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Czechoslovak Mathematical Journal, Vol. 32 (1982), No. 3, 373–376

Persistent URL: `http://dml.cz/dmlcz/101812`

Czechoslovak Mathematical Journal, 32 (107) 1982, Praha

ON A RESULT OF ETGEN AND LEWIS

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(Received January 9, 1981)

1. Let $\mathscr{H}$ be a Hilbert space, let $\mathscr{B} = \mathscr{B}(\mathscr{H}, \mathscr{H})$ be the B^*-algebra of bounded linear operators from $\mathscr{H}$ to $\mathscr{H}$ with the uniform topology, and let $\mathscr{S}$ be the subset of $\mathscr{B}$ consisting of the selfadjoint operators. We consider the second order, self-adjoint differential equation

$$(1) \qquad [P(x)\, Y']' + Q(x)\, Y = 0$$

on $X = [x_0, \infty)$, where $P, Q : X \to \mathscr{S}$ are continuous and $P(x)$ is positive definite for all $x \in X$.

The purpose of this paper is to improve the recent results of Etgen and Lewis [2] and to give a very general comparison theorem of Hille-Wintner type for (1).

Appropriate discussions of the concepts of differentiation and integration of $\mathscr{B}$-valued functions, as well as treatments of the existence and uniqueness of solutions $Y : X \to \mathscr{B}$ of (1), can be found in Hille's book [4, Chapters 4, 6 and 9].

In the study of the equation (1) from the point of view of oscillation on X many results have been found in the last several years. Most of these results are covered by the fundamental comparison theorem of Etgen and Pawlowski [3], where a detailed discussion of the definition of oscillation of (1) can be found. The comparison theorem mentioned above involves the set of positive functionals $\mathscr{G}$ on the B^*-algebra $\mathscr{B}$. A linear functional g on $\mathscr{B}$ is positive if $g(A^*A) \geqq 0$ for all $A \in \mathscr{B}$. It is a well known fact that g is the zero functional if and only if $g(I) = 0$, where I denotes the identity operator in $\mathscr{B}$. Our oscillation criteria will be given by comparing the equation (1) with second order scalar equations

$$(2) \qquad (p(x)\, y')' + q(x)\, y = 0$$

on X, where $p, q : X \to \mathscr{R}$ (the reals) are continuous and $p(x) > 0$ on X. We assume that the reader is familiar with the appropriate definitions and the main results concerning the oscillation of solutions of (2).

2. **Results.** The following result of Etgen and Pawlowski [3] will play the main role in the sequel.

*) Supported in part by SIZ for Scientific Work of BiH.

Theorem A [3]. *If there exists a* $g \in \mathscr{G}$ *such that the scalar equation*

$$(g[P(x)]\, y')' + g[Q(x)]\, y = 0 \tag{3}$$

is oscillatory, then the equation (1) *is oscillatory.*

One of the most useful comparison theorems in the study of the oscillatory behavior of the second order linear equations (2) and

$$(p_1(x)\, y')' + q_1(x)\, y = 0\,, \tag{4}$$

aside from the Sturm comparison theorem, is the so called Taam-Hille-Wintner comparison theorem:

Theorem B [6], [7]. *Let* p_1 *be bounded from above on* X. *Let*

$$\mathscr{P}(x) = \int_x^\infty q(t)\,\mathrm{d}t\,,\quad \mathscr{P}_1(x) = \int_x^\infty q_1(t)\,\mathrm{d}t$$

exist and let $0 < p_1(x) \leqq p(x)$, $0 < |\mathscr{P}(x)| \leqq \mathscr{P}_1(x)$ *for all* x *in* X. *Then if the equation* (2) *is oscillatory, then the equation* (4) *is oscillatory.*

The boundedness assumption on p_1 required in Theorem B was recently removed by Butler [1] and Kulenović [5] and Theorem B has been extended as follows:

Theorem C [1], [5]. *Theorem* B *holds without the assumption that* p_1 *is a function bounded from above.*

The main result in [2] is the following theorem:

Theorem D [2]. *Assume that there exists a number* $a \in X$ *such that on* $[a, \infty)$ *the following conditions are satisfied*:

(i) $p(x)\, I - P(x) \geqq 0$,
(ii) *there exists a positive constant* k *such that* $kI - P(x) \geqq 0$,
(iii) *there exists a* $g \in \mathscr{G}$, $g \neq 0$, *such that the integral* $\int_x^\infty g[Q(t)]\,\mathrm{d}t$ *converges, possibly just conditionally,*
(iv) *the integral* $\int_x^\infty q(t)\,\mathrm{d}t$ *converges, possibly just conditionally, and*
(v) $\int_x^\infty g[Q(t)]\,\mathrm{d}t \geqq g(I)\,\left|\int_x^\infty q(t)\,\mathrm{d}t\right|$.

If the equation (2) *is oscillatory, then the equation* (1) *is oscillatory.*

The following result is an improvement of Theorem D.

Theorem 1. *Suppose that there exist a number* $a \in X$ *and* $g \in \mathscr{G}$, $g \neq 0$ *such that on* $[a, \infty)$ *the following conditions hold*:

(i') $g[p(x)\, I - P(x)] \geqq 0$,
(ii') *the integrals* $\int_x^\infty g[Q(t)]\,\mathrm{d}t$ *and* $\int_x^\infty q(t)\,\mathrm{d}t$ *converge, possibly just conditionally, and the condition* (v) *holds.*

If the equation (2) *is oscillatory, then* (1) *is oscillatory.*

Proof. Since $g \neq 0$, $g(I) = \|g\| > 0$ and

$$(g(I)\, p(x)\, y')' + g(I)\, q(x)\, y = 0 \tag{5}$$

is oscillatory iff (2) is. Comparing the equations (3) and (5) by use of Theorem C we obtain (by (i′) and (ii′)) that the equation (3) is oscillatory, which by Theorem A implies that (1) is oscillatory.

Remark 1. Theorem D can be obtained immediately from Theorem A and Theorem B by applying the method used in the proof of Theorem 1. In the equations (1) and (3) let $P(x) \equiv I$ on X, and in the equation (2) let $p(x) \equiv 1$ on X. Then the equations (1), (3) and (2) become

$$Y'' + Q(x)\, Y = 0\,, \tag{6}$$

$$g(I)\, y'' + g[Q(x)]\, y = 0\,, \tag{7}$$

$$y'' + q(x)\, y = 0\,, \tag{8}$$

respectively.

We introduce the following sequences of iterated integrals $\{\mathscr{I}_n(x)\}_{n \geqq 1}$ and $\{\mathscr{K}_n(x)\}_{n \geqq 1}$ defined as follows:

$$\mathscr{I}_0(x) = \int_x^\infty q(t)\, \mathrm{d}t\,,$$

$$\mathscr{I}_n(x) = \int_x^\infty (\mathscr{I}_{n-1}(s))^2\, I_{n-1}(s, x)\, \mathrm{d}s\,,$$

where $I_{n-1}(s, x) = \exp\left(2 \int_x^s (\mathscr{I}_0(u) + \ldots + \mathscr{I}_{n-1}(u))\, \mathrm{d}u\right)$ for $n = 1, 2, \ldots,$

$$\mathscr{K}_0(x) = \int_x^\infty \frac{g[Q(t)]}{g(I)}\, \mathrm{d}t\,,$$

$$\mathscr{K}_n(x) = \int_x^\infty (\mathscr{K}_{n-1}(s))^2\, K_{n-1}(s, x)\, \mathrm{d}s\,,$$

where $K_{n-1}(s, x) = \exp\left(2 \int_x^s (\mathscr{K}_0(u) + \ldots + \mathscr{K}_{n-1}(u))\, \mathrm{d}u\right)$ for $n = 1, 2, \ldots.$

Theorem 2. *Suppose that there exist a number $a \in X$ and $g \in \mathscr{G}$, $g \neq 0$ such that for some $n = 1, 2, \ldots,$ the following conditions hold:*

$$\mathscr{K}_n(x) \geqq \mathscr{I}_n(x) \quad \text{and} \quad \sum_{m=0}^{n} \mathscr{K}_m(x) \geqq \sum_{m=0}^{n} \mathscr{I}(x) \quad \text{on} \quad [a, \infty)\,.$$

If the equation (8) is oscillatory, then the equation (6) is oscillatory.

Proof. Since $g \neq 0$,

$$y'' + \frac{g[Q(x)]}{g(I)}\, y = 0 \tag{9}$$

is oscillatory if and only if (7) is. Using Theorem 3 [5] we conclude that (9) is oscillatory. Thus, (7) is oscillatory which by Theorem A implies that (6) is oscillatory.

Remark 2. Theorem 2 is a generalization and an improvement of Theorem 1 in the case when $P(x) \equiv I$ and $p(x) \equiv 1$ on X. Theorem 2 is an improvement of Theorem 4 of Etgen and Lewis [2].

Remark 3. Theorem 2 does not involve any restrictions on the sign of $\int_x^\infty g[Q(t)]\,\mathrm{d}t$, and can be applied in some situations in which Theorem 1 fails.

Remark 4. Theorems 5 and 6 [2] are immediate consequences of Theorem A. Moreover, the condition that $\int_x^\infty g[Q(t)]\,\mathrm{d}t$ converges which appeared in the above theorems is redundant.

Remark 5. Using some of many possible oscillation invariant transformations of variables, one can easily transfer Theorem 2, which compares the equations (8) and (6), to the general selfadjoint equations (2) and (1). The most effective transformation ot this kind is that used by Kulenović [5] and Willett [8] in the following form:

$$s = \int_{x_0}^{x} \frac{\mathrm{d}u}{p(u)}, \quad z(s) = y(x).$$

In this case Theorem 2 with all differentials $\mathrm{d}t$ replaced by $\mathrm{d}t/p(t)$ in $\{\mathscr{I}_n(x)\}_{n\geqq 1}$ and by $\mathrm{d}t/g[P(t)]$ in $\{\mathscr{K}_n(x)\}_{n\geqq 1}$ and $q(t)$ replaced by $p(t)\,q(t)$ in $\{\mathscr{I}_n(x)\}_{n\geqq 1}$ and $g[Q(t)]$ replaced by $g[P(t)]\,g[Q(t)]$ in $\{\mathscr{K}_n(x)\}_{n\geqq 1}$ is valid for the equations (2) and (3) or (1).

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