Francis E. Masat

Idempotents and inverses in conventional semigroups

Czechoslovak Mathematical Journal, Vol. 32 (1982), No. 3, 384–388

Persistent URL: `http://dml.cz/dmlcz/101814`

Czechoslovak Mathematical Journal, 32 (107) 1982, Praha

IDEMPOTENTS AND INVERSES IN CONVENTIONAL SEMIGROUPS

F. E. MASAT*), Glassboro

(Received January 21, 1981)

1. INTRODUCTION AND PRELIMINARIES

A conventional semigroup S is a regular semigroup in which E, the set of idempotents, is self-conjugate; that is, $cEc' \subseteq E$ for each $c \in S$ and for each inverse c' of c. This property was derived from orthodox semigroups wherein the set of idempotents, being a subsemigroup, are inherently self-conjugate. Conventional semigroups were first developed by Masat in [4] and stemmed directly from generalizations of Meakin's work in [6].

Viewed as classes, the following relationships hold with all the inclusions being proper (see [4; Proposition, p. 398]): inverse semigroups $\subset$ orthodox semigroups $\subset$ conventional semigroups $\subset$ regular semigroups. This paper investigates inverses and the set of idempotents in a conventional semigroup. In particular, in a conventional semigroup the set E can be decomposed into its orthodox elements and non-orthodox elements. For each element $c \in S$ of a regular semigroup S, we define $V(c) = \{x \in S : cxc = c \text{ and } xcx = x\}$ to be the set of inverses of c. For S conventional, $V(E)$ is described in terms of E and those $\mathscr{H}$-classes of S which are not subgroups of S. Lastly, the set of inverses of A, $V(A)$, is described, where A is the kernel of a congruence on S.

The results of the next section use the notation of Clifford and Preston, [1]. Moreover, if R is a congruence relation on S, then $R^{\#}$ will denote the homomorphism on S induced by R; if A is a subset of S, then $\hat{A}$ will denote the subsemigroup of S generated by A. For a general development where $A = E$ in a regular semigroup, see [2].

2. E AND $V(E)$ FOR A CONVENTIONAL SEMIGROUP

We begin this section with a result that provides an additional way of recognizing conventional semigroups.

*) This research was supported by a Glassboro State College Faculty Research Grant.

Lemma 2.1. *In a regular semigroup S, the following conditions are equivalent:*
(i) $cEc' \subseteq E$ *for all* $c \in S$ *and* $c' \in V(c)$;
(ii) $eEe \subseteq E$ *for all* $e \in E$.

Proof. It is clear that condition (i) implies condition (ii) since $e \in V(e)$. Conversely, let $c \in S$, $c' \in V(c)$, and $e \in E$. By condition (ii) $c'cec'c \in E$ and therefore

$$\begin{aligned}(cec')(cec') &= (cc'cec')(cec'cc'), \\ &= c(c'cec'cec'c)\,c', \\ &= c(c'cec'c)(c'cec'c)\,c', \\ &= c(c'cec'c)\,c' = cec'.\end{aligned}$$

Thus, condition (ii) implies condition (i).

We now investigate the structure of E in a conventional semigroup and give a result that generalizes the description of E for all orthodox semigroups.

Theorem 2.2. *A regular semigroup S is conventional if and only if there exists a decomposition of E such that $E = \bigcup_{\alpha \in Y} E_\alpha$, where the union is disjoint, each E_α is a rectangular band, and $E_\alpha E_\beta E_\alpha \subseteq E_{\alpha\beta\alpha}$ for each $\alpha, \beta \in Y$.*

Proof. If $e \in E_\alpha$, $f \in E_\beta$ then $efe \in E_\alpha E_\beta E_\alpha \subseteq E_{\alpha\beta\alpha}$ implies that $efe \in E$. Applying Lemma 2.1, S is conventional.

Conversely, if S is a conventional semigroup then for each $e \in E$, $\{e\}$ is a rectangular band. Indexing each singleton subset of E with a set Y implies that for $\alpha, \beta \in Y$, $E_\alpha E_\beta E_\alpha \subseteq E_{\alpha\beta\alpha}$ since by Lemma 2.1 $efe \in E$.

Note that in the above theorem, Y may or may not be a semigroup. If it is, we have the following.

Corollary 2.3. [1: Exercise 1, p. 129] *A regular semigroup is orthodox if and only if $E = \bigcup_{\alpha \in Y} E_\alpha$, where the union is disjoint, each E_α is a rectangular band, and $E_\alpha E_\beta \subseteq E_{\alpha\beta}$ for each $\alpha, \beta \in Y$.*

In seeking generalizations of orthodox semigroups, we parallel previous results obtained for regularity, ideals, and so on; i.e., we investigate the notion of left or right orthodox.

For a regular semigroup S, let $e \in E$ and call e *left orthodox* if $eE \subseteq E$. Dually, e will be called *right orthodox* if $Ee \subseteq E$. An element $e \in E$ will be called *orthodox* if it is both left and right orthodox. An element e in E will be called non-orthodox if one of eE or Ee is not contained in E. We then have the following result.

Proposition. *A regular semigroup is orthodox if and only if each idempotent is orthodox.*

Proof. If each idempotent is orthodox, then E is a subsemigroup; the converse is self evident.

Taking the above notion further, we decompose the set of idempotents of a regular semigroup into its orthodox members and its non-orthodox members. This leads us to ask about the interaction between orthodox and non-orthodox idempotents. The results are presented in the next theorem.

Theorem 2.4. *Let S be a regular semigroup with set of idempotents E. Denote the set of orthodox [non-orthodox] idempotents of S by $E_0[E_0^*]$. If $\hat{E}$ and $\hat{E}_0^*$ denote the subsemigroups of S generated by E and E_0^* respectively, then:*

(i) *E_0 is a subsemigroup of S,*

(ii) *$E_0 E_0^* \subseteq E$ and $E_0^* E_0 \subseteq E$,*

(iii) *$\hat{E} = E_0 \cup \hat{E}_0^*$.*

Proof. For (i), if $e, f \in E_0$ then $efE \subseteq eE \subseteq E$ and similarly $feE \subseteq E$; i.e., ef is orthodox and E_0 is a subsemigroup.

To show (ii), let $e \in E_0$ and $f \in E_0^*$. Then $ef \in eE \subseteq E$ and dually $fe \in E$.

Part (iii) follows directly from (i) and (ii).

It should be noted that in part (iii) of the preceding theorem the union may not be disjoint. For example, in the full transformation semigroup on $\{1, 2, 3, 4\}$, (1211) and $(1331) \in E_0^*$ and $(1331)(1211) = (1111) \in \hat{E}_0^*$ and E_0. It should also be obvious that if $E_0^* = \emptyset$, then S is orthodox.

In a conventional semigroup, the set E may be used to describe various congruences on the semigroup. For example, the minimum group congruence on a conventional semigroup is given in [4: Theorem 3.1, p. 396] as $\{(a, b) \in S \times S\colon eae = ebe$ for some $e \in E\}$. And, in [5: Corollary 3.7], the minimum orthodox congruence on a conventional semigroup is generated by $\{(ef, efef)\colon e, f \in E\}$. Other similar results also appear in [5].

We now turn to inverses in a conventional semigroup and begin by investigating $V(E)$. We will let H_a denote the $\mathscr{H}$-class of a.

Theorem 2.5. *Let S be a conventional semigroup with set of idempotents E. Then:*

(i) *For all $a \in V(E)$, $a^2 = a^3$.*

(ii) *$V(E) \subseteq E \cup H$, where $H = \bigcup H_a$ for $a \in S$ such that $H_a \cap E = \emptyset$.*

Proof. For (i), recall that $V(E) = E^2$ in regular semigroups and let $a \in V(E) = E^2$. Thus $a = ef$ for some $e, f \in E$. From Lemma 2.1, $a^2 = (efe)f = (efe)^2 f = (ef)^3 = a^3$.

For (ii), we begin by showing that $V(E) \subseteq E \cup H_a$ such that $H_a \cap E = \emptyset$. If $a \in V(E)$, then $a \in E^2$ and therefore $a = ef$ for some $e, f \in E$. It follows that $a = ae = fa$. If $a \in E$, we are done; if $a \notin E$, then $a \in H_a$. Proceeding indirectly, suppose $H_a \cap E \neq \emptyset$, say $u \in H_a \cap E$. Since H_a is a subgroup of S, $a = ua = au$ and there exists $a' \in V(a) \cap H_a$. Part (i) then implies that $a = a(u) = (a)(aa') = a^2 a' = a^3 a' = a^2(aa') = a(au) = a^2$, a contradiction. It follows that $V(E) \subseteq E \cup H$.

As a corollary, we have the following result of J. C. Meakin.

Corollary 2.6 [6: Lemma 1.3, p. 323] *If S is an orthodox semigroup, then $V(E) \subseteq E$.*

Proof. From the proof of the theorem, if $a \in V(g)$, then $a = fe$. But since S is orthodox, $fe \in E$.

For T_3, the full transformation semigroup on three symbols, it is straightforward to verify that T_3 is conventional and $V(E) = E \cup H$. It should be noted that for the preceding theorem, the bicyclic semigroup provides an example wherein $V(E) \subsetneqq E \cup H$. For regular semigroups, it is not the case that in general $V(E) = E \cup H$; eg, in T_4, $(1212) \in E$, but $(1414) \in V((1212))$ is also in a subgroup of T_4.

Since E is by definition contained in the kernel of a congruence on a regular semigroup, we next consider the set of inverses of the kernel of a congruence on a conventional semigroup.

Theorem 2.7. *Let ϱ be a congruence on a conventional semigroup S. If $\mathscr{A} = \{A_i\colon i \in I\}$ denotes the kernel of ϱ and $\theta = p^{\#}$, then $V(A) \subseteq A \cup B$ where $A = \bigcup A_i$, $V(A) = \bigcup V(a)$: $a \in A$, and $B = (H\theta)\,\theta^{-1}$ where $H = \bigcup H_a$ for $a \in S$ such that $H_a \cap E = \emptyset$.*

Proof. If $a \in A$ and $a' \in V(a)$, then $a'\theta \in V(a\theta)$ where $\theta = \varrho^{\#}$. Since $\mathscr{A}$ is the kernel of ϱ, $(a\theta)^2 = a\theta$. By Theorem 2.5, $a'\theta \in V(a\theta) \subseteq V(E\theta) \subseteq E\theta \cup H\theta$, and therefore $a'\theta \in E\theta$ or $a'\theta \in H\theta$. It follows that $a' \in A \cup B$.

In the orthodox case, $V(E) \subseteq E$ and we have the following result.

Corollary 2.8 [6: Lemma 2.3, p. 325] *Let S be an orthodox semigroup, ϱ a congruence on S, and $\mathscr{A} = \{A_i\colon i \in I\}$ the kernel of ϱ. Then $V(A) \subseteq A$ where $A = \bigcup A_i$ and $V(A) = \bigcup\{V(a);\ a \in A\}$.*

A theorem related to Theorem 2.7 follows as our last result.

Theorem 2.9. *If S is a regular semigroup, ϱ an orthodox congruence on S, and $\mathscr{A} = \{A_i\colon i \in I\}$ the kernel of ϱ, then $V(A) \subseteq A$.*

Proof. Let $a \in e\varrho$ and $a' \in V(a)$. For $\theta = \varrho^{\#}$, ϱ orthodox implies that $((a'a) \, . \, (aa'))\,\theta \in E\theta$. But $a^2 \varrho a$ and we therefore have $(a'a)(aa')\,\theta = (a'\theta)(a^2\theta)(a'\theta) = (a'aa')\,\theta = a'\theta$; i.e., $a'\theta \in E\theta$, so $a' \in A$.

The preceding corollary 2.8 could also be stated as a corollary of the above theorem. The general case of S regular is presented by D. G. Fitz-Gerald in [3].

ACKNOWLEDGEMENT

The author gratefully thanks the referee for suggesting part (i) of Theorem 2.5 and the subsequent shorter proof of the theorem.

References

[1] *A. H. Clifford* and *G. B. Preston:* The Algebraic Theory of Semigroups I, A.M.S., Providence, 1961.

[2] *C. Eberhart, W. Williams* and *L. Kinch:* Idempotent—generated regular semigroups, J. Austral. Math. Soc., Vol. *15* (1973), 27—34.

[3] *D. G. Fitz-Gerald:* On inverse of products of idempotents in regular semigroups, J. Austral. Math. Soc., Vol. *13* (1972), 335—337.

[4] *F. E. Masat:* Right group and group congruences on a regular semigroup, Duke Math. J., Vol. *40* (1973), 393—402.

[5] *F. E. Masat:* Congruences on conventional semigroups, Czechoslovak Math. J., Vol. *31* (*106*), (1981), 199—205.

[6] *J. C. Meakin:* Congruences on orthodox semigroups, J. Austral. Math. Soc., Vol. *12* (1971), 323—341.

Author's address: Glassboro State College, Glassboro, NJ 08028, U.S.A.