $B \in L_q(\Gamma)$ we have for n integer

$$n^q \int_n^\infty \mathrm{d}m(E_\lambda) \leqq \int_0^\infty \lambda^q \, \mathrm{d}m(E_\lambda) = \|B\|_q^q < \infty$$

and therefore $\int_n^\infty \mathrm{d}m(E_\lambda)$ is finite. Moreover $\int_n^\infty \mathrm{d}m(E_\lambda)$ is nonzero, because we assume, a regular gage space and $\|B\|_\infty = \|A\|_\infty = \infty$. Let further $p \geqq q$, then

$$m(B^p) = \int_0^\infty \lambda^p \, \mathrm{d}m(E_\lambda) \geqq n^p \int_n^\infty \mathrm{d}m(E_\lambda)$$

and therefore

$$(m(B^p))^{1/p} \geqq n \left(\int_n^\infty \mathrm{d}m(E_\lambda) \right)^{1/p},$$

whence we obtain

$$\liminf_{p\to\infty} (m(B^p))^{1/p} \geqq n$$

for every integer n and thus $\lim\limits_{p\to\infty} \|A\|_p = \infty$.

Part 2. $\|A\|_\infty < \infty$. Without loss of generality we can assume that $A \neq 0$. Denote $\mu = \|A\|_\infty = \|B\|_\infty$ and $R = \|B\|_q^q / \mu^q$. Since $B \in L_q(\Gamma)$, $B \neq 0$, it follows that

$$m(B^q) = \mu^q \int_0^\mu \left(\frac{\lambda}{\mu} \right)^q \mathrm{d}m(E_\lambda) = \mu^q R$$

and R is finite and nonzero. Let further $p \geqq q$, then

$$m(B^p) = \mu^p \int_0^\mu \left(\frac{\lambda}{\mu}\right)^p \mathrm{d}m(E_\lambda) \leqq \mu^p \int_0^\mu \left(\frac{\lambda}{\mu}\right)^q \mathrm{d}m(E_\lambda)$$

and so we obtain

$$(m(B^p))^{1/p} \leqq \mu R^{1/p} . \tag{1}$$

Choose now $\varepsilon \in (0, \mu)$, then

$$m(B^q) = \int_0^\mu \lambda^q \, \mathrm{d}m(E_\lambda) \geqq (\mu - \varepsilon)^q \int_{\mu-\varepsilon}^\mu \mathrm{d}m(E_\lambda) .$$

Denote $S_\varepsilon = \int_{\mu-\varepsilon}^\mu \mathrm{d}E_\lambda$, then

$$m(S_\varepsilon) = \int_{\mu-\varepsilon}^\mu \mathrm{d}m(E_\lambda) \leqq \frac{1}{(\mu - \varepsilon)^q} \int_{\mu-\varepsilon}^\mu \lambda^q \, \mathrm{d}m(E_\lambda)$$

and, consequently, $m(S_\varepsilon)$ is finite and, since $\mu \in \sigma(B)$ (the spectrum of the operator B), $m(S_\varepsilon)$ is nonzero. For $p \geqq q$ we therefore obtain

$$m(B^p) = \int_0^\mu \lambda^p \, \mathrm{d}m(E_\lambda) \geqq (\mu - \varepsilon)^p \, m(S_\varepsilon)$$

whence

$$(m(B^p))^{1/p} \geqq (\mu - \varepsilon) (m(S_\varepsilon))^{1/p} . \tag{2}$$

From the relations (1) and (2) we obtain for $p \to \infty$

$$\mu - \varepsilon \leqq \liminf_{p \to \infty} (m(B^p))^{1/p} \leqq \limsup_{p \to \infty} (m(B^p))^{1/p} \leqq \mu$$

for every $\varepsilon \in (0, \mu)$ so that, finally,

$$\lim_{p \to \infty} \|A\|_p = \|A\|_\infty .$$

Corollary 3.2. *Let $\Gamma = (H, \mathscr{A}, m)$ be a regular gage space. Let there exist $q \in \langle 1, \infty)$ such that $A \in L_q(\Gamma)$ and let A be a normal operator. Then*

$$\lim_{k \to \infty} (m((A^*A)^{2^{k-1}}))^{2^{-k}} = r(A)$$

where $r(A)$ is the spectral radius of the operator A.

If we are able, in a particular case, to calculate the operators $A^*A, (A^*A)^2, (A^*A)^4 \ldots$ etc. and their integrals, Corollary 3.2 can be looked upon as an assertion about the approximation of the spectral radius of a normal operator A. It is therefore quite natural to examine the error of this approximation.

Theorem 3.3. *Let $\Gamma = (H, \mathscr{A}, m)$ be a regular gage space and let A be a nonzero operator. Assume that there exists $q \in \langle 1, \infty)$ such that $A \in L_q(\Gamma) \cap \mathscr{A}$ and $\|A\|_\infty \in$*

$\in P_\sigma(|A|)$ (*the point spectrum of the operator* $|A|$). *Then for* $p \geqq q$

$$\big|\,\|A\|_p - \|A\|_\infty\big| \leqq \frac{1}{p}\,\{\|A\|_\infty . \max(|\ln m(S)|,\,|\ln R|) . \max(1, R^{1/p})\}\,,$$

where S *is the projection onto the eigenspace corresponding to the eigenvalue* $\|A\|_\infty$ *of the operator* $|A|$ *and* $R = \|A\|_q^q/\|A\|_\infty^q$.

Proof. Let $A = WB$ be a canonical polar decomposition of the operator A and $B = \int \lambda \,\mathrm{d}E_\lambda$ the spectral resolution of the operator B. In the same way as in the proof of Theorem 3.1. we denote

$$R = \frac{\|B\|_q^q}{\|B\|_\infty^q} = \frac{\|A\|_q^q}{\|A\|_\infty^q} \quad \text{and} \quad S_\varepsilon = \int_{\mu-\varepsilon}^{\mu} \mathrm{d}E_\lambda$$

for $\varepsilon \in (0, \mu)$. First we shall prove an auxiliary

Lemma 3.4. *Let* $\Gamma = (H, \mathscr{A}, m)$ *be a regular gage space and let* A *be a nonzero operator. Assume further that there exists* $q \in \langle 1, \infty)$ *such that* $A \in L_q(\Gamma) \cap \mathscr{A}$. *Then, for sufficiently small* $\varepsilon > 0$ *and* $p \geqq q$,

$$\big|\,\|A\|_p - \|A\|_\infty\big| \leqq \varepsilon + \frac{1}{p}\,\{\|A\|_\infty . \max(|\ln m(S_\varepsilon)|,\,|\ln R|) . \max(1, R^{1/p}, (m(S_\varepsilon))^{1/p})\}\,.$$

Proof. From the relation (1) in the proof of Theorem 3.1 we obtain for $p \geqq q$

$$\|B\|_p \leqq \mu\left(1 + \frac{\ln R}{p}\, e^{\xi_p}\right),$$

where ξ_p lies between 0 and $p^{-1} \ln R$, so that

$$\|B\|_p - \mu \leqq \mu \frac{|\ln R|}{p} \max(1, R^{1/p})\,. \tag{3}$$

Further, from the relation (2) in the proof of Theorem 3.1 we obtain for $\varepsilon \in (0, \mu)$ and $p \geqq q$

$$\|B\|_p \geqq (\mu - \varepsilon)\left(1 + \frac{\ln m(S_\varepsilon)}{p}\, \mathrm{e}^{\eta_p}\right),$$

where η_p lies between 0 and $p^{-1} \ln m(S_\varepsilon)$, so that

$$\|B\|_p - \mu \geqq -\varepsilon - \mu \frac{|\ln m(S_\varepsilon)|}{p} \max(1, (m(S_\varepsilon))^{1/p})\,. \tag{4}$$

From the relations (3) and (4) the assertion of Lemma 3.4. follows.

To finish the proof of Theorem 3.3 it is sufficient to show that $m(S)$ is finite and nonzero and that $m(S) = \lim_{\varepsilon \to 0+} m(S_\varepsilon)$. Clearly, $m(S) > 0$ because the gage space is regular, $m(S)$ is finite because $m(S_\varepsilon)$ is finite and $S \leqq S_\varepsilon$. Finally, using [11: Th. 10] we get

$$\lim_{\varepsilon \to 0+} m(S_\varepsilon) = \lim_{\varepsilon \to 0+} m(I - E_{\mu-\varepsilon}) = m(I - E_{\mu-}) = m(S);$$

and since

$$R = \int_0^\mu \left(\frac{\lambda}{\mu}\right)^q \mathrm{d}m(E_\lambda) \geqq \left(\frac{\mu - \varepsilon}{\mu}\right)^q m(S_\varepsilon),$$

it follows that $R \geqq m(S)$.

Corollary 3.5. *Let $\Gamma = (H, \mathscr{A}, m)$ be a regular gage space and let A be a normal nonzero operator. Assume further that there exists $q \in \langle 1, \infty)$ such that $A \in L_q(\Gamma) \cap \mathscr{A}$ and $r(A) = \|A\|_\infty \in P_\sigma(|A|)$. Then for $k \geqq \ln q/\ln 2$,*

$$|(m((A^*A)^{2^{k-1}}))^{2^{-k}} - r(A)| \leqq \frac{1}{2^k} \{r(A) \,.\, \max(|\ln m(S)|, |\ln R|) \,.\, \max(1, R^{2^{-k}})\}\,.$$

Remark 3.6. The assumption $\|A\|_\infty \in P_\sigma(|A|)$ which occurs in Theorem 3.3 and in Corollary 3.5 is fulfilled e.g. when A is a compact operator.

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