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Milan Kučera

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# BIFURCATION POINTS OF VARIATIONAL INEQUALITIES

MILAN KUČERA, Praha

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## 0. INTRODUCTION

Let us consider a real Hilbert space $H$ and a closed convex cone $K$ in $H$ with its vertex at the origin. The inner product and the corresponding norm in $H$ will be denoted by $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$, respectively. In the whole paper, we shall suppose that $A : H \to H$ is a linear (in general nonsymmetric) completely continuous operator in $H$ and $N : \mathbb{R} \times H \to H$ is a nonlinear completely continuous mapping satisfying the condition

$$\text{(N)} \qquad \lim_{\|v\| \to 0} \frac{N(\mu, v)}{\|v\|} = 0 \quad \text{uniformly on bounded subsets of } \mathbb{R} .$$

We shall study the bifurcation problem for the variational inequality

$$\text{(I)} \qquad v \in K ,$$

$$\text{(II)} \qquad \langle v - \mu A v + N(\mu, v), w - v \rangle \geqq 0 \quad \text{for all} \quad w \in K .$$

A pair $[\mu_0, 0] \in \mathbb{R} \times H$ is a *bifurcation point of* (I), (II) (with respect to the line of trivial solutions $\{[\mu, 0]; \mu \in \mathbb{R}\}$) if in every neighbourhood of $[\mu_0, 0]$ in $\mathbb{R} \times H$ there exists $[\mu, v]$ satisfying (I), (II), $\|v\| \neq 0$.

For the proof of existence of bifurcation points of (I), (II), we shall develop the method used in [4], [5] (cf. also [2], [3]) for the study of eigenvalues of the variational inequality

$$\text{(I)} \qquad u \in K ,$$

$$\text{(II}_\text{L}\text{)} \qquad \langle u - \mu A u, w - u \rangle \geqq 0 \quad \text{for all} \quad w \in K .$$

We shall consider the equation with the penalty

$$\text{(b)} \qquad v - \mu A v + N(\mu, v) + \varepsilon \beta v = 0$$

with the norm condition

$$\text{(a)} \qquad \|v\|^2 = \frac{\delta \varepsilon}{1 + \varepsilon} ,$$

where $\beta$ is a suitable penalty operator corresponding to $K$ (see Section 2), $\varepsilon, \delta$ are real parameters. Couples of simple characteristic values $\mu^{(0)}, \mu^{(1)}$ $(0 < \mu^{(0)} < \mu^{(1)})$ of $A$ having eigenvectors $u^{(0)}, u^{(1)}$ in the interior of $K$ (with $-u^{(0)}, -u^{(1)} \notin K$) will be studied. The main idea is to prove that for an arbitrary small $\delta > 0$ there exists a closed connected set $C_\delta^+$ of triplets $[\mu, v, \varepsilon] \in \mathbb{R} \times H \times \mathbb{R}$ satisfying (a), (b) and containing $[\mu^{(0)}, 0, 0]$, whose first component $\mu$ lies in $(\mu^{(0)}, \mu^{(1)})$ (except for the point $[\mu^{(0)}, 0, 0]$), the second component $v$ lies outside of $K$ (except for some isolated points) and which is unbounded in the third component $\varepsilon$. By the limiting process $\varepsilon \to +\infty$ along such a branch (for $\delta$ fixed), we obtain at least one solution $\mu(\delta), v(\delta)$ of (I), (II) with $\mu(\delta) \in (\mu^{(0)}, \mu^{(1)})$, $\|v(\delta)\|^2 = \delta$, $v(\delta)$ lying on the boundary of $K$. The limit points of $\mu(\delta)$ for $\delta \to 0$ (which are bifurcation points of (I), (II)) lie in $(\mu^{(0)}, \mu^{(1)})$ again. In this way, the existence of at least one bifurcation point $[\mu_0, 0]$ of (I), (II) with $\mu_0 \in (\mu^{(0)}, \mu^{(1)})$ is proved under certain assumptions. The obtained bifurcating solutions $\mu(\delta), v(\delta)$ are not simultaneously solutions of the equation

$$\text{(E)} \qquad v - \mu A v + N(\mu, v) = 0\,.$$

The existence of the branches $C_\delta^+$ will be proved on the basis of a global bifurcation result of E. N. Dancer [1] (Section 4) by using some special properties of the equation with the penalty studied in Section 3. The main results are formulated in Theorems 2.1, 2.2, 2.3 and the main idea of the proof (which is rather complicated in details) is explained after Theorem 2.3.

Analogously as in [3], [4] for the problem (I), $(\mathrm{II}_\mathrm{L})$, we shall consider simple characteristic values $\mu^{(0)}, \mu^{(1)}$ of $A$ only. It was shown in [5] that the existence of characteristic values of (I), (II) lying between multiple characteristic values of $A$ can be obtained by aproximating $A$ by operators for which $\mu^{(0)}, \mu^{(1)}$ are simple. But the situation is more complicated if we want to obtain bifurcation points of (I), (II) and it is not clear at the first sight if this approximation method can be used.

Let us recall that E. Miersemann [7], [8] has investigated a similar problem proving the existence of $n$ bifurcation points of a variational inequality, where $n$ is a finite number determined by the character of $K$ and the characteristic values of $A$. Its method is based on a sup-min principle and it is in a certain sense more general than the present one. (For example, the multiplicities of characteristic values of $A$ do not play any role.) On the other hand, using the present topological method, we can consider the general non-potential case. Particularly, operator $A$ can be non-symmetric. (In [2], [3], [4], symmetric operators $A$ were studied, but this was unnecessary at least in the case of simple eigenvalues of $A$.) Moreover, under certain assumptions the present theory gives an infinite sequence of characteristic values of (I), $(\mathrm{II}_\mathrm{L})$ (see [3], [4], [5]) and bifurcation points of (I), (II) with the corresponding eigenvectors and bifurcating solutions, which are not simultaneously eigenvectors of $A$ and solutions of (E).

A result concerning the existence of an infinite sequence of eigenvalues of a variational inequality was proved also in [6], but it is in a certain sense formal with the exception of the case when $K$ is a halfspace. Further papers which are in some connection with the bifurcation problem for variational inequalities were mentioned in [4].

## 1. NOTATION. CLASSIFICATION OF CHARACTERISTIC VALUES AND BIFURCATION POINTS

Let $H, K, A$ be the same as in Introduction. We shall denote by $\partial K$ and $K^0$ the boundary and the interior of $K$, respectively, and suppose that $K^0 \neq \emptyset$. The strong convergence and the weak convergence will be denoted by $\to$ and $\rightharpoonup$, respectively. The set of all real characteristic values of the operator $A$ and of the variational inequality (I), ($\text{II}_\text{L}$) will be denoted by $r_A$ and $r_V$, respectively, i.e.

$$r_A = \{\mu \in \mathbb{R};\ u - \mu A u = 0 \text{ for some } u \in H,\ \|u\| \neq 0\},$$

$$r_V = \{\mu \in \mathbb{R}; \text{(I), (II}_\text{L}) \text{ are fulfilled for some } u \in H,\ \|u\| \neq 0\}.$$

Further, denote by $E_A(\mu)$ and $E_V(\mu)$ the set of all eigenvectors of $A$ and of (I), ($\text{II}_\text{L}$), respectively, corresponding to $\mu$. That means $E_V(\mu) = \{u \in H;\ \|u\| \neq 0, \text{(I), (II}_\text{L})$ is fulfilled$\}$. Set $E_A = \bigcup_{\mu \in r_A} E_A(\mu)$, $E_V = \bigcup_{\mu \in r_V} E_V(\mu)$.

**Definition 1.1.** We shall write

$$\begin{aligned}
&\mu \in r_{A,i} && \text{if} && \mu \in r_A && \text{and} && E_A(\mu) \cap K^0 \neq \emptyset\,;\\
&\mu \in r_{A,b} && \text{if} && \mu \in r_A \setminus r_{A,i} && \text{and} && E_A(\mu) \cap \partial K \neq \emptyset\,;\\
&\mu \in r_{A,e} && \text{if} && \mu \in r_A && \text{and} && E_A(\mu) \cap K = \emptyset\,;\\
&\mu \in r_{V,b} && \text{if} && \mu \in r_V && \text{and} && E_V(\mu) \subset \partial K\,.
\end{aligned}$$

We shall say that the elements of $r_{A,i}$, $r_{A,b}$ and $r_{A,e}$ are the *interior characteristic values, boundary characteristic values* and *external characteristic values*, respectively, of $A$. The elements of $r_{V,b}$ will be called the *boundary characteristic values of* (I), ($\text{II}_\text{L}$).

Remark 1.1. The basic properties of the sets $r_{A,i}$, $r_{A,b}$, $r_{A,e}$, $r_{V,b}$ and the relations between them are explained in [4, Remark 1.2]. (Its assertion holds also for non-symmetric operators.) Let us remember only that $\mu \in r_{A,i}$ if and only if $\mu \in r_V$ with $E_V(\mu) \cap K^0 \neq \emptyset$. This follows from the fact that $u \in K^0$ is a solution of (I), ($\text{II}_\text{L}$) if and only if $u - \mu A u = 0$. Hence, we can also speak about *interior characteristic values of* (I), ($\text{II}_\text{L}$) but they coincide with interior characteristic values of $A$ (cf. Remark 1.4).

The following lemma is a modification of Lemma 1.1 from [4] which was proved for symmetric operators only.

**Lemma 1.1.** *If* $\mu \in r_{A^*,i}$, *then* $E_V(\mu) = E_A(\mu) \cap K$.

Proof. Let $u_1 \in E_V(\mu) \cap \partial K$ be arbitrary. It is sufficient to show that $u_1 \in E_A(\mu)$ (see Remark 1.1). There exists $u_0^* \in E_{A^*}(\mu) \cap K^0$ and we have

$$\langle u_1 - \mu A u_1, u_0^* \rangle = \langle u_0^* - \mu A^* u_0, u_1 \rangle = 0 .$$

If $u_1 - \mu A u_1 \neq 0$, then there exists $z \in H$ such that

$$\langle u_1 - \mu A u_1, z \rangle < 0 \quad \text{and} \quad u_0^* + z \in K , \quad \text{i.e.}$$

$$\langle u_1 - \mu A u_1, w - u_1 \rangle < 0 \quad \text{for} \quad w = u_1 + u_0^* + z \in K$$

and this contradicts to the assumption $u_1 \in E_V(\mu)$. Hence, $u_1 \in E_A(\mu)$.

Remark 1.2. If $[\mu_0, 0]$ is a bifurcation point of (E) then $\mu_0 \in r_A$. Moreover, if $\mu_n, v_n$ satisfy (E), $\mu_n \to \mu_0$, $\|v_n\| \to 0$, $v_n/\|v_n\| \rightharpoonup u_0$, then $v_n/\|v_n\| \to u_0 \in E_A(\mu_0)$. This is well-known and easy to see. Analogously, if $[\mu_0, 0]$ is a bifurcation point of (I), (II) then $\mu_0 \in r_V$; if $\mu_n, v_n$ satisfy (I), (II), $\mu_n \to \mu_0$, $\|v_n\| \to 0$, $v_n/\|v_n\| \rightharpoonup u_0$, then $v_n/\|v_n\| \to$ $\to u_0 \in E_V(\mu_0)$. Let us prove this assertion. Setting $u_n = v_n/\|v_n\|$, (II) can be written as

$$(1.1) \qquad \left\langle u_n - \mu_n A u_n + \frac{N(\mu_n, v_n)}{\|v_n\|}, w - u_n \right\rangle \geqq 0 \quad \text{for all} \quad w \in K .$$

We have $v_n \in K$ by (I) and therefore also $u_n \in K$ and $u_0 \in K$ (a closed convex set is weakly closed). Hence, (1.1) implies

$$\left\langle u_n - \mu_n A u_n + \frac{N(\mu_n, v_n)}{\|v_n\|}, u_0 \right\rangle \geqq 0 ,$$

$$\left\langle u_n - \mu_n A u_n + \frac{N(\mu_n, v_n)}{\|v_n\|}, u_n \right\rangle = 0 .$$

Using the complete continuity of $A$, the assumption (N) and $u_n \rightharpoonup u_0$, we obtain from here

$$0 \leqq \lim_{n \to +\infty} \langle u_n, u_0 \rangle - \lim_{n \to +\infty} \langle u_n, u_n \rangle = \|u_0\|^2 - \lim_{n \to +\infty} \|u_n\|^2 .$$

Thus, $u_n \to u_0$ and $\|u_0\| = 1$. Passing to the limit for $n \to +\infty$ in (1.1) (using (N) again) we obtain ($\mathrm{II_L}$) for $\mu = \mu_0$, $u = u_0$.

Remark 1.3. Analogously as in the case of equations, a characteristic value of (I), ($\mathrm{II_L}$) need not be a bifurcation point of (I), (II).

Remark 1.4. If $v \in K^0$, then $v$ satisfies (I), (II) (with a fixed $\mu$) if and only if it satisfies (E). Of course, if $v \in \partial K$ fulfils (E), then $v$ also satisfies (I), (II), but a solution of (I), (II) lying in $\partial K$ need not satisfy (E).

Remark 1.5. It follows from Remark 1.4 that $[\mu_0, 0]$ is a bifurcation point of (I), (II) with the corresponding solutions $\mu_n, v_n$ satisfying

$$\text{(ib)} \qquad v_n \in K^0 , \quad \mu_n \to \mu_0 , \quad \|v_n\| \to 0$$

if and only if $[\mu_0, 0]$ is a bifurcation point of (E) (with the same solutions $\mu_n, v_n$). We can say that it is an *interior bifurcation point of* (E) and *of* (I), (II). These bifurcations are not interesting from the point of view of variational inequalities.

Remark 1.6. Now, let us consider that

$$(\mathrm{bb_V}) \begin{cases} \text{there are solutions } \mu_n, v_n \text{ of (I), (II) with } v_n \in \partial K,\ \|v_n\| \to 0,\ \mu_n \to \mu_0, \\ \text{there is no solution } \mu, v \text{ of (I), (II) with } |\mu - \mu_0| < \delta,\ \|v\| < \delta,\ v \in K^0 \\ \text{(for some } \delta > 0\text{)}. \end{cases}$$

In this case we can say that $[\mu_0, 0]$ is a *boundary bifurcation point of* (I), (II). It can be simultaneously a *boundary bifurcation point of* (E) (i.e. there exist solutions $\mu_n, v_n$ satisfying simultaneously (E) and $(\mathrm{bb_V})$) or an *external bifurcation point of* (E) (i.e. a bifurcation point of (E) with solutions lying outside of $K$ only near $[\mu_0, 0]$). Of course, $[\mu_0, 0]$ need not be a bifurcation point of (E). In the case of an interior or boundary bifurcation point of (E) we obtain $\mu_0 \in r_{A,i} \cup r_{A,b}$ (see Remark 1.2). Hence, if we know that $\mu_0 \notin r_{A,i} \cup r_{A,b}$, then $[\mu_0, 0]$ is either an external bifurcation point of (E) or it is not a bifurcation point of (E); in both cases, there is no solution of (E) near $[\mu_0, 0]$ which is simultaneously a solution of (I), (II). This situation will be our main point of interest.

Remark 1.7. We shall say that $\mu \in r_A$ is *simple* if its algebraic multiplicity is one. (The algebraic multiplicity is the dimension of $\bigcup_{k=1}^{\infty} \operatorname{Ker}(I - \mu A)^k$, where $\operatorname{Ker} B$ denotes the null-space of $B$.) If $\mu \in r_A$ is simple, then $\dim E_A(\mu) = \dim E_{A^*}(\mu) = 1$ ($A^*$ denotes the operator adjoint to $A$) and $\langle u, u^* \rangle \neq 0$ for $u \in E_A(\mu)$, $u^* \in E_{A^*}(\mu)$, $\|u\| \neq 0 \neq \|u^*\|$ (see [11]).

## 2. BOUNDARY BIFURCATION POINTS OF VARIATIONAL INEQUALITIES AND BRANCHES OF SOLUTIONS OF THE EQUATION WITH PENALTY

In the following, we shall assume that the closed convex cone $K$ is such that there exists a nonlinear completely continuous operator $\beta : H \to H$ with the following properties:

(P) $\beta u = 0$ if and only if $u \in K$; $\langle \beta u, u \rangle > 0$ for all $u \notin K$ (i.e. $\beta$ is the penalty operator corresponding to $K$);

(H) $\beta(tu) = t\beta u$ for all $t > 0$, $u \in H$ (i.e. $\beta$ is positive homogeneous);

(M) $\langle \beta u - \beta v, u - v \rangle \geqq 0$ for all $u, v \in H$ (i.e. $\beta$ is monotone);

(β, K) if $u \in K^0$, $v \notin K$, then $\langle \beta v, u \rangle < 0$.

Remark 2.1. The assumptions (P), (H), (M) were used also in [4], (β, K) is a slight modification of (β, K$^0$) from [4]. In [4], still further assumptions (CC), (SC′) were considered, but it is explained in [5] that they were useless. Examples of penalty operators satisfying our assumptions were given in [4], [5] and will be discussed in Section 5.

Remark 2.2. If $\mu \in r_{A,i} \cap r_{A^*,i}$ is simple, then there exist unique $u \in E_A(\mu) \cap (-K)$, $u^* \in E_{A^*}(\mu) \cap (-K)$ with $\|u\| = \|u^*\| = 1$ and we have $\langle u, u^* \rangle \neq 0$ (see Remark 1.7). In the sequel, we shall consider couples of simple characteristic values $\mu^{(0)}, \mu^{(1)} \in r_{A,i} \cap r_{A^*,i}$ such that

$$\text{(U, U*)} \qquad \operatorname{sign} \langle u^{(0)}, u_0^* \rangle = \operatorname{sign} \langle u^{(1)}, u_1^* \rangle$$

$$\text{for} \quad u^{(j)} \in E_A(\mu^{(j)}) \cap (-K), \quad u_j^* \in E_{A^*}(\mu^{(j)}) \cap (-K) \quad (j = 0, 1).$$

Of course, in the case of a symmetric operator $A$ we have $u^{(j)} = u_j^*$ and (U, U*) is automatically fulfilled for each couple $\mu^{(0)}, \mu^{(1)} \in r_{A,i} \cap r_{A^*,i}$.

**Theorem 2.1.** *Let $\mu^{(0)}, \mu^{(1)} \in r_{A,i} \cap r_{A^*,i}$ be simple and let (U, U*) be fulfilled. Suppose that (N) is fulfilled and there exists a completely continuous operator $\beta$ satisfying (P), (H), (M), ($\beta$, K). Then there exists a bifurcation point $[\mu_\infty, 0]$ of (I), (II) with $\mu_\infty \in (\mu^{(0)}, \mu^{(1)})$.*

Remark 2.3. If there is a bifurcation point $[\mu_0, 0]$ of (E) with $\mu_0 \in (\mu^{(0)}, \mu^{(1)})$ and with the corresponding solutions $\mu_n, v_n$ of (E), $v_n \in K$, $\|v_n\| \to 0$, $\mu_n \to \mu_0$, then the assertion of Theorem 2.1 is trivial (see Remark 1.4). If there is no such a bifurcation point of (E), then

(2.1) for each $\tilde{\mu} \in (\mu^{(0)}, \mu^{(1)}) \cap r_A$ there exists $\delta > 0$ such that there is no solution $\mu, v$ of (E) with $|\mu - \tilde{\mu}| < \delta$, $0 < \|v\| < \delta$.

In this case the assertion of Theorem 2.1 follows from Theorems 2.2, 2.3 formulated below. Moreover, Theorems 2.2, 2.3, describe more precisely the character of the bifurcation point under the consideration and explain how it can be obtained from the branch of the solutions of the equation with the penalty. Particularly, it is given the existence of a boundary bifurcation point of (I), (II) (in the sense of Remark 1.6) which is neither a boundary nor an interior bifurcation point of (E) (Remark 1.5), because only external bifurcation points of (E) can lie in $(\mu^{(0)}, \mu^{(1)})$ under the assumption (2.1). Let us remark that (2.1) is ensured for example if $\mu((^{(0)}, \mu^{(1)}) \cap r_A \subset r_{A,e}$.

**Definition 2.1.** For each $\delta > 0$ fixed we shall denote by $C_\delta$ the closure (in $\mathbb{R} \times H \times \mathbb{R}$) of the set of all triplets $[\mu, v, \varepsilon] \in \mathbb{R} \times H \times \mathbb{R}$ satisfying the conditions $\varepsilon \neq 0$ and

$$\text{(a)} \qquad \|v\|^2 = \frac{\delta\varepsilon}{1 + \varepsilon},$$

$$\text{(b)} \qquad v - \mu A v + N(\mu, v) + \varepsilon\beta v = 0.$$

Remark 2.4. The condition (a) cannot be fulfilled with $\varepsilon \in (-1, 0)$. It is clear from here that if $C_{\delta,0}$ is an arbitrary connected subset of $C_\delta$ containing a point of the type $[\mu, 0, 0]$, then $\varepsilon \geqq 0$ for all $[\mu, v, \varepsilon] \in C_{\delta,0}$.

**Theorem 2.2.** *Let the assumptions of Theorem* 2.1 *and* (2.1) *from Remark* 2.3 *be fulfilled with* $\langle u^{(j)}, u_j^* \rangle > 0$ *in the assumption* (U, U*). *Then for each* $\delta \in (0, \delta_0\rangle$ (*with some* $\delta_0 > 0$ *fixed*) *there exists an unbounded closed connected subset* $C_{\delta,0}^+$ *of* $C_\delta$ *containing* $[\mu^{(0)}, 0, 0]$ *such that the following implications are true for all* $[\mu, v, \varepsilon] \in C_{\delta,0}^+$:

(c) *if* $[\mu, v, \varepsilon] \neq [\tilde{\mu}, 0, 0]$ *for all* $\tilde{\mu} \in r_A$, *then* $v \notin K$;

(d) *if* $[\mu, v, \varepsilon] \neq [\mu^{(0)}, 0, 0]$, *then* $\mu \in (\mu^{(0)}, \mu^{(1)})$.

*If* $\{\mu_n, v_n, \varepsilon_n\} \subset C_{\delta,0}^+$, $\varepsilon_n \to +\infty$, $\mu_n \to \mu(\delta)$, $v_n \rightharpoonup v(\delta)$,*) *then* $v_n \to v(\delta)$, $\mu(\delta) \in (\mu^{(0)}, \mu^{(1)})$, $\|v(\delta)\|^2 = \delta$, $v(\delta) \in \partial K$, $\mu(\delta)$, $v(\delta)$ *satisfy* (I), (II) *and do not satisfy* (E). *The limit points of* $\mu(\delta)$ *for* $\delta \to 0$ *lie in* $(\mu^{(0)}, \mu^{(1)}) \cap r_{V,b}$.

**Theorem 2.3.** *Let the assumptions of Theorem* 2.1 *and* (2.1) *from Remark* 2.3 *be fulfilled with* $\langle u^{(j)}, u_j^* \rangle < 0$ *in the assumption* (U, U*). *Then for each* $\delta \in (0, \delta_0\rangle$ (*with some* $\delta_0 > 0$ *fixed*) *there exists a subset* $C_{\delta,0}^+$ *of* $C_\delta$ *with the same properties as in Theorem* 2.2 *but with* (d) *replaced by*

(d$_1$) *if* $[\mu, v, \varepsilon] \neq [\mu^{(1)}, 0, 0]$, *then* $\mu \in (\mu^{(0)}, \mu^{(1)})$,

*and containing* $[\mu^{(1)}, 0, 0]$ *instead of* $[\mu^{(0)}, 0, 0]$.

Proof of Theorems 2.2, 2.3 will be given in Section 4 on the basis of a global bifurcation result. But first, an investigation of some properties of sets of solutions of (b) is necessary and this is the subject of Section 3.

The main ideas of the proof of Theorem 2.2 are the following (for the precise proof see Section 4). It follows from a Dancer's global bifurcation result (Theorem 4.1) that for each $\delta > 0$ there exist closed connected subsets $C_{\delta,0}^+$ and $C_{\delta,0}^-$ of $C_\delta$ starting from $[\mu^{(0)}, 0, 0]$ in the direction $u^{(0)} \notin K$ and $-u^{(0)} \in K^0 \cap E_A(\mu^{(0)})$, respectively, and these sets either meet each other at a point different from $[\mu^{(0)}, 0, 0]$ or they are both unbounded. Our aim will be to show that the first case cannot occur for $\delta$ sufficiently small. This will be done by proving that all the points from $C_{\delta,0}^+$ fulfil the implications (c), (d) and that $v \in K^0$ for all $[\mu, v, \varepsilon] \in C_{\delta,0}^-$ with $\mu \in \langle \mu^{(0)}, \mu^{(1)} \rangle$, $[\mu^{(0)}, 0, 0] \neq [\mu, v, \varepsilon] \neq [\mu^{(1)}, 0, 0]$. The proof of (c), (d) is based on the following principles:

(i) for an arbitrary $\delta > 0$, the values $\mu$ are locally increasing along $C_{\delta,0}^+$ near $\mu = \mu^{(0)}$, $\varepsilon = 0$ and $\mu = \mu^{(1)}$, $\varepsilon = 0$; this is the sense of Lemma 3.2;

(ii) for $\delta > 0$ sufficiently small, $C_{\delta,0}^+$ cannot intersect the boundary of $K$ with $\varepsilon > 0$ as long as $\mu \in \langle \mu^{(0)}, \mu^{(1)} \rangle$ (this is a consequence of the assumption (2.1)) and

*) The existence of such a sequence follows from the fact that $C_{\delta,0}^+$ is unbounded, from (c), (d) and Remark 2.4.

simultaneously $C_{\delta,0}$ cannot intersect the lines $\mu = \mu^{(0)}$, $\mu = \mu^{(1)}$ as long as $v \notin K$ (this will follow from Lemma 3.1).

When the existence of an unbounded set $C^+_{\delta,0}$ satisfying (c), (d) is proved, then it suffices to show that by the limiting proces $\varepsilon \to +\infty$ along $C^+_{\delta,0}$, solutions of (I), (II) with the announced properties can be obtained. This will follow from Lemma 3.3, which is a modification of the usual penalty method. The solutions obtained will lie on $\partial K$ and this together with the assumption that $\mu^{(0)}, \mu^{(1)} \in r_{A,i}$ are simple and the implication (d) will ensure that $\mu(\delta) \in (\mu^{(0)}, \mu^{(1)})$ for $\delta$ sufficiently small and that the limit points of $\mu(\delta)$ $(\delta \to 0)$ are in $(\mu^{(0)}, \mu^{(1)})$.

The main ideas of the proof of Theorem 2.3 are the same as that of Theorem 2.2, but $\mu$ are locally decreasing along $C_\delta$ near $\mu = \mu^{(0)}$, $\varepsilon = 0$ and $\mu = \mu^{(1)}$, $\varepsilon = 0$ and this is the reason for the use of a branch $C^+_{\delta,0}$ starting from $[\mu^{(1)}, 0, 0]$ instead of $[\mu^{(0)}, 0, 0]$.

Remark 2.5. The basic principle of the proof of Theorems 2.2, 2.3 is similar to that of Theorem 2.3 in [4]. However, in [4] the case $N \equiv 0$ was considered and the operators in the penalty in the equation (b) were homogeneous. This made it possible to work with a more agreeable norm condition (a). The nonlinear term $N$ forces us to consider the branches of solutions of (b) containing small $v$ only, because the branches containing great $v$ need not have properties necessary for our pourpose (implications (c), (d)). Moreover, it was sufficient to have a unique branch of solutions of (b) in [4] for obtaining an eigenvalue and eigenvector of (I), ($II_L$), while in the present situation we need to obtain a continuum $C_{\delta,0}$ $(\delta \in (0, \delta_0\rangle)$ of suitable branches for obtaining a system of bifurcating solutions. Of course, Theorems 2.1–2.3 from [4] can be obtained from the present Theorems 2.1–2.3 by choosing $N \equiv 0$ and by a suitable transformation between the branches $S$ and $C$. In this case, $\delta > 0$ can be chosen arbitrarily; we obtain $\mu_\delta = \mu_\infty$, $v(\delta) = \sqrt{\delta} \,.\, u_\infty$.

Remark 2.6. In some cases it is possible to show that there exist infinitely many of couples $\mu^{(0)}, \mu^{(1)}$ satisfying the assumptions of Theorem 2.2 (see Section 5). Then we obtain the existence of infinitely many boundary bifurcation points of (I), (II) which are neither interior nor boundary bifurcation points of (E) (cf. Remark 2.3).

## 3. PROPERTIES OF THE EQUATION WITH PENALTY

In this section the equation (b) (Definition 2.1) with a penalty operator of the type described in Section 2 will be studied.

Remark 3.1. If $\varepsilon_n > 0$, $v_n \in H$ $(n = 1, 2, \ldots)$ and $\varepsilon_n \beta v_n \rightharpoonup f$, then (P), (β, K) imply immediately

$$(3.1) \qquad \langle f, u \rangle = \lim \varepsilon_n \langle \beta v_n, u \rangle \leqq 0 \quad \text{for all} \quad u \in K\,.$$

If $f \neq 0$, then even

$$(3.2) \qquad \langle f, u \rangle < 0 \quad \text{for all} \quad u \in K^0\,.$$

Indeed, if the last assertion is not true, then there exist $u_1 \in H$, $u_2 \in K^0$ such that $\langle f, u_1 \rangle > 0$, $\langle f, u_2 \rangle = 0$. Hence, $\langle f, u_2 + tu_1 \rangle > 0$ for all $t > 0$, but this contradicts (3.1) because $u_2 + tu_1 \in K$ for $t$ sufficiently small.

Remark 3.2. If $\varepsilon_n > 0$, $u_n \in H$, $u \in K$ $(n = 1, 2, \ldots)$, $u_n \rightharpoonup u$, $\varepsilon_n \beta u_n \rightharpoonup f$, then (P), (β, K) imply $\varepsilon_n \langle \beta u_n, u_n \rangle \geqq 0$, $\langle f, u \rangle = \lim \varepsilon_n \langle \beta u_n, u \rangle \leqq 0$. Particularly,

$$\langle f, u \rangle \leqq \liminf \varepsilon_n \langle \beta u_n, u_n \rangle . \tag{3.3}$$

Remark 3.3. It is well-known that if $u_n \rightharpoonup u$, $\beta u_n \to 0$ and (M), (P) are fulfilled, then $u \in K$. Indeed, for an arbitrary $v \in H$ we have

$$\langle \beta v, v - u \rangle = \lim \langle \beta v - \beta u_n, v - u_n \rangle \geqq 0$$

by (M). Setting $v = u + tw$, we obtain

$$\langle \beta(u + tw), w \rangle \geqq 0 \quad \text{for all} \quad w \in H , \quad t > 0 .$$

Passing to the limit for $t \to 0+$, we obtain the last inequality for $t = 0$ and for all $w \in H$. This is equivalent to $\beta u = 0$, i.e. $u \in K$ by (P).

**Lemma 3.1.** *Let $\mu_0 \in r_{A^*,i}$ and let the assumptions* (P), (β, K) *be fulfilled. If $\varepsilon_n > 0$, $v_n \in H$, $\varepsilon_n \beta v_n \rightharpoonup f$ with $f \neq 0$, then*

$$u - \mu_0 A u + f \neq 0 \quad \textit{for all} \quad u \in H . \tag{3.4}$$

Proof. If (3.4) is not true, then we have

$$\begin{aligned} u - \mu_0 A u + f &= 0 , \\ u^* - \mu_0 A^* u^* &= 0 \end{aligned}$$

with some $u \in H$ and $u^* \in K^0 \cap E_{A^*}(\mu_0)$. Using the relation $\langle Au, u^* \rangle = \langle A^* u^*, u \rangle$, we obtain from here $\langle f, u^* \rangle = 0$. This contradicts (3.2) from Remark 3.1.

**Lemma 3.2.** *Suppose that* (H), (N) *are fulfilled. Let $[\mu_n, v_n, \varepsilon_n] \in \mathbb{R} \times H \times \mathbb{R}$, $\varepsilon_n > 0$,*

$$v_n - \mu_n A v_n + N(\mu_n, v_n) + \varepsilon_n \beta v_n = 0 \tag{b\~{}}$$

$(n = 1, 2, \ldots)$, $[\mu_n, v_n, \varepsilon_n] \to [\mu_0, 0, 0]$ in $\mathbb{R} \times H \times \mathbb{R}$, $\mu_0 \neq 0$, $v_n / \|v_n\| = u_n \rightharpoonup u_0$. *Then $u_n \to u_0$, $\mu_0 \in r_A$, $u_0 \in E_A(\mu_0)$ and*

$$\langle u_0, u^* \rangle \lim \frac{\mu_n - \mu_0}{\varepsilon_n} = \mu_0 \langle \beta u_0, u^* \rangle \quad \textit{for each} \quad u^* \in E_{A^*}(\mu_0), \quad u^* \neq 0 . \tag{3.6}$$

*Particularly, if* (β, K) *is fulfilled, $\mu_0 > 0$, $u_0 \notin K$, $\langle u_0, u^* \rangle > 0$ for some $u^* \in E_{A^*}(\mu_0) \cap (-K^0)$, then*

$$\lim \frac{\mu_n - \mu_0}{\varepsilon_n} > 0 ; \tag{3.7}$$

*if* $\mu_0 > 0$, $u_0 \notin K$, $\langle u_0, u^* \rangle < 0$ *for some* $u^* \in E_{A^*}(\mu_0) \cap (-K^0)$, *then*

$$\lim \frac{\mu_n - \mu_0}{\varepsilon_n} < 0 . \tag{3.7'}$$

Proof. Denote $u_n = v_n / \|v_n\|$. Then ($b^\sim$) can be written as

$$u_n - \mu_n A u_n + \frac{N(\mu_n, v_n)}{\|v_n\|} + \varepsilon_n \beta u_n = 0 . \tag{$b^\sim$}$$

This together with the assumption (N) and the complete continuity of $A$, $\beta$ implies that $u_n \to u_0$, $\mu_0 \in r_A$, $u_0 \in E_A(\mu_0)$. Further, if $u^* \in E_{A^*}(\mu_0)$, then

$$u^* - \mu_0 A^* u^* = 0 . \tag{3.8}$$

Setting $\lambda_n = 1/\mu_n$, $\lambda_0 = 1/\mu_0$, we obtain from ($b^\sim$) and (3.8)

$$(\lambda_n - \lambda_0) \langle u_n, u^* \rangle = -\lambda_n \left\langle \frac{N(\mu_n, v_n)}{\|v_n\|}, u^* \right\rangle - \lambda_n \varepsilon_n \langle \beta u_n, u^* \rangle .$$

Replacing $\lambda_n$, $\lambda_0$ by $\mu_n$, $\mu_0$ again and using (N), we obtain (3.6).

**Lemma 3.3.** (cf. [4, Lemma 2.4]). *Let* $\mu^{(0)}, \mu^{(1)} \in r_{A,i} \cap r_{A^*,i}$ *be simple,* $0 < \mu^{(0)} < \mu^{(1)}$ *and let the assumptions* (P), (M), (β, K) *be fulfilled. Suppose that* $[\mu_n, v_n, \varepsilon_n] \in \mathbb{R} \times H \times \mathbb{R}$ *satisfy* ($b^\sim$) *from Lemma* 3.2 *and the following conditions (with* $\delta > 0$ *fixed*):

$$\|v_n\|^2 = \frac{\delta \varepsilon_n}{1 + \varepsilon_n} \quad (n = 1, 2, \ldots), \quad \varepsilon_n \to +\infty ; \tag{$a^\sim$}$$

$$v_n \notin K^0 \quad (n = 1, 2, \ldots) ; \tag{$c^\sim$}$$

$$\mu_n \in (\mu^{(0)}, \mu^{(1)}) \quad (n = 1, 2, \ldots) . \tag{$d^\sim$}$$

*If* $\mu_n \to \mu(\delta)$, $v_n \rightharpoonup v(\delta)$ *for some* $\mu(\delta) \in \mathbb{R}$, $v(\delta) \in H$, *then* $v_n \to v(\delta)$, $\|v(\delta)\|^2 = \delta$, $\mu(\delta)$, $v(\delta)$ *satisfy* (I), (II) *and* $v(\delta) \in \partial K$, $\|v(\delta)\|^2 = \delta$. *If* $\delta < \delta_0$ ($\delta_0$ *sufficiently small*), *then* $\mu(\delta) \in (\mu^{(0)}, \mu^{(1)})$.

Proof. It follows from ($a^\sim$), ($b^\sim$) that $\{\varepsilon_n \beta v_n\}$ is bounded and therefore $\beta v_n \to 0$. Hence, $v(\delta) \in \partial K$ by Remark 3.3 and ($c^\sim$). Multiplying ($b^\sim$) by $v_n$ and $v(\delta)$, we obtain

$$\langle v_n, v_n \rangle - \mu_n \langle A v_n, v_n \rangle + \langle N(\mu_n, v_n), v_n \rangle + \varepsilon_n \langle \beta v_n, v_n \rangle = 0 , \tag{3.9}$$

$$\langle v_n, v(\delta) \rangle - \mu_n \langle A v_n, v(\delta) \rangle + \langle N(\mu_n, v_n), v(\delta) \rangle + \varepsilon_n \langle \beta v_n, v(\delta) \rangle = 0 . \tag{3.10}$$

Suppose that $N(\mu_n, v_n) \to h$, $\varepsilon_n \beta v_n \rightharpoonup f$. (This is true for a subsequence at least because $N$ is completely continuous and $\{\varepsilon_n \beta v_n\}$ is bounded.) We obtain from (3.9), (3.10) and ($a^\sim$) that

$$\lim \varepsilon_n \langle \beta v_n, v_n \rangle = -\delta^2 + \mu(\delta) \langle A\, v(\delta), v(\delta) \rangle - \langle h, v(\delta) \rangle ,$$

$$\langle f, v(\delta) \rangle = -\|v(\delta)\|^2 + \mu(\delta) \langle A\, v(\delta), v(\delta) \rangle - \langle h, v(\delta) \rangle .$$

Using (3.3) from Remark 3.2, we obtain from here

$$\|v(\delta)\|^2 - \lim \|v_n\|^2 = \|v(\delta)\|^2 - \delta = \lim \varepsilon_n \langle \beta v_n, v_n \rangle - \langle f, v(\delta) \rangle \geqq 0 .$$

Hence $v_n \rightharpoonup v(\delta)$, $\|v(\delta)\| \geqq \lim \|v_n\|$, that means $v_n \to v(\delta)$, $\|v(\delta)\|^2 = \delta$. (More precisely, this is proved for a subsequence; but if this were not true for the whole sequence $\{v_n\}$, then by the same procedure we could obtain another subsequence converging to $\tilde{v}(\delta) \neq v$, which is not possible becuse $v_n \rightharpoonup v(\delta)$.) Using this fact and ($b^{\sim}$), (P), (M), we obtain for an arbitrary $w \in K$

$$\langle v(\delta) - \mu(\delta) A\, v(\delta) + N(\mu(\delta), v(\delta)),\ w - v(\delta) \rangle =$$

$$= \lim \langle v_n - \mu_n A v_n + N(\mu_n, v_n),\ w - v_n \rangle =$$

$$= \lim \varepsilon_n \langle \beta w - \beta v_n,\ w - v_n \rangle \geqq 0 ,$$

i.e. $\mu(\delta)$, $v(\delta)$ satisfy (II). It is sufficient to show that neither $\mu(\delta) = \mu^{(0)}$ nor $\mu(\delta) = \mu^{(1)}$ can occur for $\delta$ sufficiently small. If $\mu(\delta_n) = \mu^{(0)}$, $\delta_n \to 0$, then we can suppose $v(\delta_n)/\|v(\delta_n)\| \to u \in E_V(\mu^{(0)})$ by Remark 1.2. Lemma 1.1 implies $u \in E_A(\mu^{(0)})$. But $v(\delta_n) \in \partial K$, i.e. $u \in \partial K$ and this contradicts the assumption that $\mu^{(0)} \in r_{A,i}$ is simple. Analogously for $\mu^{(1)}$.

Remark 3.4. In the sequel, the following modification of the situation from Lemma 3.3 will occur. We shall have $[\mu_n, v_n, \varepsilon_n] \in \mathbb{R} \times H \times \mathbb{R}$ satisfying ($b^{\sim}$) from Lemma 3.2, $\|v_n\| > 0$, $\varepsilon_n > 0$ $(n = 1, 2, \ldots)$, $\|v_n\| \to 0$, $\mu_n \to \mu$. We shall show that if $u_n = v_n/\|v_n\| \rightharpoonup u$, $\varepsilon_n \beta u_n \rightharpoonup f$, then $u_n \to u$, $\varepsilon_n \beta u_n \to f$ and

$$u - \mu A u + f = 0 . \tag{3.11}$$

First, ($b^{\sim}$) can be written as ($b^{-}$) (the proof of Lemma 3.2) again under the assumption (H), and (3.11) follows by using the limiting process and (N). If $\{\varepsilon_n\}$ is bounded, then it follows from ($b^{-}$), (N) and the complete continuity of $A$, $\beta$ that $\{u_n\}$ contains a strongly convergent subsequence. This together with the assumption $u_n \rightharpoonup u$ implies that $u_n \to u$. (In the opposite case, we would obtain by an analogous consideration another subsequence converging to $u_0 \neq u$, which is impossible.) Now, it folows from ($b^{-}$) that also $\{\varepsilon_n \beta u_n\}$ is strongly convergent. Further, let $\{\varepsilon_n\}$ be unbounded. Then there is a subsequence $\{\varepsilon_{k_n}\}$ such that $\varepsilon_{k_n} \to +\infty$. We obtain $\beta u_{k_n} \to 0$ because $\{\varepsilon_n \beta u_n\}$ is bounded by ($b^{-}$). Thus, $u \in K$ by Remark 3.3. The identity ($b^{-}$) implies

$$\langle u_n, u_n \rangle - \mu_n \langle A u_n, u_n \rangle + \left\langle \frac{N(\mu_n, v_n)}{\|v_n\|}, u_n \right\rangle + \varepsilon_n \langle \beta u_n, u_n \rangle = 0 ,$$

$$\langle u_n, u \rangle - \mu_n \langle A u_n, u \rangle + \left\langle \frac{N(\mu_n, v_n)}{\|v_n\|}, u \right\rangle + \varepsilon_n \langle \beta u_n, u \rangle = 0$$

which together with Remark 3.2 implies that

$$\|u\|^2 - \lim \|u_n\|^2 = \lim \varepsilon_n \langle \beta u_n, u_n \rangle - \langle f, u \rangle \geqq 0 .$$

We have $u_n \rightharpoonup u$, $\|u\| \geqq \lim \|u_n\|$, i.e. $u_n \to u$, $\|u\| = 1$. Now, it follows from ($b^-$) and (N) that $\{\varepsilon_n \beta u_n\}$ is strongly convergent.

## 4. USING DANCER'S GLOBAL BIFURCATION RESULT

The aim of this section is to prove Theorems 2.2, 2.3. First, we shall explain a result of E. N. Dancer, which will be the basis of our considerations.

Let $X$ be a real Hilbert space with an inner product $(\cdot, \cdot)$ and with the corresponding norm $|||\cdot|||$, $L: X \to X$ a linear completely continuous operator in $X$. Let $G$ be a nonlinear completely continuous mapping of $\mathbb{R} \times X$ into $X$ such that

$$\text{(N')} \qquad \lim_{|||x||| \to 0} \frac{G(\mu, x)}{|||x|||} = 0 \quad \text{uniformly on bounded subsets of } \mathbb{R}\,.$$

We shall study the bifurcation problem for the equation

$$\text{(B)} \qquad x - \mu L(x) + G(\mu, x) = 0\,,$$

where $\mu$ is a real parameter. This is precisely the same problem as (E) in Section 1, but for reasons which will be seen later we consider a new space and new operators. In the sequel, $X$, $L$ and $G$ will be determined by $H$, $A$, $N$ and $\beta$ from the previous sections.

Denote by $C$ the closure in $(\mathbb{R} \times X)$ of the set of all nontrivial solutions of (B), i.e.

$$C = \overline{\{[\mu, x] \in \mathbb{R} \times H;\ |||x||| \neq 0,\ \text{(B) is fulfilled}\}}\,.$$

Hence, a point $[\mu, 0]$ is a bifurcation point of (B) if and only if $[\mu, 0] \in C$. Analogously as in Remark 1.2,

$$\text{(4.1)} \qquad \text{if} \quad [\mu_n, x_n] \in C\,, \quad \mu_n \to \tilde{\mu}\,, \quad |||x_n||| \to 0\,, \quad \frac{x_n}{|||x_n|||} \rightharpoonup \tilde{x}\,,$$

$$\text{then} \quad \tilde{\mu} \in r_L\,, \quad \tilde{x} \in E_L(\tilde{\mu})\,, \quad \frac{x_n}{|||x_n|||} \to \tilde{x}\,.$$

Now, let $\mu_0 \in r_L$ be simple (see Remark 1.7). Then $[\mu_0, 0]$ is a bifurcation point of (B) and the component $C_0$ of $C$ containing $[\mu_0, 0]$ is non-empty (see [10], [1]). Moreover, $C_0$ "consists of two branches $C_0^+$ and $C_0^-$ starting from $[\mu_0, 0]$ in the direction $x_0$ and $-x_0$, respectively", where $x_0 \in E_L(\mu_0)$. This result will be essential for us and we shall formulate it precisely.

Denote by $x_0^*$ an eigenvector of the adjoint operator $L^*$ corresponding to $\mu_0$ (i.e. $x_0^* \in E_{L^*}(\mu_0)$) and suppose that it is normed so that $(x_0^*, x_0) = 1$ (see Remark 1.7). Choose $\eta \in (0, 1)$ and set

$$K_\eta = \{[\mu, x] \in \mathbb{R} \times X;\ |(x, x_0^*)| > \eta |||x|||\}\,,$$

$$K_\eta^+ = \{[\mu, x] \in K_\eta;\ (x, x_0^*) > 0\}\,, \quad K_\eta^- = K_\eta = K_\eta \setminus K_\eta^+\,.$$

There exists $R > 0$ such that

$$C \setminus \{[\mu_0, 0]\} \cap B_R(\mu_0, 0) \subset K_\eta ,$$

where $B_R(\mu_0, 0) = \{[\mu, x] \in \mathbb{R} \times X;\ |\mu - \mu_0| + |||x||| \leqq R\}$ (for the proof see [10, Lemma 1.24]). For each $r \in (0, R\rangle$ denote by $D_r^+$ and $D_r^-$ the components of the sets $\{[\mu_0, 0]\} \cup (C \cap B_r(\mu_0, 0) \cap K_\eta^+)$ and $\{[\mu_0, 0]\} \cup (C \cap B_R(\mu_0, 0) \cap K_\eta^-)$, respectively, containing $[\mu_0, 0]$. Denote by $C_{0,r}^+$ and $C_{0,r}^-$ the components of $\overline{C_0 \setminus D_r^-}$ and $\overline{C_0 \setminus D_r^+}$, respectively, containing $[\mu_0, 0]$. Set

$$C_0^+ = \overline{\bigcup_{0<r\leqq R} C_{0,r}^+}, \qquad C_0^- = \overline{\bigcup_{0<r\leqq R} C_{0,r}^-}.$$

The definition of $C_0^+$, $C_0^-$ is independent of the choice of $\eta \in (0, 1)$ (see [10, Lemma 1.24]), $C_0^+$, $C_0^-$ are connected and $C_0 = C_0^+ \cup C_0^-$ (for the proof see [10], cf. [1]). Further, it follows from the definition of $C_0^+$, $C_0^-$ and [10, Lemma 1.24] that

(4.2) $$\text{if } [\mu_n, x_n] \in C_0^+ \setminus K_\eta^- \cap B_\gamma(\mu_0, 0) \text{ for some } \gamma > 0,$$

$$\mu_n \to \mu_0, \quad |||x_n||| \to 0, \quad \text{then} \quad \frac{x_n}{|||x_n|||} \to x_0;$$

(4.3) $$\text{if } [\mu_n, x_n] \in C_0^- \setminus K_\eta^+ \cap B_\gamma(\mu_0, 0) \text{ for some } \gamma > 0,$$

$$\mu_n \to \mu_0, \quad |||x_n||| \to 0, \quad \text{then} \quad \frac{x_n}{|||x_n|||} \to -x_0.$$

**Theorem 4.1** (E. N. Dancer [1, Theorem 2)]. *Either both $C_0^+$ and $C_0^-$ are unbounded or $C_0^+ \cap C_0^- \neq \{[\mu_0, 0]\}$.*

Remark 4.1. In the sequel, we shall utilize the properties of the bifurcation branches of the equation (B) in the following special situation. Under the assumptions of Theorem 2.1, we shall set $X = H \times \mathbb{R}$ and introduce the operators $L : X \to X$ and $G_\delta : \mathbb{R} \times X \to X$ (for each fixed $\delta > 0$) by

(*) $$L(x) = L(v, \varepsilon) = [Av, 0] \quad \text{for all} \quad x = [v, \varepsilon] \in X,$$

$$G_\delta(\mu, x) = G_\delta(\mu, v, \varepsilon) = \left[N(\mu, v) + \varepsilon\beta v,\ -\frac{1+\varepsilon}{\delta}\|v\|^2\right]$$

$$\text{for all} \quad \mu \in \mathbb{R}, \quad x = [v, \varepsilon] \in X.$$

In this special case the equation (B) (for $G_\delta$ instead of $G$, $\delta$ fixed) is equivalent to the equations (a), (b) from Definition 2.1. Hence, for the case of the operators (*) the set $C$ introduced above coincides with $C_\delta$ from Definition 2.1. We shall use this original notation and write also $C_{\delta,0}$, $C_{\delta,0}^+$, $C_{\delta,0}^-$ instead of $C_0$, $C_0^+$, $C_0^-$ in the case under consideration.

Remark 4.2. It is clear that in the case of the operators (*) from Remark 4.1

we have $\mu \in r_L$, $[u, \varepsilon] \in E_L(\mu)$ if and only if $\mu \in r_A$, $\varepsilon = 0$, $u \in E_A(\mu)$. The multiplicities of $\mu$ viewed as characteristic values of $L$ and $A$ are equal to each other. Particularly, $\mu^{(0)}$ is a simple characteristic value of $L$ under the assumptions of Theorem 2.1.

Remark 4.3. If $[\mu_0, 0] \in C$ in the general situation from the beginning of this section, then $\mu_0 \in r_L$ (see Remark 1.2). The characteristic values of $L$ are isolated and therefore there exists $r > 0$ $(r = \operatorname{dist}(\mu_0, r_L \setminus \{\mu_0\}))$ such that if $[\mu, x] \in C$, $0 < |\mu - \mu_0| < r$, then $|||x||| > 0$. Particularly, if $\mu_0 \in r_A$ in the situation from Remark 4.1 (i.e. $\mu_0 \in r_L$ by Remark 4.2) then there exists $r > 0$ such that $\varepsilon > 0$, $\|v\| > 0$ for all $[\mu, v, \varepsilon] \in C_{0,\delta}$, $0 < |\mu - \mu_0| < r$, $\delta > 0$.

Remark 4.4. The implications (4.2), (4.3) can be written as follows in the situation of Remark 4.1:

(4.2') if $[\mu_n, v_n, \varepsilon_n] \in C^+_{0\,\delta} \setminus K^-_\eta \cap B_r(\mu_0, 0)$ for some $r > 0$, $\delta > 0$,

$$\mu_n \to \mu_0,\ \|v_n\| \to 0, \text{ then } \frac{\varepsilon_n}{\|v_n\|} \to 0,\ \frac{v_n}{\|v_n\|} \to u^{(0)};$$

(4.3') if $[\mu_n, v_n, \varepsilon_n] \in C^-_{0,\delta} \setminus K^+_\eta \cap B_r(\mu_0, 0)$ for some $r > 0$, $\delta > 0$,

$$\mu_n \to \mu_0,\ \|v_n\| \to 0, \text{ then } \frac{\varepsilon_n}{\|v_n\|} \to 0,\ \frac{v_n}{\|v_n\|} \to -u^{(0)}$$

(where $u^{(0)} \in E_A(\mu^{(0)}) \cap (-K^0)$ as in (U, U*) – the assumptions of Theorem 2.1 are considered). Particularly, if $[\mu_n, v_n, \varepsilon_n] \in C_{0,\delta}$, $\mu_n \to \mu_0$, $\|v_n\| \to 0$ with $v_n \notin K$, then the case (4.2') must occur because $-u^{(0)} \in K^0$. For the case of a general point $[\tilde{\mu}, 0, 0] \in$ $\in C_\gamma$, (4.1) together with Remark 4.2 give

(4.1') if $[\mu_n, v_n, \varepsilon_n] \in C_\delta$, $[\mu_n, v_n, \varepsilon_n] \to [\tilde{\mu}, 0, 0]$, $\dfrac{v_n}{\|v_n\|} \rightharpoonup \tilde{u}$,

$$\text{then } \tilde{\mu} \in r_A,\ \tilde{u} \in E_A(\tilde{\mu}),\ \frac{v_n}{\|v_n\|} \to \tilde{u}.$$

Remark 4.5. Let us consider the situation from Theorem 2.1. It follows from the assumptions (2.1), (P) that for each $\tilde{\mu} \in (\mu^{(0)}, \mu^{(1)}) \cap r_A$ there exists a neighborhood $U(\tilde{\mu}, 0, 0)$ of $[\tilde{\mu}, 0, 0]$ in $\mathbb{R} \times H \times \mathbb{R}$ such that

$$\text{if } [\mu, v, \varepsilon] \in (U(\tilde{\mu}, 0, 0) \setminus \{[\tilde{\mu}, 0, 0]\}) \cap C_\delta \text{ with } \delta > 0, \text{ then } v \notin K.$$

Remark 4.6. Under the assumptions of Theorem 2.2, Lemma 3.2 ensures that for each $\delta > 0$ there exists a neighborhood $U(\mu^{(1)}, 0, 0)$ of $[\mu^{(1)}, 0, 0]$ in $\mathbb{R} \times H \times \mathbb{R}$ such that

$$\text{if } [\mu, v, \varepsilon] \in U(\mu^{(1)}, 0, 0),\ \mu \leqq \mu^{(1)},\ v \notin K,\ \varepsilon > 0, \text{ then}$$
$$[\mu, v, \varepsilon] \notin C_\delta.$$

Indeed, in the opposite case there exist $\delta > 0$ and $[\mu_n, v_n, \varepsilon_n] \in C_\delta$ such that $\mu_n \leqq \mu^{(1)}$,

$\varepsilon_n > 0$, $v_n \notin K$ $(n = 1, 2, \ldots)$, $[\mu_n, v_n, \varepsilon_n] \to [\mu^{(1)}, 0, 0]$. We can suppose $v_n/\|v_n\| \to \to u^{(1)} \notin K$ by Remark 4.4 and we have $(\mu_n - \mu^{(1)})/\varepsilon_n \leqq 0$, which contradicts (3.7) from Remark 3.2.

**Lemma 4.1** (cf. [4, Lemma 2.3]). *Let all the assumptions of Theorem 2.2 be fulfilled. Then there exists $\delta_0 > 0$ such that if $\delta \in (0, \delta_0)$ then for all $[\mu, v, \varepsilon] \in C^+_{\delta,0}$ the implications* (c), (d) *from Theorem 2.2 hold.*

Proof. The set $C^+_{\delta,0}$ is non-empty by Theorem 4.1 and Remark 4.1. Denote by $C^+_{\delta,1}$ the component of the set

$$\{[\mu, v, \varepsilon] \in C^+_{\delta,0};\ \mu \in \langle \mu^{(0)}, \mu^{(1)} \rangle\}$$

containing $[\mu^{(0)}, 0, 0]$. It follows from the definition of $C^+_{\delta,0}$ and Remarks 4.3, 4.4 that there exist $[\mu_n, v_n, \varepsilon_n] \in C^+_{\delta,0}$ such that $[\mu_n, v_n, \varepsilon_n] \to [\mu^{(0)}, 0, 0]$, $\|v_n\| > 0$, $\varepsilon_n > 0$, $v_n/\|v_n\| \to u^{(0)} \notin K$, i.e. also $v_n \notin K$ for $n$ sufficiently large. Lemma 3.2 (the relation (3.7)) together with $\langle u^{(j)}, u_j^* \rangle > 0$ in the assumption (U, U*) implies $\mu_n > \mu^{(0)}$ (for $n$ sufficiently large). That means

(4.4) $\quad C^+_{\delta,1}$ contains points $[\mu, v, \varepsilon]$ with $v \notin K$ (for an arbitrary $\delta > 0$).

Now, we shall prove that

(C) there exists $\delta_0 > 0$ such that (c) is valid for all $[\mu, v, \varepsilon] \in C^+_{\delta,1}$ with $\delta \in (0, \delta_0\rangle$ arbitrary.

Suppose the contrary. It follows from (4.4), Remark 4.5 and from the connectedness of $C^+_{\delta,1}$ that there exist $\delta_n$ and $[\mu_n, v_n, \varepsilon_n] \in C_{\delta_n,1}$ such that

(4.5)
$$\delta_n > 0\,, \quad v_n \in \partial K\,, \quad \mu_n \in \langle \mu^{(0)}, \mu^{(1)} \rangle\,, \quad \delta_n \to 0\,,$$
$$[\mu_n, v_n, \varepsilon_n] \neq [\tilde{\mu}, 0, 0] \quad \text{for all} \quad \tilde{\mu} \in r_A \cap \langle \mu^{(0)}, \mu^{(1)} \rangle \quad (n = 1, 2, \ldots)\,.$$

The inclusion $[\mu, 0, 0] \in C_\delta$ can hold only for $\mu \in r_A$ and therefore $\|v_n\| > 0$ by (4.5). We have $\|v_n\| \to 0$ by (a) and we can suppose $\mu_n \to \tilde{\mu} \in \langle \mu^{(0)}, \mu^{(1)} \rangle \cap r_A$ by (4.1). The case $\tilde{\mu} \in (\mu^{(0)}, \mu^{(1)})$ is not possible by (2.1) and $\tilde{\mu} = \mu^{(0)}$, $\tilde{\mu} = \mu^{(1)}$ is not possible with respect to the assumption that $\mu^{(0)}, \mu^{(1)} \in r_{A,i}$ are simple and (4.1) holds.

Further, we shall prove that

(D) there exists $\delta_0 > 0$ such that (d) is valid for all $[\mu, v, \varepsilon] \in C^+_{\delta,1}$ with $\delta \in (0, \delta_0)$ arbitrary.

Suppose the contrary. Then there exist $\delta_n > 0$ and $[\mu_n, v_n, \varepsilon_n] \in C^+_{\delta_n,1}$ $(n = 1, 2, \ldots)$ such that $\delta_n \to 0$ and either

(4.7)
$$\mu_n = \mu^{(0)}\,, \quad \|v_n\| > 0$$

or

(4.8)
$$\mu_n = \mu^{(1)}\,.$$

We can suppose that (c) holds on $C_{\delta_n,1}$. Then it follows from Remark 4.6 and from the connectedness of $C^+_{\delta,1}$ that $[\mu^{(1)}, 0, 0] \notin C^+_{\delta_n,1}$. Hence, we have $\|v_n\| > 0$ also in the case (4.8). We have $\|v_n\| \to 0$ by (a). Denoting $u_n = v_n/\|v_n\|$, we can suppose $u_n \rightharpoonup u$, $\varepsilon_n \beta u_n \rightharpoonup f$. (The boundedness of $\{\varepsilon_n \beta u_n\}$ follows from (b)). We obtain from (b) applied to $[\mu_n, v_n, \varepsilon_n]$ by the considerations described in Remark 3.4 that $u_n \to u$, $\varepsilon_n \beta u_n \to f$ and

$$u - \mu^{(j)} Au + f = 0\,, \tag{4.9}$$

where $j = 0$ or $j = 1$. The case $f \neq 0$ is not possible by Lemma 3.1. That means $\varepsilon_n \beta u_n \to f = 0$. For the proof of (D), it is sufficient to show that $u \in \partial K$, because in this case (4.9) will contradict the assumption that $\mu^{(j)} \in r_{A,i}$ are simple. Suppose $u \notin K$. The case $\varepsilon_n \to 0$ is impossible because $(\mu_n - \mu^{(j)})/\varepsilon_n = 0$ and this contradicts (3.7) from Lemma 3.2. Hence, we can suppose $\varepsilon_n \geqq \varepsilon_0 > 0$. But then $\beta u_n \to 0$ and this implies $u \in K$ by Remark 3.3, i.e. $u \in \partial K$ because $u_n \notin K$ by (c) holding on $C^+_{\delta_n,1}$. Thus, (D) is proved.

For the proof of Lemma 4.1, it is sufficient to show that

$$C^+_{\delta,1} = C^+_{\delta,0} \quad \text{for each} \quad \delta > 0 \quad \text{such that (c), (d) hold for all } [\mu, v, \varepsilon] \in C^+_{\delta,1}\,. \tag{4.10}$$

Consider that $C^+_{\delta,1} \neq C^+_{\delta,0}$ for such a number $\delta > 0$. We have $C^+_{\delta,1} \cap \overline{C^+_{\delta,0} \setminus C^+_{\delta,1}} = \{(\mu^{(0)}, 0, 0]\}$ by (d). This together with the definition of $C^+_{\delta,0}$, $C^+_{\delta,1}$ implies that there exist $[\mu_n, v_n, \varepsilon_n] \in C^+_{\delta,0} \setminus C^+_{\delta,1}$ such that $\mu_n < \mu^{(0)}$, $\varepsilon_n > 0$, $n = 1, 2, \ldots$ $[\mu_n, v_n, \varepsilon_n] \to [\mu^{(0)}, 0, 0]$, $v_n/\|v_n\| \to u^{(0)} \notin K$. This contradicts (3.7) from Lemma 3.2 and the proof of Lemma 4.1 is complete.

**Lemma 4.2.** *Let all the assumptions of Theorem 2.2 be fulfilled. Then there exists $\delta_0 > 0$ such that if $\delta \in (0, \delta_0)$ then for all $[\mu, v, \varepsilon] \in C^-_{\delta,0}$ the following implication holds*:

(c$^-$) *if* $[\mu^{(0)}, 0, 0] \neq [\mu, v, \varepsilon] \neq [\mu^{(1)}, 0, 0]$, $\mu \in \langle \mu^{(0)}, \mu^{(1)} \rangle$, *then* $v \in K^0$.

Proof. By the definition, $C^-_{\delta,0}$ contains the points $\mu, v, \varepsilon$ with $v \in K^0$. If the assertion of lemma does not hold then it follows from here, the connectedness of $C^-_{\delta,0}$ and (2.1) that there exist $\delta_n \to 0$ and $\mu_n, v_n, \varepsilon_n$ satisfying (4.5) or one of the conditions (4.7), (4.8) with $v_n \in K$. This leads to the contradiction as in the proof of Lemma 4.1.

Proof of Theorem 2.2. Let $\delta_0 > 0$ be such that (c), (d) and (c$^-$) hold for all $[\mu, v, \varepsilon]$ from $C^+_{\delta,0}$ and from $C^-_{\delta,0}$, respectively, for an arbitrary $\delta \in (0, \delta_0\rangle$ (see Lemmas 4.1, 4.2). Then $C^+_{\delta,0} \cap C^-_{\delta,0} = \{[\mu^{(0)}, 0, 0]\}$ for all $\delta \in (0, \delta_0\rangle$ because (c), (d), (c$^-$) are fulfilled simultaneously only for this point. Hence, $C^+_{\delta,0}$ is unbounded by Theorem 4.1 and Remark 4.1 for all $\delta \in (0, \delta_0\rangle$. Lemma 3.3, Remark 2.4 and (21) ensure the properties of sequences from $C^+_{\delta,0}$ announced in Theorem 2.2. The limit points of $\mu(\delta)$ lie in $(\mu^{(0)}, \mu^{(1)}) \cap r_{V,b}$ by (d), Lemma 1.1 and Remark 1.2.

Proof of Theorem 2.3 is essentially the same as that of Theorem 2.2, but the roles of $[\mu^{(0)}, 0, 0]$ and $[\mu^{(1)}, 0, 0]$ are commuted. We consider the branches $C^+_{\delta,0}$, $C^-_{\delta,0}$ starting from $[\mu^{(1)}, 0, 0]$ and replace (d) by $(d_1)$ in Lemmas 4.1, 4.2. In the proof of a modified Lemma 4.1 we use (3.7′) instead fo (3.7) (Lemma 3.2), i.e. $C^+_{\delta,0}$ is locally decreasing instead of increasing in neighborhoods of $[\mu^{(1)}, 0, 0]$, $[\mu^{(0)}, 0, 0]$. All the other considerations are the same as in the proof of Theorem 2.2.

## 5. APPLICATION

Let us denote $H = \{u \in W_2^2(\langle 0, 1\rangle);\ u(0) = u(1) = 0\}$. This is a Hilbert space with the inner product

$$\langle u, v\rangle = \int_0^1 u''(x)\, v''(x)\, \mathrm{d}x\,.$$

Define operators $A : H \to H$, $N : \mathbb{R} \times H \to H$ by

$$\langle Au, v\rangle = \int_0^1 u'(x)\, v'(x)\, \mathrm{d}x \quad \text{for all} \quad u, v \in H\,,$$

$$\langle N(\mu, u), v\rangle = \int_0^1 g(\mu, u'(x))\, v'(x)\, \mathrm{d}x \quad \text{for all} \quad \mu \in \mathbb{R}\,, \quad u, v \in H\,,$$

where $g$ is a real continuous function on $\mathbb{R}^2$ satisfying the assumption

$$\lim_{t \to 0} \frac{g(\mu, t)}{t} = 0 \quad \text{uniformly on bounded } \mu\text{-intervals}\,.$$

Then $A$ is linear, completely continuous and symmetric, $N$ is completely continuous and satisfies (N). Let us consider the bifurcation problem for the variational inequality (I), (II) with these operators and with the closed convex cone

$$K = \{u \in H;\ u(x_i) \geqq 0,\ i = 1, 2, \ldots, n\}\,,$$

where $x_i \in (0, 1)$ ($i = 1, \ldots, n$, $n$ positive integer) are given numbers. By a special choice of $g$, we obtain a variational inequality describing the behaviour of a beam which is simply fixed on its ends, compressed by a force proportional to $\mu$ and supported by fixed obstacles from below at the points $x_i$ (see [9]). We can use the penalty operator $\beta$ defined by

$$\langle \beta u, v\rangle = -\sum_{i=1}^{n} u^-(x_i)\, v(x_i) \quad \text{for all} \quad u, v \in H\,,$$

where $u^-$ denotes the negative part of $u$. This operator satisfies the assumptions of Theorem 2.1. The operator $A$ has only simple characteristic values $\mu_k = k^2\pi^2$ with the corresponding eigenvectors $u_k(x) = \sin k\pi x$ ($k = 1, 2, \ldots$). The assumption (U, U*) is automatically fulfilled because $A$ is symmetric. Hence, Theorem 2.2 asserts that if $k$, $l$ are positive integers ($k < l$) such that

$$\operatorname{sign} \sin k\pi x_1 = \ldots = \operatorname{sign} \sin k\pi x_n \neq 0 ,$$
$$\operatorname{sign} \sin l\pi x_1 = \ldots = \operatorname{sign} \sin l\pi x_n \neq 0$$

(i.e. $\mu_k, u_l \in r_{A,i}$) and if

for each positive integer $m \in (k, l)$ there exist $i, j$ such that

$$\operatorname{sign} \sin m\pi x_i = - \operatorname{sign} \sin m\pi x_j \neq 0$$

(i.e. $(\mu_k, \mu_l) \cap r_A \subset r_{A,e}$), then there exists a bifurcation point $[\mu_{k,l}, 0]$ of (I), (II) with $\mu_{k,l} \in (k^2\pi^2, l^2\pi^2)$. The bifurcating solutions $[\mu, v]$ obtained near $[\mu_{k,l}, 0]$ will be such that $v(x_i) \geqq 0$ for $i = 1, \ldots, n$ and $v(x_j) = 0$ for at least one $j$ and will not satisfy (E).

For example, if $n = 2$, $x_1 = \frac{1}{4}$, $x_2 = \frac{3}{4}$, then we obtain the existence of an infinite sequence $[\tilde{\mu}_k, 0]$ of bifurcation points of (I), (II) with $\tilde{\mu}_k = \mu_{4k-3,4k-1} \in ((4k - 3)^2 \pi^2, (4k - 1)^2 \pi^2)$ (cf. [4, Section 4]).

Analogously, we could consider the cone

$$K = \{u \in H;\ u(x) \geqq 0 \text{ for all } x \in \langle x_i, y_i \rangle,\ i = 1, \ldots, n\} ,$$

where $x_i$, $y_i$ are given numbers, $0 \leqq x_1 < y_1 < \ldots < x_n < y_n \leqq 1$, and the penalty operator $\beta$ defined by

$$\langle \beta u, v \rangle = - \sum_{i=1}^{n} \int_0^1 u^-(x)\, v(x)\, \mathrm{d}x \quad \text{for all} \quad u, v \in H .$$

In this case the method gives a finite number of bifurcation points of (I), (II). The situation is analogous as in [4, Section 4], wehre it is described and illustrated in more detail. (The only difference is that here we obtain bifurcation points of (I), (II) while in [4], the eigenvalues of (I), ($II_L$) are studied.)

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*Author's address:* 115 67 Praha 1, Žitná 25, ČSSR (Matematický ústav ČSAV).