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ON CONGRUENCE RELATIONS OF MONOUNARY ALGEBRAS I

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The lattice $\operatorname{Con}(A, f)$ of all congruence relations of a monounary algebra (A, f) was studied by J. Berman [1], L. A. Skornjakov and D. P. Jegorova [2], D. P. Jegorova [3], [4] and G. Č. Kurinnoj [7], [8]; cf. also the expository article [11].

Let $A \neq \emptyset$ be a set. We denote by $E(A)$ the system of all equivalence relations on A. Let F be the system of all unary operations on A. For $f \in F$ we put

$$R(f) = \{g \in F : \operatorname{Con}(A, f) = \operatorname{Con}(A, g)\} .$$

Consider the following conditions for a monounary algebra (A, f):

(a) (A, f) has at least one connected component without cycle.

(b) Each connected component of (A, f) has a cycle of the cardinality less than 3.

In this paper it is proved that the conditions (a) and (b) can be expressed in terms of the system $\operatorname{Con}(A, f)$ (without using explicitly the operation f). Further it will be shown that if (a) or (b) holds, then all operations $g \in R(f)$ can be reconstructed by means of $\operatorname{Con}(A, f)$. From this reconstruction we obtain in particular:

(i) If (a) is valid, then card $R(f) = 1$. (This result was proved already by Kurinnoj [7].)

(ii) If (b) holds and if $\operatorname{Con}(A, f) = E(A)$, card $A > 2$, then card $R(f) = 1 +$ + card A. If (b) holds and if $\operatorname{Con}(A, f) \neq E(A)$, then card $R(f) \leqq 4$.

These results will be applied in Part II for obtaining the estimate for card $R(f)$ in the general case. Analogous questions for a type of partial monounary algebras were investigated in [5].

1. PRELIMINARIES

We start with recalling the basic notions concerning monounary algebras (cf. B. Jónsson [6], M. Novotný [9], [10]).

Let A be a nonempty set and let $F = F(A)$ be the set of all unary operations defined on A. If $f \in F(A)$, then the pair (A, f) is called a *monounary algebra*. (A, f) is said to be *connected*, if for each $x, y \in A$ there are positive integers m, n with $f^m(x) = f^n(y)$. Let N be the set of all positive integers and let $N_0 = N \cup \{0\}$. For $x \in A$ we denote

$$K_f(x) = \bigcup_{m \in N} \bigcup_{n \in N} f^{-m}(f^n(x))\,;$$

the set $K_f(x)$ is said to be the *connected component of* (A, f) containing the element x.

Again, let $x \in A$ and suppose that $f^n(x) = x$ for some $n \in N$. The set $C_0^f[x] = \{f^k(x) : k \in N\}$ is called the *cycle* of (A, f) generated by x. Further we put $C_1^f[x] = f^{-1}(C_0^f[x]) - C_0^f[x]$, and for each $m \in N$ we set $C_{m+1}^f[x] = f^{-1}(C_m^f[x])$.

Let θ be an equivalence relation on the set A. We shall often not distinguish between θ and the partition of the set A corresponding to θ; if $x \in A$, then the class of this partition containing the element x will be denoted by $x\theta$. If θ does not coincide with the identity on A, then we also write $\theta = [\theta_i : i \in I]$, where $\{\theta_i : i \in I\} = \{x\theta : x \in A \text{ and card } x\theta > 1\}$.

For a monounary algebra (A, f) we denote by $\mathrm{Con}(A, f)$ the system of all congruence relations of (A, f). For $x, y \in A$ the symbol $\theta^f(x, y)$ means the least congruence relation of (A, f) having the property that the elements x and y belong to the same class of the corresponding partition of A. We often write $\theta(x, y)$ instead of $\theta^f(x, y)$, when no ambiguity can occur.

Let the monounary algebra (A, f) be fixed. A property $p(x_1, \ldots, x_n)$ concerning an n-tuple $(x_1, \ldots, x_n)$ of elements of A will be said to be a *k-property*, if it can be expressed merely by congruence relations of (A, f) (without using explicitly the operation f). Analogously, a subset $X \subseteq A$ is called a *k-set*, if it can be defined by congruence relations of (A, f).

As an example we can mention here that (as it will be proved below) the union of all cycles C with card $C > 2$ is a k-set.

The following lemma can be easily verified.

1.1. Lemma. *Let x, y be distinct elements of A. Then $\theta(x, y) = [\{x, y\}]$ if and only if some of the following conditions is valid:* (a) $f(x) = x$ *and* $f(y) = y$; (b) $f(x) = f(y) = x$; (c) $f(x) = f(y) = y$; (d) $f(x) = f(y) \notin \{x, y\}$; (e) $f(x) = y$ *and* $f(y) = x$.

1.2. Lemma. *Let x, y be distinct elements of A, $f(x) = f(y) = y$. Then the following assertions are valid:*

(a) *By means of congruence relations of (A, f) we can determine all elements $z \in A$ such that $f(z) = z$.*

(b) *Let $z \in A$, $z \neq y$, $f(z) = z$. By means of congruence relations of (A, f) we can determine all $v \in A$ such that $f(v) = z$.*

(c) *Let $n \in N$. By means of congruence relations of (A, f) we can determine all z belonging to the set $C_n^f[y]$ and for $z \in C_n^f[y]$ we can determine $u \in A$ such that $f(z) = u$.*

(d) *By means of congruence relations of* (A,f) *we can determine all pairs* $z, z' \in A$ *such that* $\{z, z'\}$ *is a cycle of* (A,f).

Proof. (a) Let $z \in A$, $z \neq y$. Then $f(z) = z$ if and only if $\theta(x, z) = [\{x, y, z\}]$ and $\theta(y, z) = [\{y, z\}]$. Namely, the necessary condition is obvious and the sufficient condition can be obtained by means of Lemma 1.1. (if we consider elements y, z instead of x, y, and then use the relation for $\theta(x, z)$).

(b) Let $z \in A$, $z \neq y, f(z) = z$, $v \in A$, $v \neq z$. Then $f(v) = z$ if and only if $\theta(z, v) = [\{z, v\}]$ and $\theta(v, y) = [\{v, y, z\}]$; this follows from Lemma 1.1 for the elements v, y and from the relation for $\theta(v, y)$.

(c) Let $z \in A$, $z \neq x$. According to Lemma 1.1 (for x and z) we get that $z \in C_1^f[y]$ if and only if $\theta(x, z) = [\{x, z\}]$. Suppose that $m \in N$, $m > 1$ and that for each $n \in N$, $n < m$, the assertion from (c) is valid. We shall prove that for $z \notin \bigcup_{n<m} C_n^f[y]$ the following holds:

$z \in C_m^f[y]$ if and only if there exists $u \in C_{m-1}^f[y]$ such that $\theta(z, u) = [u\ \theta(u, f(u)) \cup \cup \{z\}]$ and $\theta(z, y) \neq [\{z, y\}]$. In this case $f(z) = u$.

In proving that the condition is necessary it suffices to put $u = f(z)$. Let us prove that the condition is sufficient. We have $\theta(z, u) = [u\ \theta(u, f(u)) \cup \{z\}] = [\{f^k(u) : : k \in N_0\} \cup \{z\}]$ and $z \notin \bigcup_{n<m} C_n^f[y]$, which implies that $f(z) = z$ or $f(z) = u$. If $f(z) = z$, then $\theta(z, y) = [\{z, y\}]$, which is a contradiction. Hence $f(z) = u$ and $z \in C_m^f[y]$.

(d) Let $z, z' \in A$, $z \neq z'$. It is obvious that $f(z) = z'$ and $f(z') = z$ imply $\theta(z, z') = [\{z, z'\}]$ and $\theta(y, z) = \theta(y, z')$. The converse implication can be obtained by using 1.1 and the fact $\theta(y, z) = \theta(y, z')$.

1.3. Lemma. *Let* x, y *be distinct elements of* A, $f(x) = y$ *and* $f(y) = z$. *Then the following assertions are valid*:

(a) *By means of congruence relations of* (A,f) *we can determine all elements* $z \in A$ *such that* $f(z) = z$.

(b) *Let* $z \in A, f(z) = z$. *By means of congruence relations of* (A,f) *we can determine all* $v \in A$ *such that* $f(v) = z$.

(c) *By means of congruence relations of* (A,f) *we can determine all pairs* $u, u' \in A$ *such that* $\{u, u'\}$ *is a cycle of* (A,f).

(d) *Let* $n \in N$. *By means of congruence relations of* (A,f) *we can determine all* $z \in C_n^f[x]$ *and for* $z \in C_n^f[x]$ *we can determine* $u \in A$ *such that* $f(z) = u$.

Proof. (a) For $z \in A$ the relation $f(z) = z$ holds if and only if $\theta(x, z) = \theta(y, z) = [\{x, y, z\}]$; we shall prove only that the condition is sufficient (the necessity of the condition is obvious).

Since $f(x)\ \theta(x, z) f(z)$, i.e. $y\ \theta(x, z) f(z)$, we have $f(z) \in \{x, y, z\}$. In the case $f(z) = x$ we get $\theta(y, z) = [\{z, y\}]$; if $f(z) = y$, we obtain $\theta(x, z) = [\{x, z\}]$. Thus $f(z) = z$.

(b) Let $z \in A$, $f(z) = z$, $v \in A$, $v \neq z$. Then $f(v) = z$ if and only if $\theta(z, v) = [\{z, v\}]$ and $\theta(v, x) = [\{x, y, v, z\}]$; it is obvious that the condition of this assertion is necessary, and it follows from Lemma 1.1 (for the elements z, v) and from the relation for $\theta(v, x)$ that the condition is also sufficient.

(c) Let u, u' be distinct elements of A such that $u \notin \{x, y\}$ and $u' \in \{x, y\}$. We shall show that the following relation is valid: $f(u) = u'$ and $f(u') = u$ if and only if $\theta(u, x) = [\{u, x\}, \{u', y\}]$ and $\theta(u, u') = [\{u, u'\}]$. It is obvious that the condition is necessary. Let us prove that it is also sufficient. Since $f(u)\, \theta(u, x) f(x)$, i.e. $f(u)\,.$ $.\, \theta(u, x)\, y$, it follows that $f(u) = y$ or $f(u) = u'$. If $f(u) = y$, then $\theta(u, x) = [\{u, x\}]$, which is a contradiction. Thus $f(u) = u'$. Further $f^2(u)\, \theta(u, x) f^2(x)$, i.e. $f(u')\,.$ $.\, \theta(u, x)\, x$, hence $f(u') = x$ or $f(u') = u$. If $f(u') = x$, then $\theta(u, u') \neq [\{u, u'\}]$, therefore we obtain $f(u') = u$.

(d) Let $z \in A$, $z \notin \{x, y\}$. The relation $f(z) = x$ if and only if $\theta(x, z) = [\{x, y, z\}]$ and $\theta(y, z) = [\{y, z\}]$ can be obtained by using Lemma 1.1 (for y, z). Similarly we can determine by means of congruence relations of (A, f) whether $f(z) = y$. Hence the assertion (d) holds for $n = 1$. Further let $n \in N$, $n > 1$ and suppose that for each $m \in N$, $m < n$ the assertion (d) is valid. Let $z \notin \bigcup_{m<n} C_m^f[x]$. We shall prove the following relation: $z \in C_n^f[x]$ if and only if there exists $u \in C_{n-1}^f[x]$ such that $\theta(z, u) = [u\, \theta(u, f(u)) \cup \{z\}]$ and $u\, \theta(z, x)\, y$. In this case $f(z) = u$.

In proving that the condition is necessary it suffices to put $u = f(z)$. Let us prove that condition is sufficient. Since $\theta(z, u) = [u\, \theta(u, f(u)) \cup \{z\}] = [\{f^k(u) : k \in N_0\} \cup$ $\cup \{z\}]$ and $z \notin \bigcup_{m<n} C_m^f[x]$, we obtain $f(z) = z$ or $f(z) = u$. If $f(z) = z$, then $\theta(z, x) = [\{z, x, y\}]$, which is a contradiction. Hence $f(z) = u$ and $z \in C_n^f[x]$. The proof is complete.

1.4. Corollary. *Let (A, f) be a monounary algebra such that each connected component of (A, f) possesses a cycle having the cardinality less than* 3. *Further let x, y be distinct elements of A, $f(x) = f(y) = y$. If $a \in A$, then $f(a)$ can be determined by means of congruence relations.*

Proof. Let the assumptions of the lemma hold and let $a \in A - \{x, y\}$. First suppose that the element a belongs to the same component as y, i.e., $a \in C_n^f[y]$ for some $n \in N$. From 1.2 (c) it follows that this possibility can be determined by means of $\mathrm{Con}(A, f)$ and also that $f(a)$ can be found by means of $\mathrm{Con}(A, f)$. Now let a belongs to $C_n^f[z]$ for some $n \in N \cup \{0\}$, $z \in A$, where $f(z) = z \neq y$. If $n = 0$, i.e. $f(a) = a$, it can be described by means of $\mathrm{Con}(A, f)$ in view of 1.2 (a). If $n = 1$, then $f(a)$ can be determined by means of $\mathrm{Con}(A, f)$ in view of 1.2 (b). The situation when $n > 1$ is analogous to that when $a \in C_n^f[y]$. Further consider the case when a belongs to $C_n^f[u]$ for some $n \in N \cup \{0\}$, $u \in A$, where u belongs to a cycle with the cardinality 2. From 1.2 (d) it follows that the case when $n = 0$ can be described by means of $\mathrm{Con}(A, f)$ and in this case $f(a)$ can be determined by $\mathrm{Con}(A, f)$. If $n > 1$, this case and $f(a)$ can be determined by means of $\mathrm{Con}(A, f)$ in view of 1.3 (d).

Analogously as 1.4 the following assertion can be proved:

1.5. Corollary. *Let (A, f) be a monounary algebra such that each connected* *component of (A, f) possesses a cycle having the cardinality less than* 3. *Further let x, y be distinct elements of A, $f(x) = y$ and $f(y) = x$. If $a \in A$, then $f(a)$ can be determined by means of congruence relations.*

2. (A,f) HAVING AT LEAST ONE CONNECTED COMPONENT WITHOUT CYCLE

As before, let (A, f) be a monounary algebra. Consider the following condition for the set $\mathrm{Con}(A, f)$:

(0) There are distinct elements $x, y \in A$ such that $\theta(x, y) = [x\,\theta(x, y)]$, $\mathrm{card}\, x\,\theta(x, y) = \aleph_0$, and for no $z \in A$, $z \neq x$, both the relations $\theta(x, z) < \theta(x, y)$, $\theta(x, z) = [x\,\theta(x, z)]$ are valid.

In this paragraph it will be shown that if (0) is valid, then we can reconstruct the operation f if the set $\mathrm{Con}(A, f)$ is given. From this it follows that $R(f) = \{f\}$.

At first we shall prove that the condition (0) is equivalent with the condition in the title of this paragraph.

2.1. Lemma. *Let $x \in A$. The following conditions are equivalent*:

(1) *The connected component $K_f(x)$ possesses no cycle.*

(2) *There is $y \in A$ such that $\theta(x, y) = [x\,\theta(x, y)]$, $\mathrm{card}\, x\,\theta(x, y) = \aleph_0$, and for no $z \in A$, $z \neq x$, both the relations $\theta(x, z) < \theta(x, y)$, $\theta(x, z) = [x\,\theta(x, z)]$ are valid.*

Proof. Assume that $K_f(x)$ possesses no cycle. Put $y = f(x)$; then $\theta(x, y) = [\{f^i(x) : i \in N_0\}]$, $\mathrm{card}\, x\,\theta(x, y) = \aleph_0$. Let $z \in A$ be such that $\theta(x, z) < \theta(x, y)$. Then $z = f^k(x)$ for some $k \in N$, $k > 1$ and the relation $x\,\theta(x, z) f(x)$ does not hold. Since we have $f(x)\,\theta(x, z) f(z)$, i.e. $f(x)\,\theta(x, z) f^{k+1}(x)$, it follows that the partition of the congruence relation $\theta(x, z)$ has at least two nontrivial classes, namely $x\,\theta(x, z)$ and $y\,\theta(x, z)$.

Let us suppose that the condition (2) is valid and that $K_f(x)$ possesses a cycle. If y is such that $K_f(y)$ possesses a cycle, then $x\,\theta(x, y) \subseteq \{f^i(x), f^i(y) : i \in N_0\}$, $\mathrm{card}\, x\,\theta(x, y) < \aleph_0$. Hence $K_f(y)$ has no cycle. If x does not belong to a cycle, then $\theta(x, y)$ has at least two nontrivial classes $(x\,\theta(x, y), f(x)\,\theta(x, y))$. Similarly we obtain at least two nontrivial classes in the partition corresponding to the congruence relation $\theta(x, y)$ in the case, when x belongs to a cycle having more than one element. According to (2), $K_f(x)$ contains a cycle of the form $\{x\}$ and $\theta(x, y) = [\{f^i(y) : i \in N_0\} \cup \{x\}]$. Put $z = f(y)$. Then $\theta(x, z) < \theta(x, y)$, $\theta(x, z) = [\{f^i(z) : i \in N_0\} \cup \{x\}] = [x\,\theta(x, z)]$, which is a contradiction.

The following assertions are consequences of Lemma 2.1.

2.1′. Lemma. *Let $x \in A$. Then the property*

(1) *the connected component $K_f(x)$ possesses no cycle*

is a k-property.

2.1″. *Let $g \in R(f)$, $x \in A$. Then $K_f(x)$ possesses no cycle if and only if $K_g(x)$ possesses no cycle.*

If analogous situations will occur in what follows, we shall mention mostly the result analogous to Lemma 2.1, but we shall automatically aply also results analogous to 2.1′ and 2.1″.

2.2. Lemma. *Let $x, x' \in A$ and let $K_f(x)$ and $K_f(x')$ possess no cycles. Then $K_f(x) = K_f(x')$ if and only if the partition of $\theta(x, x')$ contains $\aleph_0$ nontrivial classes.*

Proof. Let $K_f(x) \neq K_f(x')$. Then $\theta(x, x') = [\{x, x'\}, \{f(x), f(x')\}, \{f^2(x), f^2(x')\}, \ldots]$. Now suppose that $K_f(x) = K_f(x')$. Then there are $m, n \in N_0$ such that $f^m(x) = f^n(x')$, where $m - n$ is uniquely determined. Without loss of generality we can assume that $m - n > 0$ and that m is the least nonnegative integer such that $f^m(x) = f^n(x')$ for some $n \in N_0$. Then the partition corresponding to the congruence relation $\theta(x, x')$ has exactly m nontrivial classes, namely $x\,\theta(x, x'), f(x)\,\theta(x, x'), \ldots$ $\ldots, f^{m-1}(x)\,\theta(x, x')$.

Now we shall introduce the following denotations. Let $u, v \in A$ be such that $K_f(u) = K_f(v)$ possesses no cycle. We set $u <_f v$, if $v = f^n(u)$ for some $n \in N$. Further we put $u\,\sigma^f\,v$, if there is $n \in N_0$ such that $f^n(u) = f^n(v)$. If $u\,\sigma^f\,v$ and n is the least nonnegative integer such that $f^n(u) = f^n(v)$, we shall write $u\,\sigma^f_n\,v$. The following property will be denoted as $(c(u, v))$:

$(c(u, v))$ There exists $\theta \in \mathrm{Con}(A, f)$ such that card $v\theta = \aleph_0$, $u\theta = \{u\}$, and if $w \in v\theta$, then the relation $w\,\sigma^f\,v$ is not valid.

2.3. Lemma. *Let $x, u, v \in A$. Assume that (i) $K_f(x)$ possesses no cycle, and (ii) $u, v \in K_f(x)$. Then the condition*

$$(\mathrm{n})\; u\,\sigma^f_n\,v$$

is a k-property for each $n \in N$.

Proof. At first let us recall that according to 2.1 and 2.2 (i) and (ii) are k-properties. Let the assumption of the lemma hold. From 1.1 it follows that $u\,\sigma^f_1\,v$ if and only if $\theta(u, v) = [\{u, v\}]$, hence (1) is a k-property. Suppose that $n \in N$, $n > 1$ and that (m) is a k-property for each $m \in N$, $m < n$. We shall prove the following assertion: $u\,\sigma^f_n\,v$ if and only if the relation $u\,\sigma^f_m\,v$ is not valid for any $m \in N$, $m < n$ and if there exist $u_i, v_i \in A$, $u_i \neq v_i$ for $i = 1, 2, \ldots, n - 1$ such that

$$u_i\,\sigma^f_i\,v_i \quad \text{for} \quad i = 1, \ldots, n - 1$$

and that

$$\theta(u, v) = [\{u_1, v_1\}, \{u_2, v_2\}, \ldots, \{u_{n-1}, v_{n-1}\}, \{u, v\}]\,.$$

Let $u\,\sigma^f_n\,v$. Since n is the least positive integer with $f^n(u) = f^n(v)$, it is obvious that the relation $u\,\sigma^f_m\,v$ holds for no $m \in N$, $m < n$. Denote $u_i = f^{n-i}(u)$, $v_i = f^{n-i}(v)$ for each $i \in \{1, \ldots, n - 1\}$; then we have $u_i \neq v_i$, $u_i\,\sigma^f_i\,v_i$ and $\theta(u, v) = [\{v_1, u_1\}, \ldots$ $\ldots, \{u_{n-1}, v_{n-1}\}, \{u, v\}]$. Now let us prove that the condition of the above assertion is sufficient. We have $f(u)\,\theta(u,v)\,f(v), \ldots, f^{n-1}(u)\,\theta(u,v)\,f^{n-1}(v)$. Since for no $m \in N$, $m < n$ the relation $u\ \sigma_m^f\ v$ is valid, it follows that $f(u) \neq f(v), \ldots, f^{n-1}(u) \neq f^{n-1}(v)$. If $f(u) = u_i, f(v) = v_i$ for some $i \in \{1, \ldots, n-1\}$, then $f^{i+1}(u) = f^i(u_i) = f^i(v_i) = f^{i+1}(v)$, and therefore $i = n - 1$. The case when $f(u) = v_i$, $f(v) = u_i$ is analogous. Hence $u\ \sigma_n^f\ v$. Thus (n) is a k-property.

2.4. Lemma. *Let $x \in A$, $u, v \in K_f(x)$ and let $K_f(x)$ possess no cycle. Under these assumptions $u <_f v$ if and only if $(c(u, v))$ is valid and $(c(v, u))$ fails to hold.*

Proof. Suppose that $u <_f v$ and let n be the positive integer such that $f^n(u) = v$. Put $\theta = \theta(v, f(v))$; then $\operatorname{card} v\theta = \operatorname{card}\{f^i(v) : i \in N_0\} = \aleph_0$, $u\theta = \{u\}$ and for no $i \in N$ the relation $f^i(v)\,\sigma\,v$ is valid. Hence $(c(u, v))$ holds. Now let $\theta' \in \operatorname{Con}(A, f)$ be such that $\operatorname{card} u\theta' = \aleph_0$. Denote by $\{u_i : i \in N\}$ the set such that $u \notin \{u_i : i \in N\}$, $u\theta' = \{u\} \cup \{u_i : i \in N\}$. We have $f^n(u)\,\theta'\,f^n(u_i)$, i.e. $v\,\theta'\,f^n(u_i)$ for each $i \in N$. If $v\theta' = \{v\}$, then $f^n(u_i) = v = f^n(u)$, therefore $u\ \sigma_n^f\ u_i$ for each $i \in N$. Thus the condition $(c(v, u))$ is not valid.

Now assume that the relation $u <_f v$ does not hold. If $v <_f u$, then $(c(v, u))$ is valid and $(c(u, v))$ is not valid. Let $v <_f u$ does not hold. Put $\theta = \theta(v, f(v))$, $\theta' = \theta(u, f(u))$. Then from the definition of θ or θ', respectively, it follows that $(c(u, v))$ and $(c(v, u))$ are valid.

2.4′. Lemma. *Let $x \in A$, $u, v \in K_f(x)$ and let $K_f(x)$ possess no cycle. Then the property*

(1) $$u <_f v$$

is a k-property.

2.4″. *Let $x \in A$, $u \in K_f(x)$ and $K_f(x)$ possess no cycle. If $g \in R(f)$, then $g(u) = f(u)$.*

Proof. The relation $f(u) = g(u)$ follows from the fact that $u <_f v$ if and only if $u <_g v$ (cf. Lemma 2.4′).

2.5. Lemma. *Let $x \in A$ and let $K_f(x)$ possess no cycle. If $u, z \in A$, $z \notin K_f(x)$, then the following conditions are equivalent:*

(1) $f(z) = u$.

(2) $u \in f(x)\,\theta(z, x) - K_f(x)$.

Proof. Let the assumption of the lemma hold. Suppose that $f(z) = u$. Then $u \notin K_f(x)$ and $u\,\theta(z, x)\,f(x)$, i.e., $u \in f(x)\,\theta(z, x) - K_f(x)$. Now let (2) be valid. If $K_f(z)$ possesses no cycle, we obtain $\theta(z, x) = [\{z, x\}, \{f(z), f(x)\}, \ldots]$, therefore $f(x)\,\theta(z, x) - K_f(x) = \{f(z)\}$. Assume that $K_f(z)$ possesses a cycle C. If $z \notin C$, then $f(x)\,\theta(z, x) = \{f(z), f(x)\}$; if $z \in C$, $\operatorname{card} C = n$, then $f(x)\,\theta(z, x) = \{f(z), f(x), f^{n+1}(x), f^{2n+1}(x), \ldots\}$, hence $f(x)\,\theta(z, x) - K_f(x) = \{f(z)\}$.

2.6. Theorem. *Let $x \in A$ and let $K_f(x)$ possess no cycle. Then the operation f can be determined by means of congruence relations of (A, f).*

Proof. The assertion follows from Lemmas 2.2, 2.4 and 2.5.

2.7. Corollary. (Cf. [7], Theorem 6.) *Let $x \in A$ and $K_f(x)$ possess no cycle. Then* $R(f) = \{f\}$.

Let us remark that by proving Thm. 6 in [7] and in the corresponding lemmas ([7], pp. 13–39, 45–50) there is used the operation f itself, i.e., the author does not work merely with the system $\mathrm{Con}(A, f)$.

3. THE CASE OF SMALL CYCLES; AUXILIARY RESULTS

In this section some auxiliary results will be established which will be applied in 4 and in Part II.

By a *small cycle* of an algebra (A, f) we shall understand a cycle C with card $C \leqq 2$. A cycle C_1 with card $C_1 > 2$ is called *large*.

3.1. Lemma. *Let $C \subseteq A$,* card $C > 2$. *Then C is a cycle of (A, f) if and only if the following conditions are satisfied:*

(a) *If $t' \in C$, $z \in A$, then there is $t \in C$, $t \neq t'$ such that* card $t'\,\theta(z, t) > 1$.

(b) *If $t_1 \in A$ and if for each $z \in A$ there is $t \in C$, $t \neq t_1$ with* card $t_1\,\theta(z, t) > 1$, *then $t_1 \in C$.*

(c) *If x, y, z are distinct elements of C, then* card $z\,\theta(x, y) > 1$.

Proof. It can be easily verified that if C is a cycle, then the conditions (a)–(c) are satisfied. Let us suppose that (a)–(c) are valid. If there are $z, y \in C$ with $K_f(z) \neq K_f(y)$, then there is $x \in C$ such that either $x \notin K_f(z)$ or $z \notin K_f(y)$. Suppose that $x \notin K_f(z)$. Then $z\,\theta(x, y) = \{z\}$, which is a contradiction with (c). Hence we obtain that $C \subseteq K_f(x)$ for some $x \in A$. At first assume that there is $t' \in C$ such that t' does not belong to any cycle. Denote $z = f(t')$. From the condition (a) it follows that there is $t \in C$, $t \neq t'$ with card $t'\,\theta(z, t) > 1$. therefore $t' = f^n(t)$ for some $n \in N$. Similarly, using the condition (a) for the elements $t \in C$ and $f(t) \in A$ we obtain that there exists $s \in C$ such that $s \in \bigcup_{m \in N} f^{-m}(t)$. Then we have $s\,\theta(t, t') = \{s\}$, which is a contradiction with (c). Thus C is a subset of some cycle $C_0^f[y]$, $y \in A$. Let $t_1 \in C_0^f[y]$ and let $z \in A$. If $z \notin K_f(y)$, $t \in C$, $t \neq t_1$, then card $t_1\,\theta(z, t) > 1$. If $z \in C_0^f[y]$, $t \in C$, $t \neq t_1$, $t \neq z$ (such t exists, since card $C > 2$), then card $t_1\,\theta(z, t) > 1$. If $z \in C_n^f[y]$ for $n \in N$ and if $k, m \in N$ are such that $k = m$ card $C_0^f[y] - n \geqq 0$, then there exists $t \in C$, $t \neq t_1$, $t \neq f^k(z)$ and then card $t_1\,\theta(z, t) > 1$. Hence the condition (b) implies that $t_1 \in C$ and the proof is complete.

From Lemma 3.1 it follows that by means of $\mathrm{Con}(A, f)$ we can find all large cycles of (A, f).

In what follows (in Part I) we shall assume that each connected component of (A, f) contains a cycle the cardinality of which is less than 3. In the following Lemmas 3.2–3.18 we suppose that distinct letters $x, y, z, \ldots$ denote distinct elements. Moreover, we shall not prove the implications (1) $\Rightarrow$ (2) in 3.2–3.18; it can be easily verified.

The figure that corresponds to the following Lemma 3.2 is denoted as Fig. 3.2, and similarly for other lemmas in 3. If the same figure is related also to some lemma of 4, then we denote it also by the number of the corresponding lemma from 4. In the figures we use the following denotations:

• a elements with the property $f(a) = a$;

○ b a pair of elements $b, c \in A$ with $f(c) = b$
↑
○ c (the possibility $f(b) = b$ being not excluded).

3.2. Lemma. Let $x, y, x', y' \in A$. The following conditions are equivalent:
(1) $f(x) = y$, $f(y) = x$, $f(x') = y'$, $f(y') = x'$.
(2) $\theta(x, y) = [\{x, y\}]$, $\theta(x, x') = \theta(y, y') = [\{x, x'\}, \{y, y'\}]$, $\theta(x, y') = \theta(x', y) = [\{x, y'\}, \{x', y\}]$, $\theta(x', y') = [\{x', y'\}]$.

Proof. Let us suppose that the condition (2) is satisfied. Since $\theta(x, y) = [\{x, y\}]$, according to 1.1 we obtain (a) $f(x) = x$, $f(y) = y$, or (b) $f(x) = f(y) = x$, or (c) $f(x) = f(y) = y$, or (d) $f(x) = f(y) \notin \{x, y\}$, or (e) $f(x) = y, f(y) = x$. The situation is analogous for the elements x', y'; denote the corresponding cases (a')–(e').

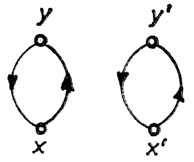


Fig. 3.2, 4.1(a)

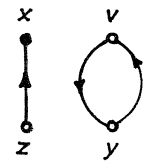
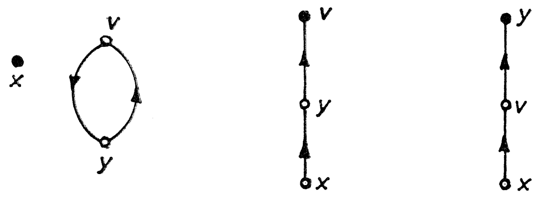


Fig. 3.4, 4.1(b)



(a) (b) (c)

Fig. 3.3, 4.5

Suppose that $f(x) = x$. Then $x\ \theta(x, x') f(x')$, hence according to (2) we have $f(x') = x'$ or $f(x') = x$, thus $\theta(x, x') = [\{x, x'\}]$, which is a contradiction with (2). We get $f(x) \neq x$, and analogous relations (with respect to the symmetry) are valid for the elements y, x', y'. Therefore no from the cases (a), (b), (c), (a'), (b'), (c') can occur. Further we have $f(x)\, \theta(x, x') f(x')$ and (2) implies that (i) $f(x) = f(x')$, or

(ii) $\{f(x, f(x')\} = \{x, x'\}$, or (iii) $\{f(x), f(x')\} = \{y, y'\}$. From (i) or (ii) it follows that $\theta(x, x') = [\{x, x'\}]$, a contradiction with (2). Suppose that (iii) and (d) is valid. Then $f(x) = f(y) = y'$, hence $f(x') = y$ and (e') does not hold, therefore (d') holds and $f(y') = f(x') = y$. In this case we obtain $\theta(y, y') = [\{y, y'\}]$, which is a contradiction with (2), thus (e) must be satisfied. Analogously it can be verified that (e') must hold.

3.3. Lemma. *Let $x, y, v \in A$. The following conditions are equivalent*:

(1) (a) $f(x) = x$, $f(y) = v$, $f(v) = y$, *or* (b) $f(x) = y$, $f(y) = f(v) = v$, *or* (c) $f(x) = v$, $f(v) = f(y) = y$.

(2) $\theta(y, v) = [\{y, v\}]$, $\theta(x, y) = \theta(x, v) = [\{x, y, v\}]$.

Proof. Suppose that (2) is valid. Since $\theta(y, v) = [\{y, v\}]$, from 1.1 it follows that (a') $f(y) = y, f(v) = v$, or (b') $f(y) = f(v) = v$, or (c') $f(y) = f(v) = y$, or (d') $f(v) = f(y) \notin \{v, y\}$, or (e') $f(y) = v, f(v) = y$. In the case (d') we have $f(x)\,\theta(x, y)\,f(y)$, hence either $f(x) = f(y)$ or $\{f(x), f(y)\} \subseteq \{x, y, v\}$. If $f(x) = f(y)$, then $\theta(x, y) = [\{x, y\}]$, which is a contradiction. By considering the second possibility we obtain that either $f(y) = x$, $f(x) = y$, or $f(x) = y$, thus either $\theta(x, y) = [\{x, y\}]$ or $\theta(x, v) = [\{x, v\}]$; a contradiction with (2). Further, since $\theta(x, y) \neq [\{x, y\}]$, we get $f(x)\,\theta(x, y)\,f(y)$ and $f(x) \neq f(y)$. Analogously (with respect to the symmetry we obtain that $f(x) \neq f(v)$. Next from the relation $\theta(x, y) = [\{x, y, v\}]$ it follows that $f(x) \in \{x, y, v\}$. Let (a') hold. If $f(x) \in \{x, y\}$, then $\theta(x, y) = [\{x, y\}]$, and if $f(x) = v$, then $\theta(x, v) = [\{x, v\}]$, which is a contradiction. If (b') is valid and $f(x) \in \{x, v\}$, then $\theta(x, v) = [\{x, v\}]$, a contradiction, thus if (b') holds, then $f(x) = y$, and we have the case (b). The case (c') is analogous to (b') and we obtain that the condition (c) is satisfied. Now let (e') be valid. We have shown that $f(x) \neq f(v) = y$, $f(x) \neq f(y) = v$, thus $f(x) = x$, and this is the case (a).

3.4. Lemma. *Let $x, y, v, z \in A$. The following conditions are equivalent*:

(1) $f(x) = f(z) = x$, $f(y) = v$, $f(v) = y$.

(2) $\theta(y, v) = [\{y, v\}]$, $\theta(x, y) = \theta(x, v) = [\{x, y, v\}]$, $\theta(x, z) = [\{x, z\}]$, $\theta(z, y) = [\{x, y, z, v\}]$.

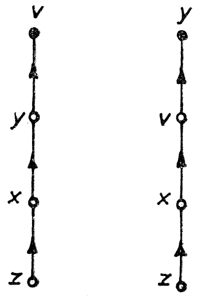
Proof. Let (2) be valid. According to 3.3 we obtain that some of the following conditions (a)–(c) is satisfied: (a) $f(x) = x$, $f(y) = v$, $f(v) = y$; (b) $f(x) = y$, $f(y) = f(v) = v$; (c) $f(x) = v$, $f(v) = f(y) = y$. Since $\theta(x, z) = [\{x, z\}]$, from 1.1 it follows, that if (b) is valid, then $f(z) = f(x) = y$, if (c) is valid, then $f(z) = f(x) = v$, and if (a) holds, then $f(z) = x$ or $f(z) = z$. In the cases (b) and (c) we obtain $\theta(z, y) = [\{z, y, v\}]$, which is a contradiction. Consider the case (a). If $f(z) = z$, then $\theta(z, y) = [\{z, y, x\}]$, a contradiction, hence we get that $f(z) = x$.

3.5. Lemma. *Let $x, y, v, z \in A$. The following conditions are equivalent*:

(1) (a) $f(z) = x$, $f(x) = y$, $f(y) = f(v) = v$, *or* (b) $f(z) = x$, $f(x) = v$, $f(y) = f(v) = y$.

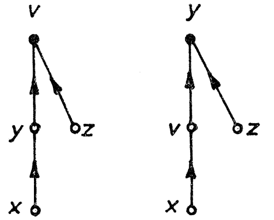
(2) $\theta(y, v) = [\{y, v\}]$, $\theta(x, y) = \theta(x, v) = [\{x, y, v\}]$, $\theta(z, x) = \theta(z, v) = [\{z, x, y, v\}]$.

Proof. Suppose that (2) is valid. From 3.3 it follows that (a') $f(x) = x, f(y) = v, f(v) = y$, or (b') $f(x) = y$, $f(y) = f(v) = v$, or (c') $f(x) = v$, $f(v) = f(y) = y$. Since $f(z)\,\theta(z, x) f(x)$, in view of (2) we obtain that $f(z) \in \{z, x, y, v\}$. If (a') and



(a) (b)

Fig. 3.5, 4.2



(a) (b)

Fig. 3.6, 4.3

$f(z) \in \{z, y, v\}$, then $\{x\} \in \theta(z, v)$, and if (a') and $f(z) = x$, then $\theta(z, x) = [\{z, x\}]$, which is a contradiction. In the cases (b') and (c') we get $\{x\} \in \theta(z, v)$, if we suppose that $f(z) \in \{z, y, v\}$. Hence $f(z) = x$. In the case (b') we have (a) and in the case (c') we have (b) from the condition (1).

3.6. Lemma. *Let $x, y, v, z \in A$. The following conditions are equivalent*:

(1) (a) $f(x) = y$, $f(y) = f(v) = f(z) = v$, *or* (b) $f(x) = v$, $f(y) = f(v) = f(z) = y$.

(2) $\theta(y, v) = [\{y, v\}]$, $\theta(x, y) = \theta(x, v) = [\{x, y, v\}]$, $\theta(x, z) = [\{x, z\}, \{y, v\}]$.

Proof. Let (2) be valid. From 3.3 it follows that (a') $f(x) = x, f(y) = v, f(v) = y$, or (b') $f(x) = y, f(y) = f(v) = v$, or (c') $f(x) = v, f(v) = f(y) = y$. In the case (a') we have $x\,\theta(x, z) f(z)$, hence $f(z) \in \{x, z\}$, but then $\theta(x, z) = [\{x, z\}]$, a contradiction. In the case (b') we get $y\,\theta(x, z) f(z)$ and $\theta(x, z) \neq [\{x, z\}]$, thus from (2) for $\theta(x, z)$ it follows that $f(z) = v$; we have obtained (a) in (1). If (c') is valid, then $v\,\theta(x, z) f(z)$, hence we get $f(z) = y$, which is the case (b) in (1).

3.7. Lemma. *Let $y, x, v, z, u \in A$. The following conditions are equivalent*:

(1) $f(x) = y$, $f(y) = f(v) = f(z) = v$, $f(u) = z$.

(2) $\theta(y, v) = [\{y, v\}]$, $\theta(x, y) = \theta(x, v) = [\{x, y, v\}]$, $\theta(x, z) = [\{x, z\}, \{y, v\}]$, $\theta(u, x) = [\{u, x\}, \{z, y\}]$.

Proof. Let (2) be valid. From 3.6 it follows that either (a) $f(x) = y$, $f(y) = f(v) = f(z) = v$, or (b) $f(x) = v$, $f(y) = f(v) = f(z) = y$. In the case (b) we get

$f(u)\,\theta(u, x)\,v$, hence $f(u) = v$, but then $\theta(u, x) = [\{u, x\}]$, which is a contradiction. If (a) is valid, then $f(u)\,\theta(u, x)\,y$, and since $\theta(u, x) = [\{u, x\}, \{z, y\}]$, we get $f(u) = z$.

3.8. Lemma. *Let $x, y, v, z, u \in A$. The following conditions are equivalent*:

(1) $f(x) = y,\ f(y) = f(v) = f(z) = v,\ f(u) = u$.

(2) $\theta(y, v) = [\{y, v\}],\quad \theta(x, y) = \theta(x, v) = [\{x, y, v\}],\quad \theta(x, z) = [\{x, z\}, \{y, v\}]$, $\theta(u, z) = [\{u, z, v\}]$, $\theta(u, y) = [\{u, y, v\}]$.

Proof. Let (2) be valid. According to 3.6 we have either (a) $f(x) = y$, $f(y) = f(v) = f(z) = v$, or (b) $f(x) = v$, $f(y) = f(v) = f(z) = y$. First consider the case (b). We have $f(u)\,\theta(u, z)\,y$ and $\theta(u, z) = [\{u, z, v\}]$, hence $f(u) = y$, and then $\theta(u, y) = [\{u, y\}]$, which is a contradiction. Suppose that (a) holds. Then $f(u)\,.\,\theta(u, z)\,v$, thus $f(u) \in \{u, z, v\}$. If $f(u) = z$, then $\theta(u, y) = [\{u, y\}, \{z, v\}]$; if $f(u) = v$, then $\theta(u, y) = [\{u, y\}]$. This is a contradiction, therefore $f(u) = u$ and the proof is complete.

3.9. Lemma. *Let $x, y, v, z \in A$, The following conditions are equivalent*:

(1) (a) $f(x) = y, f(y) = f(v) = v, f(z) = z$, or (b) $f(x) = x, f(y) = v, f(v) = f(z) = y$.

(2) $\theta(y, v) = [\{y, v\}]$, $\theta(x, y) = \theta(x, v) = [\{x, y, v\}]$, $\theta(z, v) = [\{z, v\}]$, $\theta(z, y) = [\{z, y, v\}]$.

Proof. Let (2) be valid. According to 3.3 we have (a') $f(x) = x$, $f(y) = v$, $f(v) =$

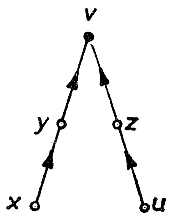


Fig. 3.7, 4.1 (c)

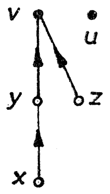


Fig. 3.8, 4.1 (d)

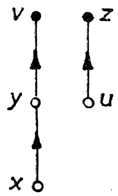
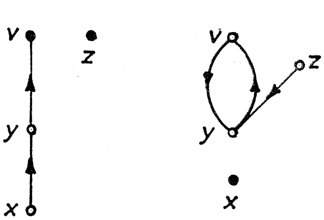


Fig. 3.10, 4.1 (e)



(a) (b)

Fig. 3.9, 4.4

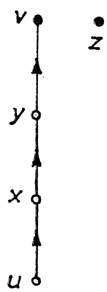


Fig. 3.11, 4.1 (f)

$= y$, or (b') $f(x) = y$, $f(y) = f(v) = v$, or (c') $f(x) = v$, $f(v) = f(y) = y$. In the case (c') we get (according to 1.1 and $\theta(z, v)$) that $f(z) = f(v) = y$. But then $\theta(z, y) = [\{z, y\}]$, which is a contradiction. If (a') or (b') holds, then $f(y) = v$, hence $f(z)\,\theta(z, y)\,v$. If $f(z) = z$ in the case (a') or if $f(z) = y$ in the case (b'), then $\theta(z, v) = [\{z, v, y\}]$, a contradiction. If $f(z) = v$ in the case (a') or (b'), then $\theta(z, y) = [\{z, y\}]$, a contradiction. Thus we have got that (1) is valid.

3.10. Lemma. *Let* $x, y, v, z, u \in A$. *The following conditions are equivalent:*

(1) $f(x) = y$, $f(y) = f(v) = v$, $f(u) = f(z) = z$.

(2) $\theta(y, v) = [\{y, v\}]$, $\theta(x, y) = \theta(x, v) = [\{x, y, v\}]$, $\theta(z, v) = [\{z, v\}]$, $\theta(z, y) = [\{z, y, v\}]$, $\theta(u, y) = [\{u, y\}, \{z, v\}]$, $\theta(u, v) = [\{u, v, z\}]$.

Proof. Let (2) be valid. According to 3.9 we have either (a) $f(x) = y$, $f(y) = f(v) = v$, $f(z) = z$, or (b) $f(x) = x$, $f(y) = v$, $f(v) = f(z) = y$. Then $f(u)\,\theta(u, y)\,v$, hence $f(u) \in \{v, z\}$. If $f(u) = v$, then $\theta(u, y) = [\{u, y\}]$, a contradiction, thus $f(u) = z$. In the case (b) we get $\theta(u, v) = [\{u, v, y, z\}]$, which is a contradiction. Therefore the condition (1) is satisfied.

3.11. Lemma. *Let* $x, y, v, z, u \in A$. *The following conditions are equivalent:*

(1) $f(u) = x$, $f(x) = y$, $f(y) = f(v) = v$, $f(z) = z$.

(2) $\theta(y, v) = [\{y, v\}]$, $\theta(x, y) = \theta(x, v) = [\{x, y, v\}]$, $\theta(z, v) = [\{z, v\}]$, $\theta(z, y) = [\{z, y, v\}]$, $\theta(u, x) = [\{u, x, y, v\}]$, $\theta(u, z) = [\{x, y, u, v, z\}]$.

Proof. Let (2) be valid. From 3.9 it follows that either (a) $f(x) = y$, $f(y) = f(v) = v$, $f(z) = z$, or (b) $f(x) = x$, $f(y) = v$, $f(v) = f(z) = y$ holds. Since $f(u)\,\theta(u, x)\,.\,f(x)$, we have $f(u) \in \{u, x, y, v\}$. Consider the case (b). If $f(u) \in \{u, y, v\}$, then $\{x\} \in \theta(u, z)$, and if $f(u) = x$, then $\theta(u, z) = [\{u, z\}, \{x, y, v\}]$, which is a contradiction. Hence (a) is valid. If $f(u) \in \{u, y, v\}$, then $\{x\} \in \theta(u, z)$, and from this contradiction it follows that $f(u) = x$.

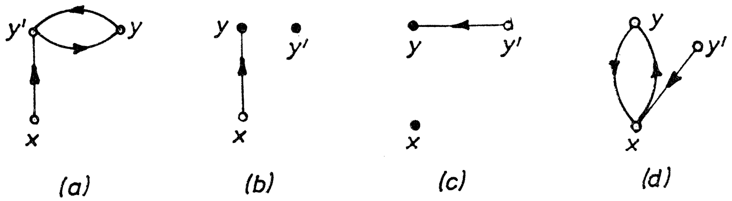


Fig. 3.12, 4.9

3.12. Lemma. *Let* $x, y, y' \in A$. *The following conditions are equivalent:*

(1) (a) $f(x) = f(y) = y'$, $f(y') = y$, *or* (b) $f(x) = f(y) = y$, $f(y') = y'$, *or* (c) $f(x) = x$, $f(y) = f(y') = y$, *or* (d) $f(x) = y$, $f(y) = f(y') = x$.

(2) $\theta(x, y) = [\{x, y\}]$, $\theta(y, y') = [\{y, y'\}]$, $\theta(x, y') = [\{x, y, y'\}]$.

Proof. Suppose that (2) is valid. Since $\theta(x, y) = [\{x, y\}]$, according to 1.1 we have (a′) $f(x) = x$, $f(y) = y$, or (b′) $f(x) = f(y) = y$, or (c′) $f(x) = f(y) = x$, or (d′) $f(x) = f(y) \notin \{x, y\}$, or (e′) $f(x) = y$, $f(y) = x$. Similarly, since $\theta(y, y') = [\{y, y'\}]$, we have (a″) $f(y) = y$, $f(y') = y'$, or (b″) $f(y) = f(y') = y'$, or (c″) $f(y) = f(y') = y$, or (d″) $f(y) = f(y') \notin \{y, y'\}$, or (e″) $f(y) = y'$, $f(y') = y$. It is obvious that (a′) and (d″) cannot hold in the same time (because $f(y) = y$ in (a′) and $f(y) \neq y$ in (d″)). After excluding other analogical pairs which obviously cannot occur, we shall consider only the remaining cases, i.e. the pairs (a′)–(a″), (a′)–(c″), (b′)–(a″), (b′)–(c″), (c′)–(d″), (d′)–(b″), (d′)–(d″), (d′)–(e″), (e′)–(d″). In the cases (a′)–(a″), (b′)–(c″), (c′)–(d″), (d′)–(b″) and (d′)–(d″) we have $\theta(x, y') = [\{x, y'\}]$, which is a contradiction. Hence we have (a′)–(c″), i.e. (c); or (b′)–(*a*″), i.e. (b); or (d′)–(e″), i.e. (a); or (e′)–(d″), i.e. (d).

3.13. Lemma. *Let* $x, y, y', z \in A$. *The following conditions are equivalent*:

(1) $f(x) = f(y) = y'$, $f(y') = y$, $f(z) = x$.

(2) $\theta(x, y) = [\{x, y\}]$, $\theta(y, y') = [\{y, y'\}]$, $\theta(x, y') = [\{x, y, y'\}]$, $\theta(z, x) = [\{z, x, y', y\}]$, $\theta(z, y') = [\{z, y'\}, \{x, y\}]$.

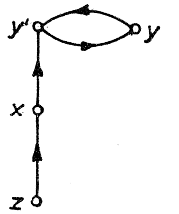


Fig. 3.13, 4.1(g)

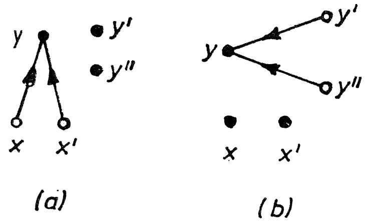


Fig. 3.15, 4.7

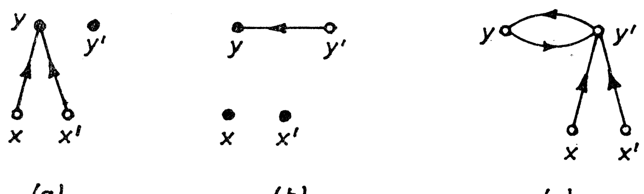


Fig. 3.14, 4.8

Proof. Assume that (2) is valid. From 3.12 it follows that some of the following conditions is satisfied: (a) $f(x) = f(y) = y'$, $f(y') = y$; (b) $f(x) = f(y) = y$, $f(y') = y'$; (c) $f(x) = x$, $f(y) = f(y') = y$; (d) $f(x) = y$, $f(y) = f(y') = x$. In the case (a) we have $f(z)\, \theta(z, y')\, y$ and $\theta(z, y') \neq [\{z, y'\}]$, hence $f(z) \neq y$ and $f(z) = x$, because of $\theta(z, y') = [\{z, y'\}, \{x, y\}]$. We shall show that in the other cases we get a contradiction. If (b) is valid, then $f(z)\, \theta(z, y')\, y'$, hence $f(z) \in \{z, y'\}$, but then

$\theta(z, y') = [\{z, y'\}]$, which is a contradiction. If (c) holds, then $f(z)\,\theta(z, y')\,y$, $\theta(z, y') \neq [\{z, y'\}]$, thus $f(z) \neq y$ and from (2) it follows that $f(z) = x$. But in this case $\theta(z, x) = [\{z, x\}]$, and we have a contradiction. In the case (d) we have $f(z)\,\theta(z, y')\,x$, $\theta(z, y') \neq [\{z, y'\}]$, thus $f(z) = y$, $\theta(z, x) = [\{z, x\}]$, which is a contradiction as well.

3.14. Lemma. *Let $x, x', y, y' \in A$. The following conditions are equivalent:*

(1) (a) $f(x) = f(x') = f(y) = y$, $f(y') = y'$, *or* (b) $f(x) = x$, $f(x') = x'$, $f(y) = f(y') = y$, *or* (c) $f(x) = f(x') = f(y) = y'$, $f(y') = y$.

(2) $\theta(x, y) = [\{x, y\}]$, $\theta(y, y') = [\{y, y'\}]$, $\theta(x, y') = [\{x, y, y'\}]$, $\theta(x', y) = [\{x', y\}]$, $\theta(x', y') = [\{x', y, y'\}]$.

Proof. Let (2) be valid. According to 3.12 we obtain that (a′) $f(x) = f(y) = y'$, $f(y') = y$, or (b′) $f(x) = f(y) = y$, $f(y') = y'$, or (c′) $f(x) = x$, $f(y) = f(y') = y$, or (d′) $f(x) = y$, $f(y) = f(y') = x$ holds. Further, from 3.12 (for the elements x', y, y' instead of x, y, y') we get analogous conditions denoted by (a″)–(d″). We have to investigate only cases (a′)–(a″), (b′)–(b″), (c′)–(c″), because the remaining cases are impossible. Then (a′)–(a″) gives (c), (b′)–(b″) gives (a) and (c′)–(c″) gives (b).

3.15. Lemma. *Let $x, x', y, y', y'' \in A$. The following conditions are equivalent:*

(1) (a) $f(x) = f(x') = f(y) = y$, $f(y') = y'$, $f(y'') = y''$, *or* (b) $f(x) = x$, $f(x') = x'$, $f(y) = f(y') = f(y'') = y$.

(2) $\theta(x, y) = [\{x, y\}]$, $\theta(y, y') = [\{y, y'\}]$, $\theta(x, y') = [\{x, y, y'\}]$, $\theta(x', y) = [\{x', y\}]$, $\theta(x', y') = [\{x', y, y'\}]$, $\theta(y, y'') = [\{y, y''\}]$, $\theta(y', y'') = [\{y', y''\}]$.

Proof. Let (2) be valid. From 3.14 it follows that either (a′) $f(x) = f(x') = f(y) = y$, $f(y') = y'$, or (b′) $f(x) = x$, $f(x') = x'$, $f(y) = f(y') = y$, or (c′) $f(x) = f(x') = f(y) = y'$, $f(y') = y$ holds. Then in view of the relations for $\theta(y, y'')$ and for $\theta(y', y'')$ and according to 1.1 we get that if (a′) is satisfied, then $f(y'') = y''$, i.e. (a); if (b′) is valid, then $f(y'') = f(y') = y$, i.e. (b); if (c′) is valid, we get a contradiction. Thus the condition (1) is satisfied.

3.16. Lemma. *Let $x, x', y, y' \in A$. The following conditions are equivalent:*

(1) (a) $f(x) = f(y) = y$, $f(x') = f(y') = y'$, *or* (b) $f(x) = f(y) = y'$, $f(x') = f(y') = y$.

(2) $\theta(x, y) = [\{x, y\}]$, $\theta(y, y') = [\{y, y'\}]$, $\theta(x, y') = [\{x, y, y'\}]$, $\theta(x', y') = [\{x', y'\}]$, $\theta(x, x') = [\{x, x'\}, \{y, y'\}]$.

Proof. Let (2) be valid. According to 3.12 we have (a′) $f(x) = f(y) = y'$, $f(y') = y$, or (b′) $f(x) = f(y) = y$, $f(y') = y'$, or (c′) $f(x) = x$, $f(y) = f(y') = y$, or (d′) $f(x) = y$, $f(y) = f(y') = x$. Further, from 1.1 (for x', y' instead of x, y) it follows that if (a′), then $f(x') = y$; if (b′), then $f(x') = y'$ or $f(x') = x'$; if (c′), then $f(x') = y$; if (d′), then $f(x') = x$. Hence if (a′) is valid, then (b) holds. In the case (b) we get either (a) or, if $f(x') = x'$, a contradiction, since then $\theta(x, x') =$

$= [\{x, x', y\}]$. If either (c′) or (d′) holds, then $\theta(x, x') = [\{x, x', y\}]$, which is a contradiction.

3.17. *Let $x, x', y, y', z \in A$. The following conditions are equivalent:*

(1) $f(x) = f(y) = y$, $f(x') = f(y') = y'$, $f(z) = z$.

(2) $\theta(x, y) = [\{x, y\}]$, $\theta(y, y') = [\{y, y'\}]$, $\theta(x, y') = [\{x, y, y'\}]$, $\theta(x', y') = [\{x', y'\}]$, $\theta(x, x') = [\{x, x'\}, \{y, y'\}]$, $\theta(z, x) = [\{z, x, y\}]$, $\theta(z, y) = [\{z, y'\}]$.

Proof. Suppose that (2) holds. From 3.16 it follows that either (a) $f(x) = f(y) = y$, $f(x') = f(y') = y'$, or (b) $f(x) = f(y) = y'$, $f(x') = f(y') = y$ is valid. In the case (b) we have $f(z)\, \theta(z, x)\, y'$, hence $f(z) = y'$, but then $\theta(z, x) = [\{z, x\}]$, which is a contradiction. Thus (a) holds and we obtain $f(z)\, \theta(z, x)\, y$, therefore $f(z) \in \{z, x, y\}$. If $f(z) = x$, then $\theta(z, y') = [\{z, x, y, y'\}]$; if $f(z) = y$, then $\theta(z, y') = [\{z, y, y'\}]$. We have a contradiction in these both cases, hence $f(z) = z$ and the condition (1) is satisfied.

(a) (b)

Fig. 3.16, 4.6

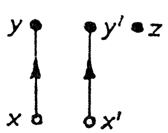


Fig. 3.17, 4.1 (h)

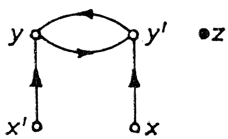


Fig. 3.18, 4.1 (i)

3.18. Lemma. *Let $x, x', y, y', z \in A$. The following conditions are equivalent:*

(1) $f(x) = f(y) = y'$, $f(x') = f(y') = y$, $f(z) = z$.

(2) $\theta(x, y) = [\{x, y\}]$, $\theta(y, y') = [\{y, y'\}]$, $\theta(x, y') = [\{x, y, y'\}]$, $\theta(x', y') = [\{x', y'\}]$, $\theta(x, x') = [\{x, x'\}, \{y, y'\}]$, $\theta(z, y) = \theta(z, y') = [\{z, y, y'\}]$.

Proof. Let (2) be valid. Frof 3.16 it follows that either (a) $f(x) = f(y) = y$, $f(x') = f(y') = y'$, or (b) $f(x) = f(y) = y'$, $f(x') = f(y') = y$ holds. In the case (a) we have $f(z)\, \theta(z, y)\, y$, hence $f(z) \in \{z, y, y'\}$. If $f(z) \in \{z, y\}$, then $\theta(z, y) = [\{z, y\}]$, which is a contradiction. If $f(z) = y'$, then $\theta(z, y') = [\{z, y'\}]$, a contradiction. Thus (b) holds and we obtain $f(z)\, \theta(z, y)\, y'$, $f(z) \in \{z, y, y'\}$. If $f(z) = y$, then $\theta(z, y') = [\{z, y'\}]$ and if $f(z) = y'$, then $\theta(z, y) = [\{z, y\}]$, which is a contradiction. Therefore $f(z) = z$.

Remark. Kurinnoj [7] investigated pairs of monounary algebras (A, f), (A, g) with $\mathrm{Con}(A, f) = \mathrm{Con}(A, g)$ and he searched to characterize those monounary algebras (A, f) which fulfil the condition $R(f) = \{f\}$. Lemma 3.13 shows that the assertion of Theorem 7 in [7] is not correct (namely, from Thm. 7 [7] it would follow that for the algebra (A, f), where $A = \{x, y, y', z\}$ and the condition (1) from 3.13 holds, we should have $R(f) \neq \{f\}$, contradicting 3.13).

4. THE CASE OF SMALL CYCLES; MAIN RESULTS

From Lemma 3.1 it follows that if C is a cycle with card $C > 2$, then C can be determined by means of $\mathrm{Con}(A, f)$. Further, according to the results from 2, the case when each connected component of (A, f) possesses a cycle, can be described by $\mathrm{Con}(A, f)$ as well. In this paragraph we shall assume that each connected component of (A, f) possesses a cycle C with card $C \leqq 2$.

At first we notice that the conditions (2) in 3.2–3.18 are expressed merely by the properties of $\mathrm{Con}(A, f)$, without using explicitly the operation f itself. Again let us remark that the figure that is related to some of the following lemmas is denoted by the same number as the corresponding lemma.

4.1. Lemma. *Let there exist distinct elements in A fulfilling the condition* (2) *from some of the lemmas* 3.2, 3.4, 3.7, 3.8, 3.10, 3.11, 3.13, 3.17 *and* 3.18. *Then f is uniquely determined by* $\mathrm{Con}(A, f)$.

Proof. (Cf. Fig. 4.1 (a)–(i)) The assertion follows from the corresponding lemmas and from 1.4 and 1.5.

We shall introduce the following notions. Let (A_1, f_1) be a monounary algebra, (B_1, f_1) be a subalgebra of (A_1, f_1) and let $T \subseteq B_1$, $b \in B_1$. We shall say that (A_1, f_1) is a $c'_b(T)$-extension of a monounary algebra (B_1, f_1), if for each $a \in A_1 - B_1$ some of the following two conditions is satisfied:

(i) there exists $t \in T$ such that either $f(a) = a$, $f(t) = t$, or $a \neq f(a) = f(t) \neq t$ (we shall say also that a and t behave in the same way);

(ii) there exists $n \in N$ such that $f^n(a) = b$.

If (A_1, f_1) is a $c'_b(T)$-extension of (B_1, f_1) such that for each $a \in A_1 - B_1$ the condition (i) is valid, then we shall say that (A_1, f_1) is a $c'(T)$-extension of (B_1, f_1). Further, if $T = \{t_1, \ldots, t_m\}$, we shall write also a $c'_b(t_1, \ldots, t_m)$-extension instead of $c'_b(\{t_1, \ldots, t_m\})$-extension, and similarly with $c'(T)$-extension.

Let the assumption of 4.1 be not satisfied.

4.2. Lemma. *Let there exist distinct elements $x, y, v, z \in A$ fulfilling the condition* (2) *from* 3.5. *Then $R(f)$ consists of two elements and they can be described by means of* $\mathrm{Con}(A, f)$. *The algebra (A, f) is a $c'_y(y)$-extension resp. $c'_v(v)$-extension of the algebra given in* Fig. 4.2 (a) *resp.* 4.2 (b).

Proof. The elements $x, y, v, z \in A$ fulfil the condition (1) from 3.5. We suppose that the assumption of 4.1 is not satisfied, thus there are no subalgebras of (A, f) isomorphic with those pictured in Fig. 3.4, 3.7 and 3.11. Hence (A, f) is connected and it can be only a $c'_y(y)$-extension resp. a $c'_v(v)$-extension of the algebras given in Fig. 3.5 (a) resp. 3.5 (b). Consider the case (a); in this case $f(y) = v$. Let $s, t \in A$, $s \neq t$. Then

(i) if $s, t \notin \{y, v\}$, then either $y\ \theta(s, t)\ v$ or $\{y\} \in \theta(s, t)$ and $\{v\} \in \theta(s, t)$;

(ii) if $s \notin \{v, y\}$ and s behaves in the same way as y, then $\theta(s, y) = [\{s, y\}]$ and $\theta(s, v) = [\{s, v\}]$;

(iii) if $s \notin \{v, y\}$ and s does not behave in the same way as y, then $y\ \theta(s, y)\ v$ and $y\ \theta(s\ v)\ v$.

From this follows that if the roles of y and v are interchanged, then the system of all congruence relations does not change. Hence card $R(f) \geqq 2$. The same assertion we obtain in the case (b). Since 1.4 implies that the pair $y, v \in A$ such that $f(y) = f(v) = v$ determines uniquely the operation f on A, we infer that card $R(f) \leqq 2$, hence card $R(f) = 2$. We have already verified that both the operations belonging to $R(f)$ can be described by means of $\text{Con}(A, f)$.

Let the assumptions of 4.1 and 4.2 be not satisfied.

4.3. *Let there exist distinct elements $x, y, v, z \in A$ fulfilling the condition* (2) *from* 3.6. *Then $R(f)$ consists of two elements and they can be described by means of* $\text{Con}(A, f)$. *The algebra (A, f) is a $c'(z, x)$-extension of some of the algebras given in* Fig. 4.3 (a), 4.3 (b).

Proof. The elements $x, y, v, z \in A$ fulfil the condition (1) from 3.6, i.e., we have either (a) $f(x) = y, f(y) = f(v) = f(z) = v$, or (b) $f(x) = v, f(v) = f(y) = f(z) = y$. Let $a \in A - \{x, y, v, z\}$. Consider the case (a). Since the assumptions of 4.1 and 4.2 do not hold, we obtain that (A, f) is connected (cf. Fig. 4.1 (d), 4.1 (b)) and that the element a behaves in the same way as z or as x. The case (b) is analogous, a behaves in the same way as z or as x, too. Let X resp. Z be the set of all $a \in A - \{x, y, v, z\}$ which behave in the same way as x resp. as z. Then

$$X = \{a \in A - \{x\} : \theta(a, x) = [\{a, x\}]\},$$

$$Z = \{a \in A - \{y, v, z\} : \theta(a, z) = [\{a, z\}]\},$$

$A = \{x, y, v, z\} \cup X \cup Z$ and either (a) $f(x) = f(x_1) = y,\ f(y) = f(v) = f(z) = f(z_1) = v$, or (b) $f(x) = f(x_1) = v,\ f(v) = f(y) = f(z) = f(z_1) = y$ for each $x_1 \in X$ and each $z_1 \in Z$. It is obvious that these two cases cannot be distinguished by means of $\text{Con}(A, f)$.

Let the assumptions of 4.1–4.3 be not satisfied.

4.4. Lemma. *Let there exist distinct elements $x, y, v, z \in A$ fulfilling the condition* (2) *from* 3.9. *Then $R(f)$ consists of two elements and they can be described* *by means of* Con(A,f). *The algebra* (A,f) *is a* $c'(x, z)$-*extension of some of the algebras given in* Fig. 4.4 (a), 4.4 (b).

Proof. The elements $x, y, v, z \in A$ fulfil the condition (1) from 3.9, hence we have either (a) $f(x) = y$, $f(y) = f(v) = v$, $f(z) = z$, or (b) $f(x) = x$, $f(y) = v$, $f(v) = f(z) = y$. Consider the case (a). The assumptions of 4.1–4.3 are not valid, thus, for each $a \in A - \{x, y, v, z\}$, we obtain that a behaves in the same way as x or in the same way as z (cf. Fig. 4.1 (b), 4.1(e), 4.1 (d), 4.1 (f)). The same we obtain considering the case (b) (cf. Fig. 4.1 (g), 4.1 (a), 4.1 (b)). Let X resp. Z be the system of all $a \in A - \{x, y, v, z\}$ such that a and x resp. a and z behave in the same way. Then

$$X = \{a \in A - \{x\} : \theta(a, x) = [\{a, x\}]\},$$
$$Z = \{a \in A - \{z\} : \theta(a, z) = [\{a, z\}]\},$$

$A = \{x, y, v, z\} \cup X \cup Z$ and either (a) $f(x) = f(x_1) = y$, $f(y) = f(v) = v$, $f(z) = z$, $f(z_1) = z_1$, or (b) $f(x) = x$, $f(x_1) = x_1$, $f(y) = v$, $f(v) = f(z) = f(z_1) = y$ is valid for each $x_1 \in X$ and $z_1 \in Z$. It is obvious that these cases cannot be distinguished by means of Con(A,f).

Let the assumptions of 4.1–4.4 be not valid.

4.5. Lemma. *Let there exist distinct elements* $x, y, v \in A$ *fulfilling the condition* (2) *from* 3.3. *Then* $R(f)$ *consists of three elements and they can be described by means of* Con(A,f). *The algebra* (A,f) *is a* $c'(x)$-*extension of some of the algebras given in* Fig. 4.5 (a)–(c).

Proof. The elements $x, y, z \in A$ fulfil the condition (1) from 3.3, i.e., we have (a) $f(x) = x$, $f(y) = v$, $f(v) = y$, or (b) $f(x) = y$, $f(y) = f(v) = v$, or (c) $f(x) = v$, $f(v) = f(y) = y$. Let $a \in A - \{x, y, v\}$. The assumptions of 4.1–4.4 are not satisfied, hence if (a) is valid, then x and a behave in the same way (cf. Fig. 4.4 (b),4.1(a), 4.1 (b)). Analogously, if (b) or (c) is valid, then x and a behave in the same way as well (cf. Fig. 4.4 (a), 4.3 (a), 4.2 (a)). Let X be the set of all $a \in A - \{x, y, v\}$ which behave in the same way as x. Then

$$X = \{a \in A - \{x\} : \theta(a, x) = [\{a, x\}]\},$$

$A = \{x, y, v\} \cup X$ and we have (a) $f(x) = x$, $f(x_1) = x_1$, $f(y) = v$, $f(v) = y$, or (b) $f(x) = f(x_1) = y$, $f(y) = f(v) = v$, or (c) $f(x) = f(x_1) = v$, $f(v) = f(y) = y$ for each $x_1 \in X$. Obviously, these cases cannot be distinguished by means of Con(A,f).

Let the assumptions of 4.1–4.5 be not valid.

4.6. Lemma. *Let there exist distinct elements* $x, x', y, y' \in A$ *fulfilling the condition* (2) *from* 3.16. *Then* $R(f)$ *consists of two elements and they can be described by means of* Con(A,f). *The algebra is a* $c'(x, x')$-*extension of some of the algebras given in* Fig. 4.6 (a), 4.6 (b).

Proof. The elements $x, x', y, y' \in A$ fulfil the condition (1) from 3.16, thus we have either (a) $f(x) = f(y) = y, f(x') = f(y') = y'$, or (b) $f(x) = f(y) = y', f(x') = f(y') = y$. Let $a \in A - \{x, x', y, y'\}$. The assumptions of 4.1–4.5 are not satisfied, hence if (a) is valid, then a behaves in the same way as x or as x' (cf. Fig. 4.1 (h), 4.1 (b), 4.5 (b)). The same assertion holds if (b) is valid (cf. Fig. 4.5 (a), 4.1 (a), 4.1 (g)). Let X resp. X' be the set of all $a \in A - \{x, x', y, y'\}$ such that a and x resp. a and x' behave in the same way. Then

$X = \{a \in A - \{x, y\} : \theta(a, x) = [\{a, x\}]\}$, $X' = \{a \in A - \{y'\} : \theta(a, x') = [\{a, x'\}]\}$, $A = \{x, x', y, y'\} \cup X \cup X'$ and either (a) $f(x) = f(x_1) = f(y) = y$, $f(x') = f(x'_1) = f(y') = y'$, or (b) $f(x) = f(x') = f(y) = y', f(x') = f(x'_1) = f(y') = y$ for each $x_1 \in X$ and each $x'_1 \in X'$. These cases cannot be distinguished by means of congruence relations.

Let the assumptions of 4.1–4.6 be not valid.

4.7. Lemma. *Let there exist distinct elements $x, x', y, y', y'' \in A$ fulfilling the condition* (2) *from* 3.15. *Then $R(f)$ consists of two elements and they can be described by means of* Con(A, f). *The algebra (A, f) is a $c'(x, y')$-extension of some of the algebras given in* Fig. 4.7 (a), 4.7 (b).

Proof. The elements $x, x', y, y', y'' \in A$ fulfil the condition (1) from 3.5, hence we have (a) $f(x) = f(x') = f(y) = y$, $f(y') = y'$, $f(y'') = y''$, or (b) $f(x) = x$, $f(x') = x'$, $f(y) = f(y') = f(y'') = y$. Let $a \in A - \{x, x', y, y', y''\}$. The assumptions of 4.1–4.6 are not satisfied, thus if (a) holds, then a behaves in the same way as x or as y' (cf. Fig. 4.3 (a), 4.6 (a), 4.1 (b)). If (b) is valid, then a behaves in the same way as x or as y', too. Let X resp. Y' be the set of all $a \in A - \{x, y, y'\}$ which behave in the same way as x resp. as y'. Then

$$X = \{a \in A - \{x, y\} : \theta(a, x) = [\{a, x\}]\},$$
$$Y' = \{a \in A - \{y, y'\} : \theta(a, y') = [\{a, y'\}]\},$$

$A = \{x, y, y'\} \cup X \cup Y'$ and either (a) $f(x) = f(x_1) = f(y) = y$, $f(y'_1) = y'_1$, or (b) $f(x) = x$, $f(x_1) = x_1$, $f(y) = f(y'_1) = y$ for each $x_1 \in X$ and each $y'_1 \in Y_1$. These two cases cannot be distinguished by means of Con(A, f).

Let the assumptions of 4.1–4.7 be not valid.

4.8. Lemma. *Let there exist distinct elements $x, x', y, y' \in A$ fulfilling the condition* (2) *from* 3.14. *Then $R(f)$ consists of three elements and they can be described by means of* Con(A, f). *The algebra (A, f) is a $c'(x)$-extension of some of the algebras given in* Fig. 4.8 (a)–(c).

Proof. The elements $x, x', y, y' \in A$ fulfil the condition (1) from 3.14, i.e., we have either (a) $f(x) = f(x') = f(y) = y$, $f(y') = y'$, or (b) $f(x) = x$, $f(x') = x'$, $f(y) = f(y') = y$, or (c) $f(x) = f(x') = f(y) = y'$, $f(y') = y$. Let $a \in A - \{x, x', y, y'\}$. Then a behaves in the same way as x in each of the cases (a)–(c) (for the cases (a) and (b) cf. Fig. 4.7 (a), 4.5 (a), 4.5 (b) and 4.6 (a); for the case (c) cf. Fig.4.1 (a), 4.4 (b), 4.1 (g) and 4.6 (b)). Let X be the set of all $a \in A - \{x, y\}$ which behave in the same way as x. Then

$$X = \{a \in A - \{x, y\} : \theta(a, x) = [\{a, x\}]\},$$

$A = \{x, y, y'\} \cup X$ and one of the following cases is valid: (a) $f(x) = f(x_1) = f(y) = y$, $f(y') = y'$, (b) $f(x) = x$, $f(x_1) = x_1$, $f(y) = f(y') = y$, (c) $f(x) = f(x_1) = f(y) = y'$, $f(y') = y$, for each $x_1 \in X$. These cases cannot be distinguished by means of congruence relations.

Let the assumptions of 4.1–4.8 be not valid.

4.9. Lemma. *Let there exist distinct elements $x, y, y' \in A$ fulfilling the condition* (2) *from* 3.12. *Then $R(f)$ consists of four elements and they can be described by means of* $\mathrm{Con}(A, f)$. *The algebra (A, f) is some of the algebras given in* Fig. 4.9 (a)–(d).

Proof. The elements $x, y, y' \in A$ fulfil the condition (1) from 3.12, i.e. some of the following possibilities is valid: (a) $f(x) = f(y) = y'$, $f(y') = y$, (b) $f(x) = f(y) = y$, $f(y') = y'$, (c) $f(x) = x$, $f(y) = f(y') = y$, (d) $f(x) = y$, $f(y) = f(y') = x$. The assumptions of 4.1–4.8 are not satisfied, hence $A = \{x, y, y'\}$ (for the cases (a) and (d) cf. Fig. 4.8 (c), 4.6 (b), 4.1 (g), 4.1 (a) and 4.4 (b); for the cases (b) and (c) cf. Fig. 4.6 (a), 4.8 (a), 4.8 (b), 4.5 (a), 4.5 (b)). The cases (a)–(d) cannot be distinguished by means of $\mathrm{Con}(A, f)$.

Let the assumptions of 4.1–4.9 be not valid. Then there are the following possibilities: (a) (A, f) consists of one two-element cycle; (b) (A, f) consists of some one-element cycles; (c) (A, f) is connected, possesses a cycle $\{a\}$ and $f(x) = a$ for each $x \in A$ (cf. Fig. 4.9 (a), 4.9 (b), 4.5 (a), 4.5 (b)). Hence the following assertion is obvious:

4.10. Lemma. (i) $\mathrm{Con}(A, f) = E(A)$.

(ii) *If* card $A = 2$, *then $R(f)$ consists of four elements (all unary operations which can be defined on A). If* card $A > 2$, *then either* (a) $f(x) = x$ *for each* $x \in A$, *or* (b) *there exists $a \in A$ such that $f(x) = a$ for each $x \in A$, and then* card $R(f) = 1 +$ card A.

From now do not assume that some of the conditions of the lemmas 4.1–4.9 fails to be valid.

4.10.1. Lemma. $\mathrm{Con}(A, f) = E(A)$ *if and only if* (ii) *from* 4.10 *holds.*

Proof. Obviously (ii) ⇒ (i). Let (i) be valid. Then $\theta(x, y) = [\{x, y\}]$ for each $x, y \in A$, $x \neq y$. Hence each connected component of (A, f) contains a cycle with the cardinality less than 3. If (A, f) has more than one component, then these components consist of one-element cycles. If (A, f) has only one component, then either (A, f) is a two-element cycle or the condition (b) from (ii) is valid. Thus (i) ⇒ (ii).

Now suppose that the following condition $(*)$ is valid:

$(*)$ each connected component of (A, f) contains a cycle with the cardinality less than 3.

Let us introduce the following two notions (modifying the concept of $c_b'(T)$-extension resp. of $c'(T)$-extension). Let (B_1, g_1) and (A_1, f_1) be monounary algebras and let $T \subseteq B_1$, $b \in B_1$. We shall say that (A_1, f_1) is a $c_b(T)$-extension of (B_1, g_1), if there is an isomorphic φ of (B_1, g_1) into (A_1, f_1) such that (A_1, f_1) is a $c'_{\varphi(b)}(\varphi(T))$-extension of $(\varphi(B_1), f_1 \mid \varphi(B_1))$. The notion of $c(T)$-extension is defined analogously.

The above considerations performed in this paragraph can be summarized as follows:

By using merely the system $\mathrm{Con}(A, f)$ (without using explictly the operation f) we can decide whether or not $(*)$ is valid. If $(*)$ holds, then we can describe all unary operations g on A such that $\mathrm{Con}(A, f) = \mathrm{Con}(A, g)$.

In particular, from 4.1—4.10 we obtain the following propositions:

4.11. Proposition. *Let (A, f) be a monounary algebra such that $(*)$ is valid.*

(i) *If* $\mathrm{con}(A, f) = E(A)$, *then* $\mathrm{card}\, R(f) = 1 + \mathrm{card}\, A$ *whenever* $\mathrm{card}\, A > 2$, *and if* $\mathrm{card}\, A = 2$, *then* $\mathrm{card}\, R(f) = 4$.

(ii) *Let* $\mathrm{Con}(A, f) \neq E(A)$. *Then we have*:

(a) $\mathrm{card}\, R(f) = 2$ *if and only if (A, f) is a $c_y(y)$-extension of the monounary algebra given in* Fig. 4.2 (a), *a $c(x, z)$-extension of some of the algebras given in* Fig. 4.3 (a), 4.4 (a), 4.4 (b), *a $c(x, x')$-extension of some of the algebras given in* Fig. 4.6 (a), 4.6 (b) *or a $c(x, y')$-extension of the algebra given in* Fig. 4.7 (a).

(b) $\mathrm{card}\, R(f) = 3$ *if and only if (A, f) is a $c(x)$-extension of some of the algebras given in* Fig. 4.5 (a), 4.5 (b), 4.8 (a), 4.8 (b) *and* 4.8 (c).

(c) $\mathrm{card}\, R(f) = 4$ *if and only if (A, f) is isomorphic with some of the algebras given in* Fig. 4.9 (a) *and* 4.9 (b).

4.12. Proposition. *Let (A, f) be a monounary algebra such that $(*)$ is valid and* $\mathrm{Con}(A, f) \neq E(A)$. *Then* $\mathrm{card}\, R(f) \leqq 4$.

4.13. Proposition. *Let A be a set*, $\mathrm{card}\, A \geqq 4$. *Then for each $i \in \{1, 2, 3, 4\}$ there exists a unary operation f_i on A such that (A, f_i) fulfil $(*)$ and* $\mathrm{card}\, R(f_i) = i$.

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