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THE LATTICE OF EQUATIONAL THEORIES PART III: DEFINABILITY AND AUTOMORPHISMS

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0. INTRODUCTION

The present Part III is a continuation of the papers [1] and [2]. Its aim is to prove the following four theorems.

Theorem 1. *For any type Δ, the set of one-based equational theories of type Δ is definable in the lattice $\mathscr{L}_\Delta$.*

Theorem 2. *For any type Δ, the set of finitely based equational theories of type Δ is definable in $\mathscr{L}_\Delta$.*

Theorem 3. (i) *If Δ is either the type $\{F\}$ or the type $\{F, o\}$ for some unary symbol F and some nullary symbol o, then the automorphism group of $\mathscr{L}_\Delta$ is isomorphic to the group of permutations of an infinite countable set.*

(ii) *If Δ is any other type then the lattice $\mathscr{L}_\Delta$ has no automorphisms besides the obvious "syntactically defined" ones.*

Theorem 4. *For any type Δ, any finitely based equational theory of type Δ is definable up to automorphisms in $\mathscr{L}_\Delta$.*

These four theorems solve a problem formulated by A. Tarski in [9] and Problems 1 and 3 and Conjecture I formulated by R. McKenzie in [8].

For a more detailed formulation of these results see Section 13.

The terminology and notation remain the same as in [1] and [2]. If $(a_1, b_1), \ldots$ $\ldots, (a_n, b_n)$ is a finite sequence of equations then $\mathrm{Cn}((a_1, b_1), \ldots, (a_n, b_n))$ denotes the equational theory generated by $(a_1, b_1), \ldots, (a_n, b_n)$. An equational theory T is said to be one-based if $T = \mathrm{Cn}(a, b)$ for some equation (a, b); it is said to be finitely based if $T = \mathrm{Cn}((a_1, b_1), \ldots, (a_n, b_n))$ for some finite sequence $(a_1, b_1), \ldots$ $\ldots, (a_n, b_n)$ of equations.

In order to be able to precise what we mean by the obvious "syntactically defined" automorphisms of $\mathscr{L}_\Delta$, we introduce the following notation.

Let Δ be an arbitrary type. We denote by H_Δ the group $S_{\Delta_0} \times S_\Delta^{(1)}$ where S_{Δ_0} is the group of all permutations of Δ_0 and $S_\Delta^{(1)}$ is the group of permutations f of $\Delta^{(1)}$

with the following two properties: if $f(F, i) = (G, j)$ then $n_F = n_G$; if $f(F, i) = (G, j)$ and $f(F, k) = (H, l)$ then $G = H$. Notice that if Δ is not a large unary type then H_Δ coincides with the group G_Δ defined in Section 7 of [2].

For every pair $(c, f) \in H_\Delta$ define a permutation $P_{c,f}$ of W_Δ as follows: if $t \in V$, put $P_{c,f}(t) = t$; if $t \in \Delta_0$, put $P_{c,f}(t) = c(t)$; if $t = F(t_1, \ldots, t_n)$ where $F \in \Delta_n$, $n \geqq 1$ and $f(F, 1) = (G, i(1)), \ldots, f(F, n) = (G, i(n))$, put $P_{c,f}(t) = G(P_{c,f}(t_{i^{-1}(1)}), \ldots \ldots, P_{c,f}(t_{i^{-1}(n)}))$.

For every pair $(c, f) \in H_\Delta$ and every $T \in \mathscr{L}_\Delta$ put $Q_{c,f}(T) = \{(P_{c,f}(u), P_{c,f}(v)); (u, v) \in T\}$. It is easy to see that $Q_{c,f}(T)$ is an equational theory; for any $(c, f) \in H_\Delta$, $Q_{c,f}$ is an automorphism of $\mathscr{L}_\Delta$. It is easy to verify that the mapping $(c, f) \mapsto Q_{c,f}$ is a homomorphism of the group H_Δ into the automorphism group of $\mathscr{L}_\Delta$; this homomorphism is injective if Δ is not the type consisting of two nullary symbols. The automorphisms $Q_{c,f}$ with $(c, f) \in H_\Delta$ are just the obvious "syntactically defined" automorphisms of $\mathscr{L}_\Delta$.

1. DEFINABILITY OF C_Δ AND E_Δ

Let Δ be an arbitrary type. We define three equational theories C_Δ, E_Δ, B_Δ of type Δ as follows:

$(u, v) \in C_\Delta$ iff either $u = v$ or u, v are not variables;

$(u, v) \in E_\Delta$ iff $\text{var}(u) = \text{var}(v)$;

$(u, v) \in B_\Delta$ iff either $u = v$ or there are nullary symbols $H, K \in \Delta$ such that $H \leqq u$ and $K \leqq v$.

Evidently, C_Δ and E_Δ are coatoms of $\mathscr{L}_\Delta$, i.e. maximal elements of $\mathscr{L}_\Delta$ different from $W_\Delta \times W_\Delta$.

Similarly as in [2], we introduce abbrevations for some special formulas and explain their meaning in the lattice $\mathscr{L}_\Delta$.

Definition. (i) $\psi_1(X, Y) \equiv X < Y \,\&\, \neg\, \exists Z(X < Z \,\&\, Z < Y)$.

(ii) $\psi_2(X) \equiv \exists Y(\omega_1(Y) \,\&\, \psi_1(X, Y))$.

(iii) $\psi_3(X) \equiv \forall A, B\, \exists C, D, E(A \leqq B \rightarrow (C = A \vee X \,\&\, D = C \wedge B \,\&\, E = X \wedge B \,\&\, D = A \vee E))$.

1.1. Lemma. (i) $\psi_1(X, Y)$ *in* $\mathscr{L}_\Delta$ *iff* Y *covers* X *in* $\mathscr{L}_\Delta$.

(ii) $\psi_2(X)$ *in* $\mathscr{L}_\Delta$ *iff* X *is a coatom of* $\mathscr{L}_\Delta$.

(iii) $\psi_3(X)$ *in* $\mathscr{L}_\Delta$ *iff* X *is a modular element of* $\mathscr{L}_\Delta$.

Definition. (i) $\psi_4 \equiv \forall X, Y((\psi_2(X) \,\&\, \psi_2(Y)) \rightarrow X = Y)$.

(ii) $\psi_5 \equiv \exists X, Y, Z(\psi_2(X) \,\&\, \psi_2(Y) \,\&\, \psi_2(Z) \,\&\, X \neq Y \,\&\, X \neq Z \,\&\, Y \neq Z)$.

1.2. Lemma. (i) ψ_4 *in* $\mathscr{L}_\Delta$ *iff* Δ *contains only nullary symbols.*

(ii) ψ_5 *in* $\mathscr{L}_\Delta$ *iff* Δ *is large.*

Proof. It follows e.g. from [4].

Definition. (i) $\psi_6(X) \equiv \forall Y(X \leqq Y \leftrightarrow \forall Z((\forall S, T \exists U((S < T \& T < Z) \rightarrow$
$\rightarrow (\psi_1(T, U) \& U \leqq Z))) \rightarrow Z \leqq Y))$.
(ii) $\psi_7(X) \equiv \exists Y(\psi_6(Y) \& (\omega_1(Y) \rightarrow \omega_0(X)) \& (\neg \omega_1(Y) \rightarrow X = Y))$.

1.3. Lemma. *Let Δ be a small type containing a (single) unary symbol F. Then:*
(i) *If $\Delta = \{F\}$ then $\psi_6(X)$ in $\mathscr{L}_\Delta$ iff $X = W_\Delta \times W_\Delta$.*
(ii) *If $\Delta \neq \{F\}$ then $\psi_6(X)$ in $\mathscr{L}_\Delta$ iff $X = B_\Delta$.*
(iii) *$\psi_7(X)$ in $\mathscr{L}_\Delta$ iff $X = B_\Delta$.*

Proof. Denote by M the set of equational theories $Z \in \mathscr{L}_\Delta$ with the following property: if $T \in \mathscr{L}_\Delta$ and $1_{W_\Delta} \neq T \subset Z$ then T is covered in $\mathscr{L}_\Delta$ by some $U \subseteq Z$. Evidently, $\psi_6(X)$ in $\mathscr{L}_\Delta$ iff X is the join of M in $\mathscr{L}_\Delta$. If either $\Delta = \{F\}$ or $\Delta = \{F, H\}$ for some nullary symbol H, then we have a nice description of the lattice $\mathscr{L}_\Delta$ (see Theorems 3 and 4 of [3]); from this description it follows that the join of M in $\mathscr{L}_\Delta$ equals $W_\Delta \times W_\Delta$ in the case $\Delta = \{F\}$ and B_Δ in the case $\Delta = \{F, H\}$. It remains to consider the case when Δ contains at least two different nullary symbols and to prove that the join of M equals B_Δ in this case.

Suppose that there exists a $Z \in M$ with $Z \nsubseteq B_\Delta$. Put $T = Z \cap B_\Delta$, so that $1_{W_\Delta} \neq T \subset Z$. Evidently, there is no cover U of T in $\mathscr{L}_\Delta$ such that $U \subseteq Z$. We get a contradiction with $Z \in M$. This proves $J \subseteq B_\Delta$.

For every triple H, K, n such that $H, K \in \Delta_0$, $H \neq K$ and $n \geqq 0$ denote by $A_{H,K,n}$ the equational theory generated by $(H, F^n K)$. It is evident that $A_{H,K,n}$ belongs to M. Since Δ contains at least two nullary symbols, B_Δ is generated by these theories $A_{H,K,n}$ and so $B_\Delta \subseteq J$.

Definition. (i) $\psi_8(X) \equiv \psi_2(X) \& \exists Y, Z_1, Z_2, Z_3(\psi_7(Y) \& Y \leqq Z_1 \& \psi_1(Z_1, X) \&$
$\& Y \leqq Z_2 \& \psi_1(Z_2, X) \& Y \leqq Z_3 \& \psi_1(Z_3, X) \& Z_1 \neq Z_2 \& Z_1 \neq Z_3 \& Z_2 \neq Z_3)$.
(ii) $\psi_9(X) \equiv \psi_2(X) \& \neg \psi_8(X)$.

1.4. Lemma. *Let Δ be a small type containing a (single) unary symbol F. Then:*
(i) *$\psi_8(X)$ in $\mathscr{L}_\Delta$ iff $X = E_\Delta$.*
(ii) *$\psi_9(X)$ in $\mathscr{L}_\Delta$ iff $X = C_\Delta$.*

Proof. Evidently, if Δ is a small type containing a single unary symbol, then E_Δ and C_Δ are the only two coatoms of $\mathscr{L}_\Delta$ and $E_\Delta \neq C_\Delta$. There are infinitely many equational theories Z such that $B_\Delta \subseteq Z$ and Z is covered by E_Δ; for example, for any prime number p the equational theory Z generated by $B_\Delta \cup \{(x, F^p x)\}$ (where $x \in V$) has the desired properties. On the other hand, there are exactly two equational theories Z such that $B_\Delta \subseteq Z$ and Z is covered by C_Δ, namely the following ones:
$E_\Delta \cap C_\Delta$;
$(U \times U) \cup 1_{W_\Delta}$ where $U = \{F^n x;\ n \geqq 2,\ x \in V\} \cup \{F^m c;\ m \geqq 0,\ c \in \Delta_0\}$.

Definition. (i) $\psi_{10}(X) \equiv \psi_2(X) \& \psi_3(X) \& \exists A, B\, \forall Y(\psi_3(Y) \rightarrow (Y \leqq X \text{ VEL } Y =$
$= A \text{ VEL } Y = B))$.
(ii) $\psi_{11}(X) \equiv \psi_2(X) \& \psi_3(X) \& \neg \psi_{10}(X)$.

1.5. Lemma. *Let Δ be a large type. Then:*
(i) $\psi_{10}(X)$ *in* $\mathscr{L}_\Delta$ *iff* $X = C_\Delta$.
(ii) $\psi_{11}(X)$ *in* $\mathscr{L}_\Delta$ *iff* $X = E_\Delta$.

Proof. As we know, C_Δ and E_Δ are coatoms of $\mathscr{L}_\Delta$. It follows from Theorem 5.1 of [1] that C_Δ, E_Δ are modular elements of $\mathscr{L}_\Delta$ and if Y is a modular element of $\mathscr{L}_\Delta$ such that $Y \nsubseteq C_\Delta$ then either $Y = W_\Delta \times W_\Delta$ or $Y = E_\Delta$. On the other hand, it is evident that there are infinitely many modular elements of $\mathscr{L}_\Delta$ that are not contained in E_Δ.

Definition. (i) $\psi_{12}(X) \equiv \psi_2(X) \,\&\, ((\neg\psi_4 \,\&\, \neg\psi_5) \rightarrow \psi_9(X)) \,\&\, (\psi_5 \rightarrow \psi_{10}(X))$.
(ii) $\psi_{13}(X) \equiv \psi_2(X) \,\&\, ((\neg\psi_4 \,\&\, \neg\psi_5) \rightarrow \psi_8(X)) \,\&\, (\psi_5 \rightarrow \psi_{11}(X))$.
(iii) $\bar{\psi}_{13}(X) \equiv \exists Y(\psi_{13}(Y) \,\&\, X \leqq Y)$.

1.6. Lemma. *Let Δ be any type. Then:*
(i) $\psi_{12}(X)$ *in* $\mathscr{L}_\Delta$ *iff* $X = C_\Delta$.
(ii) $\psi_{13}(X)$ *in* $\mathscr{L}_\Delta$ *iff* $X = E_\Delta$.
(iii) $\bar{\psi}_{13}(X)$ *in* $\mathscr{L}_\Delta$ *iff* $X \subseteq E_\Delta$.

2. DEFINABILITY OF THE SET OF EDZ-THEORIES

An equational theory T is said to be an EDZ-theory if at most one block of T is of cardinality $\geqq 2$. The set of EDZ-theories of type Δ will be denoted by $\mathscr{Z}_\Delta$.

Recall that $\mathscr{F}_\Delta$ denotes the lattice of full subsets of W_Δ. For every $U \in \mathscr{F}_\Delta$ put $\mathrm{Z}(U) = (U \times U) \cup 1_{W_\Delta}$. Evidently, T is an EDZ-theory of type Δ iff $T = \mathrm{Z}(U)$ for some $U \in \mathscr{F}_\Delta$.

For every subset U of W_Δ put $\mathrm{Z}(U) = \mathrm{Z}(U^*)$; for every term t put $\mathrm{Z}(t) = \mathrm{Z}(\{t\})$.

2.1. Proposition. *$\mathscr{Z}_\Delta$ is a complete lattice with respect to inclusion. If Δ contains not only nullary symbols then $U \mapsto \mathrm{Z}(U)$ is an isomorphism of $\mathscr{F}_\Delta$ onto $\mathscr{Z}_\Delta$ (and consequently $\mathscr{Z}_\Delta$ is distributive). If either Δ contains no nullary symbols or Δ is strictly large then $\mathscr{Z}_\Delta$ is a complete sublattice of $\mathscr{L}_\Delta$.*

2.2. Lemma. *Let Δ contain only nullary symbols and let $T \in \mathscr{L}_\Delta$. Then T is an EDZ-theory iff T is a modular element of $\mathscr{L}_\Delta$.*

Proof. It follows from Theorem 4.1 of [1].

Definition. $\psi_{14}(X) \equiv \forall Y(Y \leqq X \leftrightarrow \forall Z, U((\psi_8(U) \,\&\, \psi_3(Z) \,\&\, X \leqq$
$\leqq Z \,\&\, \neg Z \leqq U) \rightarrow Y \leqq Z))$.

2.3. Lemma. *Let Δ be a small type containing a (single) unary symbol F. Then $\psi_{14}(X)$ in $\mathscr{L}_\Delta$ iff X is an EDZ-theory.*

Proof. For every $X \in \mathscr{L}_\Delta$ denote by M_X the set of modular elements $Z \in \mathscr{L}_\Delta$ such that $X \subseteq Z$ and $Z \nsubseteq E_\Delta$. Evidently, $\psi_{14}(X)$ in $\mathscr{L}_\Delta$ iff X is the intersection of M_X.

It follows from Theorem 4.2 of [1] that if Z is a modular element of $\mathscr{L}_\Delta$ and $Z \nsubseteq E_\Delta$ then Z is an EDZ-theory. Since the intersection of any system of EDZ-theories is an EDZ-theory, it follows that if X is the intersection of M_X then X is an EDZ-theory. Conversely, let X be an EDZ-theory, $X = \mathrm{Z}(U)$. For every $n \geqq 0$ put $U_n = U \cup \{F^i y;\ i \geqq n,\ y \in V \cup \Delta_0$, so that $\mathrm{Z}(U_n) \in M_X$. Evidently, X is the intersection of the equational theories $\mathrm{Z}(U_n)$ $(n = 0, 1, 2, \ldots)$ and hence X is the intersection of M_X.

For every symbol $F \in \Delta$ of arity $n \geqq 1$ and every term t define terms $F^0(t)$, $F^1(t)$, $F^2(t), \ldots$ as follows: $F^0(t) = t$; $F^{k+1}(t) = F(F^k(t), \ldots, F^k(t))$.

2.4. Lemma. *Let $F \in \Delta$ be a symbol of arity $n \geqq 1$; let $k \geqq 1$, $x \in V$ and let u, v be two terms such that $u \neq v$, $F^k(x) \nleqq u$, $F^k(x) \nleqq v$. Then (u, v) is not a consequence of $(x, F^k(x))$.*

Proof. Denote by A the set of all the terms t such that $F^k(x) \nleqq t$. For every symbol $G \in \Delta$ of an arbitrary arity m define an m-ary operation f_G on A as follows: if $G = F$ and $t_1 = \ldots = t_m = F^{k-1}(t)$ for some term t, put $f_G(t_1, \ldots, t_m) = t$; in all other cases put $f_G(t_1, \ldots, t_m) = G(t_1, \ldots, t_m)$. It is easy to see that the algebra with the underlying set A and with the operations f_G satisfies $(x, F^k x)$ but does not satisfy (u, v).

Definition. $\psi_{15}(X) \equiv \psi_3(X) \,\&\, \exists A, B, C(\psi_{11}(B) \,\&\, \psi_{10}(C) \,\&\, X \leqq B \,\&\, \psi_3(A) \,\&$
$\& \, A < X \,\&\, \forall Y((\psi_3(Y) \,\&\, Y < X) \rightarrow Y \leqq A) \,\&\, \forall L \exists M(M = X \wedge L \,\&$
$\& \, (\neg L \leqq C \rightarrow X = A \vee M)))$.

2.5. Lemma. *Let Δ be a large type. Then $\psi_{15}(X)$ in $\mathscr{L}_\Delta$ iff there exists a term t such that $X = E_\Delta \cap \mathrm{Z}(t)$.*

Proof. Assume first that $X = E_\Delta \cap \mathrm{Z}(t)$ for some term t. By Theorem 5.1 of [1], X is a modular element of $\mathscr{L}_\Delta$; evidently, $X \subseteq E_\Delta$. Define an equational theory A as follows: $(u, v) \in A$ iff $\mathrm{var}(u) = \mathrm{var}(v)$ and either $u = v$ or $u, v > t$ or $u \sim v \sim t$. By Theorem 5.1 of [1], A is a modular element of $\mathscr{L}_\Delta$; evidently $A \subset X$. It follows from Theorem 5.1 of [1] that if Y is a modular element of $\mathscr{L}_\Delta$ and $Y \subset X$ then $Y \subseteq A$. In order to prove $\psi_{15}(X)$, it remains to show that if L is an equational theory and $L \nsubseteq C_\Delta$ then $X = A \vee (X \cap L)$. Evidently, $A \vee (X \cap L) \subseteq X$. Let $(u, v) \in X$. It remains to prove $(u, v) \in A \vee (X \cap L)$. This is evident if $(u, v) \in A$. Let $(u, v) \notin A$. We have either $u \sim t$, $v > t$ or $u > t$, $v \sim t$; it is enough to consider the case $u \sim t$, $v > t$. There is an automorphism p of W_Δ such that $p(t) = u$. Since $L \nsubseteq C_\Delta$, there exists an equation $(x, a) \in L$ such that $x \in V$, $a \notin V$ and $\mathrm{var}(a) = \{x\}$. We have $(t, \sigma_t^x(a)) \in X \cap L$, $(p(t), p\,\sigma_t^x(a)) \in X \cap L$, $(u, p\,\sigma_t^x(a)) \in X \cap L$; moreover, $(p\,\sigma_t^x(a), v) \in A$ and so $(u, v) \in A \vee (X \cap L)$.

Now assume that $\psi_{15}(X)$ is satisfied. There exists a modular element A of $\mathscr{L}_\Delta$ with the properties described in ψ_{15}.

Let us prove first that if $(a, b) \in X$ and $a \notin U_X$ (for the definition of U_X see Section 5 of [1]), then $(a, b) \in A$. Suppose, on the contrary, that $(a, b) \notin A$. Take an arbitrary symbol $F \in \Delta$ of arity $n \geqq 1$ and a variable x. Evidently, there exists an integer $k \geqq 1$ such that $F^k(x) \nleqq a$. Denote by L the equational theory generated by $(x, F^k(x))$; we have $L \nleqq C_\Delta$ and so $X = A \vee (X \cap L)$. Since $(a, b) \in X$, there exists an $A \cup \cup (X \cap L)$-proof $a_0, \ldots, a_m$ from a to b. For every $i \in \{0, \ldots, m\}$ we have $(a, a_i) \in X$ and so $a_i \sim a$. Hence if $i \in \{1, \ldots, m\}$ then it follows from 2.4 that either $a_{i-1} = a_i$ or $(a_{i-1}, a_i) \notin L$; consequently, $(a_{i-1}, a_i) \in A$. We get $(a, b) = (a_0, a_m) \in A$, a contradiction.

Suppose that there exists an equation $(c, d) \in X$ such that $c \neq d$ and $c \notin U_X$. Define an equational theory Y as follows: $(u, v) \in Y$ iff $(u, v) \in X$ and either $u = v$ or $u \nleqq c$, $v \nleqq c$. It follows from Theorem 5.1 of [1] that Y is a modular element of $\mathscr{L}_\Delta$. Moreover, we have $Y \subset X$; since $\psi_{15}(X)$ is satisfied, we get $Y \subseteq A$. Since $A \subset X$, there exists an equation $(a, b) \in X \setminus A$; as we have proved above, $a \in U_X$ and $b \in U_X$. Hence $a \nleqq c$ and $b \nleqq c$; we get $(a, b) \in Y$ by the definition of Y. However, this is a contradiction, since $Y \subseteq A$ and $(a, b) \notin A$.

By Theorem 5.1 of [1], it follows that $X = ((U_X \times U_X) \cup 1_{W_\Delta}) \cap E_\Delta$. Evidently, U_X is non-empty. The set U_X contains a minimal element t. It follows from $\forall Y((\psi_3(Y) \,\&\, Y < X) \rightarrow Y \leqq A)$ and $A < X$ that $X = E_\Delta \cap Z(t)$.

Definition $\psi_{16}(X) \equiv \exists Y, Z(\psi_{11}(Y) \,\&\, Z = X \wedge Y \,\&\, \psi_{15}(Z) \,\&$
$\& \,\forall U(Z = U \wedge Y \rightarrow U \leqq X))$.

2.6. Lemma. *Let Δ be a large type. Then $\psi_{16}(X)$ in $\mathscr{L}_\Delta$ iff $X = Z(t)$ for some term t.*

Proof. It follows from 2.5; evidently, if t is a term then $Z(t)$ is just the greatest equational theory X with the property $X \cap E_\Delta = Z(t) \cap E_\Delta$.

Definition. $\psi_{17}(X) \equiv \psi_3(X) \,\&\, \forall A((\psi_3(A) \,\&\, \forall B((\psi_{16}(B) \,\&\, B \leqq X) \rightarrow B \leqq A)) \rightarrow$
$\rightarrow X \leqq A)$.

2.7. Lemma. *Let Δ be a large type. Then $\psi_{17}(X)$ in $\mathscr{L}_\Delta$ iff X is an* EDZ-*theory.*

Proof. It follows from 2.6 and from Theorem 5.1 of [1].

Definition. $\varepsilon(X) \equiv (\psi_4 \,\&\, \psi_3(X))$ VEL $(\neg\psi_4 \,\&\, \neg\psi_5 \,\&\, \psi_{14}(X))$ VEL $(\psi_5 \,\&\, \psi_{17}(X))$.
As a consequence of 2.2, 2.3 and 2.7, we get:

2.8. Theorem. *Let Δ be an arbitrary type. Then $\varepsilon(X)$ in $\mathscr{L}_\Delta$ iff X is an* EDZ-*theory. Consequently, the set of* EDZ-*theories of type Δ is definable in $\mathscr{L}_\Delta$.*

Recall that by a formula we mean a first-order formula in the language of lattice theory. For every formula f define a formula f^0 by induction on the length of f as follows:

(1) if f is a formula without quantifiers then f^0 is the same formula as f;

(2) if f is the formula $\neg g$ (the formula $g \,\&\, h$, g VEL h, $g \to h$, $g \leftrightarrow h$, resp.) then f^0 is the formula $\neg g^0$ (the formula $g^0 \,\&\, h^0$, g^0 VEL h^0, $g^0 \to h^0$, $g^0 \leftrightarrow h^0$, resp.);
(3) if f is the formula $\forall X\, g$ then f^0 is the formula $\forall X(\varepsilon(X) \to g^0)$;
(4) if f is the formula $\exists X\, g$ then f^0 is the formula $\exists X(\varepsilon(X) \,\&\, g^0)$.

Now let $f(X_1, \ldots, X_n)$ be a formula where $X_1, \ldots, X_n$ are all the free variables in f. Then the formula $\varepsilon(X_1) \,\&\, \ldots \,\&\, \varepsilon(X_n) \,\&\, f^0(X_1, \ldots, X_n)$ will be denoted by $f^\varepsilon(X_1, \ldots \ldots, X_n)$.

2.9. Lemma. *Let Δ contain not only nullary symbols, so that $U \mapsto \mathrm{Z}(U)$ is an isomorphism of $\mathscr{F}_\Delta$ onto $\mathscr{Z}_\Delta$. Let $f(X_1, \ldots, X_n)$ be a formula and $X_1, \ldots, X_n$ be all its free variables. Then $f^\varepsilon(X_1, \ldots, X_n)$ in $\mathscr{L}_\Delta$ iff there are sets $U_1, \ldots, U_n \in \mathscr{F}_\Delta$ such that $X_1 = \mathrm{Z}(U_1), \ldots, X_n = \mathrm{Z}(U_n)$ and $f(U_1, \ldots, U_n)$ holds in $\mathscr{F}_\Delta$.*

3. PARALLEL EQUATIONS

By a parallel equation we shall mean an equation (a, b) such that $\mathrm{var}\,(a) = \mathrm{var}\,(b)$, $a \nleq b$ and $b \nleq a$.

3.1. Proposition. *Let Δ contain not only nullary symbols. Let (a, b) be a parallel equation and denote by A the least* EDZ*-theory containing (a, b). Let $T \in \mathscr{L}_\Delta$. Then $T = \mathrm{Cn}(a, \bar{p}(b))$ for some permutation p of* $\mathrm{var}(a)$ *iff the following three conditions are satisfied:*
(1) *$T \subseteq E_\Delta \cap A$;*
(2) *A is just the least* EDZ*-theory containing T;*
(3) *if $B \in \mathscr{L}_\Delta$ and $B \subset T$ then there exists an* EDZ*-theory $C \supseteq B$ such that $\mathrm{Z}(a) \nsubseteq C$ and $\mathrm{Z}(b) \nsubseteq C$.*

Proof. First assume that $T = \mathrm{Cn}(a, \bar{p}(b))$. (1) and (2) are evident. Let $B \in \mathscr{L}_\Delta$ and $B \subset T$. Evidently, $\{a, \bar{p}(b)\}$ is a block of T. Hence if $(a, u) \in B$ for some u then $u = a$; if $(\bar{p}(b), u) \in B$ for some u then $u = \bar{p}(b)$. Denote by U the set of the terms u such that $v \leqq u$ for some $(v, w) \in B$ with $v \neq w$. Evidently U is a full set, $\mathrm{Z}(U)$ is an EDZ-theory, $\mathrm{Z}(U) \supseteq B$, $\mathrm{Z}(a) \nsubseteq \mathrm{Z}(U)$ and $\mathrm{Z}(b) \nsubseteq \mathrm{Z}(U)$.

Conversely, let T satisfy (1), (2), (3). Denote by U the set of the terms u such that $v \leqq u$ for some $(v, w) \in T$ with $v \neq w$. Evidently, U is a full set and $\mathrm{Z}(U)$ is just the least EDZ-theory containing T. By (2), $\mathrm{Z}(U) = A$. Hence $a \in U$; by (1) there is a term $c \neq a$ with $(a, c) \in T$. Put $B = \mathrm{Cn}(a, c)$. We have $B \subseteq T$ and by (3) we can not have $B \subset T$. Hence $B = T$. By (2), A is just the least EDZ-theory containing (a, c). Hence $c \sim b$; since $T \subseteq E_\Delta$, $c = \bar{p}(b)$ for some permutation p of $\mathrm{var}(b) = \mathrm{var}(a)$.

3.2. Proposition. *Let Δ be a large type. Let (a, b) be a parallel equation and put $C = \mathrm{Cn}(a, b)$. Let $T \in \mathscr{L}_\Delta$. Then $T = C$ iff the following two conditions are satisfied:*
(1) *there is a permutation p of* $\mathrm{var}(a)$ *such that $T = \mathrm{Cn}(a, \bar{p}(b))$;*
(2) *whenever (c, d) is a parallel consequence of (a, b) then $(c, \bar{q}(d)) \in T$ for some permutation q of* $\mathrm{var}(c)$.

Proof. The direct implication is evident. Let (1), (2) be satisfied and let p be as in (1). It is enough to derive a contradiction from the following assumption: There exists a variable $x \in \text{var}(a)$ with $p(x) \neq x$.

Evidently, the set var(a) contains at least two elements. Consequently, Δ is a strictly large type. The rest of the proof of 3.2 will be divided into several lemmas.

3.3. Lemma. *Let* $(F, i) \in \Delta^{(2)}$, $n = n_F$ *and* $k \in \{1, \ldots, n\} \setminus \{i\}$. *Then at least one of the following two conditions is satisfied*:

(i) $a = F(y_1, \ldots, y_{i-1}, a_0, y_{i+1}, \ldots, y_n)$ *for some term* a_0 *and pairwise different variables* $y_1, \ldots, y_n$ *such that if* $l \in \{1, \ldots, n\} \setminus \{i, k\}$ *then* $y_l \notin \text{var}(a_0)$;

(ii) $b = F(y_1, \ldots, y_{i-1}, b_0, y_{i+1}, \ldots, y_n)$ *for some term* b_0 *and pairwise different variables* $y_1, \ldots, y_n$ *such that if* $l \in \{1, \ldots, n\} \setminus \{i, k\}$ *then* $y_l \notin \text{var}(b_0)$.

Proof. Suppose that neither (i) nor (ii) is satisfied. Put $z_k = x$ and let $z_1, \ldots, z_{i-1}, z_{i+1}, \ldots, z_n$ be pairwise different variables such that if $l \in \{1, \ldots, n\} \setminus \{i, k\}$ then $z_l \notin \text{var}(a)$. Put $c = F(z_1, \ldots, z_{i-1}, a, z_{i+1}, \ldots, z_n)$ and $d = F(z_1, \ldots, z_{i-1}, b, z_{i+1}, \ldots, z_n)$. Evidently, (c, d) is a consequence of (a, b) and (c, d) is a parallel equation; by (2) there exists a permutation q of var(c) such that $(c, \bar{q}(d)) \in T$. Let $u_0, \ldots, u_m$ be a minimal $(a, \bar{p}(b))$-proof from c to $\bar{q}(d)$. We have $u_0 = c$; since neither (i) nor (ii) is satisfied, there is no other possibility than $u_1 = F(z_1, \ldots, z_{i-1}, \bar{p}(b), z_{i+1}, \ldots, z_n)$; similarly, if $m \geqq 2$ than there is no other possibility than $u_2 = F(z_1, \ldots, z_{i-1}, a, z_{i+1}, \ldots, z_n) = c = u_0$. If $m \geqq 2$, we get a contradiction with the minimality of $u_0, \ldots, u_m$; hence $m = 1$ and so $\bar{q}(d) = F(z_1, \ldots, z_{i-1}, \bar{p}(b), z_{i+1}, \ldots, z_n)$. Hence $q(z_k) = z_k$ and $\bar{q}(b) = \bar{p}(b)$, so that $q(x) = x$ and $q(x) = p(x)$. We get $p(x) = x$, a contradiction.

3.4. Lemma. *Let* $F, G \in \Delta$, $F \neq G$, $a = F(a_1, \ldots, a_{n_F})$ *and* $b = G(b_1, \ldots, b_{n_G})$ *for some terms* $a_1, \ldots, a_{n_F}$, $b_1, \ldots, b_{n_G}$. *Then* $a_1, \ldots, a_{n_F}$ *are pairwise different variables*, $b_1, \ldots, b_{n_G}$ *are pairwise different variables and* $n_F = n_G \geqq 2$.

Proof. Since Δ is strictly large, there exists an at least binary symbol $H \in \Delta$. By 3.3, either $H = F$ or $H = G$. It is enough to consider the case $H = F$. Using the fact that Card(var(a)) $\geqq 2$, it follows easily from 3.3 that $a_1, \ldots, a_{n_F}$ are pairwise different variables.

Suppose $n_G = 1$. Then it follows from 3.3 that $\Delta = \Delta_0 \cup \Delta_1 \cup \{F\}$. Moreover, since $a \nleqq b$, F does not occur in b. Hence b contains no symbols of arities $\geqq 2$ and so Card(var(b)) $\leqq 1$, a contradiction with var(a) = var(b) and Card(var(a)) $\geqq 2$.

Since Card(var(b)) $\geqq 2$, we can not have $n_G = 0$. We have proved $n_G \geqq 2$. Similarly as above, this implies that $b_1, \ldots, b_{n_G}$ are pairwise different variables. Since var(a) = var(b), it follows that $n_F = n_G$.

3.5. Lemma. *Let* $F \in \Delta$ *and* $a = F(a_1, \ldots, a_{n_F})$ *for some terms* $a_1, \ldots, a_{n_F}$. *Then* $b = F(b_1, \ldots, b_{n_F})$ *for some terms* $b_1, \ldots, b_{n_F}$.

Proof. We have $b = G(b_1, \ldots, b_{n_G})$ for some $G \in \Delta$ and some terms $b_1, \ldots, b_{n_G}$.

Suppose $F \neq G$. By 3.4, $n_F = n_G \geqq 2$, $a_1, \ldots, a_{n_F}$ are pairwise different variables and $\{a_1, \ldots, a_{n_F}\} = \{b_1, \ldots, b_{n_F}\}$. Put $c = \sigma^x_{F(x,\ldots,x)}(a)$ and $d = \sigma^x_{F(x,\ldots,x)}(b)$. Then (c, d) is a parallel consequence of (a, b); by (2) we have $(c, \bar{q}(d)) \in T$ for some permutation q of $\text{var}(c) = \text{var}(a)$. There exists an $(a, \bar{p}(b))$-proof from c to $\bar{q}(d)$; evidently, every member of this proof belongs to $\{c, \sigma^x_{G(x,\ldots,x)}(a), \sigma^x_{F(x,\ldots,x)}\bar{p}(b), \sigma^x_{G(x,\ldots,x)}\bar{p}(b)\}$ and so $\bar{q}(d) = \sigma^x_{F(x,\ldots,x)}\bar{p}(b)$. But then $\bar{q}\sigma^x_{F(x,\ldots,x)}(b) = \sigma^x_{F(x,\ldots,x)}\bar{p}(b)$, $\bar{q}(F(x, \ldots, x)) = p(x)$, a contradiction.

3.6. Lemma. *There exists exactly one at least binary symbol $F \in \Delta$; we have $n_F = 2$; there are two variables y, z and two terms c, d not belonging to V such that either $a = F(c, y)$, $b = F(z, d)$ or $a = F(y, c)$, $b = F(d, z)$.*

Proof. It follows easily from 3.3 and 3.5.

We shall denote by F the only binary symbol from Δ and sometimes we shall write uv instead of $F(u, v)$. By 3.6 it is enough to consider the case when $a = cy$ and $b = zd$ for some variables, y, z and terms c, d not belonging to V.

3.7. Lemma. *Let $x \in \text{var}(a)$ and $p(x) \neq x$. Then:*
(i) *There exists a substitution f such that either $f(a) = ax$ or $f(a) = \bar{p}(b)\,x$.*
(ii) *There exists a substitution g such that either $g(\bar{p}(b)) = x\,\bar{p}(b)$ or $g(\bar{p}(b)) = xa$.*

Proof. It is enough to prove (i), since (ii) is similar. Suppose that there is no such a substitution f. Evidently, (ax, bx) is a parallel consequence of (a, b); by (2) there exists an $(a, \bar{p}(b))$-proof from ax to $\bar{q}(bx)$ for some permutation q of $\text{var}(a)$. Since (i) is not satisfied, evidently every member of this proof equals either ax or $\bar{p}(b)\,x$. Hence $\bar{q}(bx) = \bar{p}(b)\,x$. From this we get $p(x) = x$, a contradiction.

3.8. Lemma. *Let $x \in \text{var}(a)$ and $p(x) \neq x$; let f be a substitution such that $f(a) = ax$. Then $a = ((y_1y_2 \cdot y_3)\ldots)\,y_k$ for some $k \geqq 3$ and variables $y_1, \ldots, y_k$; either $y_1, \ldots, y_k$ are pairwise different or $k = 3$ and $y_2 \notin \{y_1, y_3\}$.*

Proof. It follows from $f(a) = ax$ that $a = ((y_1y_2 \cdot y_3)\ldots)\,y_k$ for some $k \geqq 3$ and variables $y_1, \ldots, y_k$. There exists a variable $y \in \text{var}(a)$ such that $y \neq x$ and $p(y) \neq y$. By 3.7, there exists a substitution g such that either $g(a) = ay$ or $g(a) = \bar{p}(b)\,y$. If $g(a) = \bar{p}(b)\,y$, then evidently $k = 3$ and $y_2 \notin \{y_1, y_3\}$. Let $g(a) = ay$. We have $f(y_1) = y_1y_2$, $f(y_2) = y_3, \ldots, f(y_{k-1}) = y_k$, $f(y_k) = x$ and $g(y_1) = y_1y_2$, $g(y_2) = y_3, \ldots, g(y_{k-1}) = y_k$, $g(y_k) = y$. From this it follows that $y_k \notin \{y_1, \ldots, y_{k-1}\}$. Since $y_i = y_j$ (where $i, j \in \{1, \ldots, k-1\}$) implies $y_{i+1} = y_{j+1}$, it follows that $y_1, \ldots, y_k$ are pairwise different.

3.9. Lemma. *Let $x \in \text{var}(a)$ and $p(x) \neq x$; let f be a substitution such that $f(a) = ax$. Then $a = y_1y_2 \cdot y_1$ for some variables y_1, y_2 with $y_1 \neq y_2$.*

Proof. Suppose that this is not true. By 3.8 we have $a = ((y_1y_2 \cdot y_3)\ldots)\,y_k$ for some $k \geqq 3$ and pairwise different variables $y_1, \ldots, y_k$. By 3.7 there exists a substitution g such that either $g(\bar{p}(b)) = x\,\bar{p}(b)$ or $g(\bar{p}(b)) = xa$.

Consider first the case $g(\bar{p}(b)) = x\ \bar{p}(b)$. By a lemma symmetrical to 3.8, we have $\bar{p}(b) = z_l(\ldots(z_3 \cdot z_2 z_1))$ for some $l \geqq 3$ and variables $z_1, \ldots, z_l$ such that either $z_1, \ldots, z_l$ are pairwise different or $l = 3$, $z_2 \notin \{z_1, z_3\}$. Since $\text{var}(a) = \text{var}(\bar{p}(b))$, it follows that $l = k$, $z_1, \ldots, z_k$ are pairwise different and $\{y_1, \ldots, y_k\} = \{z_1, \ldots, z_k\}$.

Now consider the case $g(\bar{p}(b)) = xa$. Then evidently $\bar{p}(b) = z_l(((z_1 z_2 \cdot z_3) \ldots) z_{l-1})$ for some l and variables $z_1, \ldots, z_l$; we have $3 \leqq l \leqq k + 1$ and $z_1, \ldots, z_l$ are pairwise different. There exists a variable $y \in \text{var}(a)$ such that $y \neq x$ and $p(y) \neq y$. By 3.7, there exists a substitution h such that either $h(\bar{p}(b)) = y\ \bar{p}(b)$ or $h(\bar{p}(b)) = ya$. In both these cases it is easy to see that $z_1, \ldots, z_l$ are pairwise different. Since $\text{var}(a) = \text{var}(\bar{p}(b))$, it follows that $l = k$.

We have proved: either $\bar{p}(b) = z_k(\ldots(z_3 \cdot z_2 z_1))$ or $\bar{p}(b) = z_k(((z_1 z_2 \cdot z_3) \ldots) z_{k-1})$ for some pairwise different variables $z_1, \ldots, z_k$ with $\{z_1, \ldots, z_k\} = \{y_1, \ldots, y_k\}$.

Denote by h the substitution such that $h(p(x)) = p(x)$ and h maps $V \setminus \{p(x)\}$ onto $\{x\}$. Evidently, $(h(a), h(b))$ is a parallel consequence of (a, b); by (2) there exists an $(a, \bar{p}(b))$-proof from $h(a)$ to $\bar{q}(h(b))$ for some permutation q of $\text{var}(h(a)) = \{x, p(x)\}$. Evidently, every member of this proof equals either $h(a)$ or $h(\bar{p}(b))$. Hence $\bar{q}\ h(b) = h\ \bar{p}(b)$. There exists a variable $y \in \text{var}(a) \setminus \{x, p(x)\}$; we get $q\ h(x) = h\ p(x)$ and $q\ h(y) = h\ p(y)$, i.e. $q(x) = p(x)$ and $q(x) = x$, so that $p(x) = x$, a contradiction.

3.10. Lemma. *For every $x \in \text{var}(a)$ such that $p(x) \neq x$ there exist two substitutions f, g such that $f(a) = \bar{p}(b)\, x$ and $g(\bar{p}(b)) = xa$.*

Proof. Suppose that this is not true. It follows easily from 3.7, 3.9 and a lemma symmetrical to 3.9 that there are two different variables y_1, y_2 such that $a = y_1 y_2 \cdot y_1$ and either $b = y_1 \cdot y_2 y_1$ or $b = y_2 \cdot y_1 y_2$.

Let $b = y_1 \cdot y_2 y_1$. Then $\bar{p}(b) = y_2 \cdot y_1 y_2$. Since $((y_1 \cdot y_2 y_2)\, y_1, y_1(y_2 y_2 \cdot y_1))$ is a parallel consequence of (a, b), by (2) either $((y_1 \cdot y_2 y_2)\, y_1, y_1(y_2 y_2 \cdot y_1))$ or $((y_1 \cdot y_2 y_2)\, y_1, y_2(y_1 y_1 \cdot y_2))$ is a consequence of $(y_1 y_2 \cdot y_1, y_2 \cdot y_1 y_2)$. However, this is impossible, since if t is a term such that $((y_1 \cdot y_2 y_2)\, y_1, t)$ is a consequence of $(y_1 y_2 \cdot y_1, y_2 \cdot y_1 y_2)$ then evidently either $t = (y_1 \cdot y_2 y_2)\, y_1$ or $t = (y_2 y_2) \cdot (y_1 \cdot y_2 y_2)$.

Let $b = y_2 \cdot y_1 y_2$. Then $\bar{p}(b) = y_1 \cdot y_2 y_1$. Since $(a y_2, b y_2)$ is a parallel consequence of (a, b), by (2) either $((y_1 y_2 \cdot y_1)\, y_2, (y_2 \cdot y_1 y_2)\, y_2)$ or $((y_1 y_2 \cdot y_1)\, y_2, (y_1 \cdot y_2 y_1)\, y_1)$ is a consequence of $(y_1 y_2 \cdot y_1, y_1 \cdot y_2 y_1)$; however, this is evidently impossible.

3.11. Lemma. *We have $p(x) = x$ for all $x \in \text{var}(a)$.*

Proof. Suppose $p(x) \neq x$ for some $x \in \text{var}(a)$. By 3.10, there are two substitutions f, g such that $f(a) = \bar{p}(b)\, x$ and $g(\bar{p}(b)) = xa$. For every finite sequence $t_1, \ldots, t_k$ of terms and every sequence $e_1, \ldots, e_{k-1}$ of numbers from $\{1, -1\}$ define a term $[t_1, e_1, t_2, e_2, \ldots, t_{k-1}, e_{k-1}, t_k]$ by induction on k as follows: if $k = 1$ then this term equals t_1; if $k \geqq 2$ and $e_1 = 1$ then this term equals $t_1 \cdot [t_2, e_2, \ldots, t_{k-1},$

$e_{k-1}, t_k]$; if $k \geqq 2$ and $e_1 = -1$ then this term equals $[t_2, e_2, \ldots, t_{k-1}, e_{k-1}, t_k] \,.\, t_1$. It is evident that $a = [y_1, -1, y_2, 1, y_3, -1, \ldots, y_{k-1}, (-1)^{k-1}, y_k]$ and $\bar{p}(b) = [z_1, 1, z_2, -1, z_3, 1, \ldots, z_{l-1}, (-1)^l, z_l]$ for some k, l and variables $y_1, \ldots, y_k, z_1, \ldots, z_l$; we have $k \geqq 3$, $l \geqq 3$ and $k - 1 \leqq l \leqq k + 1$. There exists a variable $y \in \text{var}(a)$ such that $y \neq x$ and $p(y) \neq y$. By 3.10, there are substitutions f_0, g_0 such that $f_0(a) = \bar{p}(b)\, y$ and $g_0(\bar{p}(b)) = ya$. Evidently,

$$f(y_1) = x\,, \quad f_0(y_1) = y\,, \quad g(z_1) = x\,, \quad g_0(z_1) = y\,,$$
$$f(y_2) = f_0(y_2) = z_1\,, \quad g(z_2) = g_0(z_2) = y_1\,,$$
$$\ldots$$
$$f(y_{k-1}) = f_0(y_{k-1}) = z_{k-2}\,, \quad g(z_{l-1}) = g_0(z_{l-1}) = y_{l-2}\,,$$
$$f(y_k) = f_0(y_k) = [z_{k-1}, \ldots, z_l]\,, \quad g(z_l) = g_0(z_l) = [y_{l-1}, \ldots, y_k]\,.$$

Since $x \neq y$, from these relations it follows that $y_1, \ldots, y_k$ are pairwise different and $z_1, \ldots, z_l$ are pairwise different. Now $\text{var}(a) = \text{var}(\bar{p}(b))$ implies $k = l$ and $\{y_1, \ldots, y_k\} = \{z_1, \ldots, z_l\}$. Now we can define a substitution h and finish the proof in the same way as in the proof of 3.9.

This completes the proof of 3.2.

4. STRICTLY LARGE TYPES, NICE EQUATIONS

An equation (a, b) is said to be nice if $\text{var}(a) = \text{var}(b)$, $a \notin V$, $b \notin V$ and there exists a pair $(F, i) \in \Delta^{(1)}$ with $n_F \geqq 2$ such that $a \notin t\begin{bmatrix} 1 \\ F, i \end{bmatrix}$ and $b \notin t\begin{bmatrix} 1 \\ F, i \end{bmatrix}$ for any term t.

4.1. Proposition. *Let (a, b) be a nice equation. Put $C = \text{Cn}(a, b)$. Then C is just the greatest element T of $\mathscr{L}_\Delta$ with the following two properties:*
(1) $T \subseteq E_\Delta$;
(2) if (u, v) is a parallel equation then $(u, v) \in T$ iff $(u, v) \in C$.

Proof. Evidently, the theory C has both these properties. Now let T be any element of $\mathscr{L}_\Delta$ with these two properties. Let $(c, d) \in T$ and $c \neq d$; we must prove $(c, d) \in C$. Let (F, i) be as above. Put $n = n_F$ and $m = 1 + \text{Max}\,(\lambda_0(a), \lambda_0(b), \lambda_0(c), \lambda_0(d))$. Let us fix a number $j \in \{1, \ldots, n\} \setminus \{i\}$. Evidently, there exist an integer $k \geqq 2$ and a mapping $(r, s) \mapsto z_{r,s}$ of the set $\{1, \ldots, km\} \times \{1, \ldots, n\}$ into V with the following two properties:
(i) $z_{r_1,s_1} = z_{r_2,s_2}$ iff either $(r_1, s_1) = (r_2, s_2)$ or $s_1 = s_2 = j$ and $\{r_1, r_2\} = \{(k-1)\,m, km\}$;
(ii) $\text{var}(c) \subseteq \{z_{m,j}, z_{2m,j}, \ldots, z_{(k-1)m,j}\}$.
We fix one such integer k and one such mapping $(r, s) \mapsto z_{r,s}$. For every term t and every $r \in \{0, 1, \ldots, km\}$ define a term $t^{(r)}$ as follows: $t^{(0)} = t$; if $r \geqq 1$ then $t^{(r)} = F(z_{r,1}, \ldots, z_{r,i-1}, t^{(r-1)}, z_{r,i+1}, \ldots, z_{r,n})$. Evidently, $\lambda_0(t^{(r)}) = \lambda_0(t) + r$.

Suppose $c^{(km)} \leqq d^{(km)}$. Then there is a substitution f such that $f(c^{(km)})$ is a subterm of $d^{(km)}$. Since $\lambda_0(c^{(km)}) \geqq km$ and $\lambda_0(d^{(0)}) < m$, we have $f(c^{(km)}) = d^{(r)}$ for some

$r > (k-1)m$; since $z_{(k-1)m,j} = z_{km,j}$, we get $r = km$, so that $f(c^{(km)}) = d^{(km)}$. Hence $f(c) = d$ and $f(z_{m,j}) = z_{m,j}$, $f(z_{2m,j}) = z_{2m,j}, \ldots, f(z_{km,j}) = z_{km,j}$; by (ii) we get $c = d$, a contradiction.

Similarly, we can not have $d^{(km)} \leqq c^{(km)}$. Thus $(c^{(km)}, d^{(km)})$ is a parallel equation; since it evidently belongs to T, by (2) it belongs to C. Let $u_0, \ldots, u_l$ be an (a, b)-proof from $c^{(km)}$ to $d^{(km)}$. Let us prove by induction on $p \in \{0, \ldots, l\}$ that $u_p = v_p^{(km)}$ for some term v_p with $(c, v_p) \in C$. For $p = 0$ it is evident. Let $p \geqq 1$ and let $u_{p-1} = v_{p-1}^{(km)}$ and $(c, v_{p-1}) \in C$. Since either (u_{p-1}, u_p) or (u_p, u_{p-1}) is an immediate consequence of (a, b), it is enough to show that if $f(a)$ is a subterm of u_{p-1} for some substitution f then $f(a)$ is a subterm of v_{p-1} and if $f(b)$ is a subterm of u_{p-1} then $f(b)$ is a subterm of v_{p-1}. We shall consider only the case when $f(a)$ is a subterm of u_{p-1}; the other case is quite analogous. Suppose that $f(a)$ is not a subterm of v_{p-1}. Then $f(a) = v_{p-1}^{(r)}$ for some $r \in \{1, \ldots, km\}$. Since $a \notin t\begin{bmatrix} 1 \\ F, i \end{bmatrix}$ for any term t, we get $v_{p-1}^{(r)} \notin t\begin{bmatrix} 1 \\ F, i \end{bmatrix}$ for any term t and so r is divisible by m; especially, $r \geqq m$. It follows from $f(a) = v_{p-1}^{(r)}$ and $\lambda_0(a) < m$ that

$$a = F(w_{r,1}, \ldots, w_{r,i-1}, F(w_{r-1,1}, \ldots, w_{r-1,i-1}, F(\ldots F(w_{r-q,1}, \ldots \\ \ldots, w_{r-q,n}) \ldots), w_{r-1,i+1}, \ldots, w_{r-1,n}), w_{r,i+1}, \ldots, w_{r,n})$$

for some $q < m$ and some variables $w_{r,1}, \ldots, w_{r-q,i-1}, w_{r-q,i}, w_{r-q,i+1}, \ldots, w_{r,n}$ such that $f(w_{r,1}) = z_{r,1}, \ldots, f(w_{r-q,i-1}) = z_{r-q,i-1}, f(w_{r-q,i}) = v_{p-1}^{(r-q)}, f(w_{r-q,i+1}) = z_{r-q,i+1}, \ldots, f(w_{r,n}) = z_{r,n}$. However, the variables $z_{r,1}, \ldots, z_{r-q,i-1}, z_{r-q,i+1}, \ldots, z_{r,n}$ are evidently pairwise different and different from $v_{p-1}^{(r-q)}$, so that the variables $w_{r,1}, \ldots, w_{r-q,i-1}, w_{r-q,i}, w_{r-q,i+1}, \ldots, w_{r,n}$ are pairwise different, too. This means $a \in w_{r-q,i}\begin{bmatrix} q \\ F, i \end{bmatrix}$, a contradiction with $a \notin t\begin{bmatrix} 1 \\ F, i \end{bmatrix}$ for any term t. The induction is thus finished. Especially, for $p = l$ we get $(c, d) \in C$.

4.2. Proposition. *Let Δ be a strictly large type and (a, b) be an equation of type Δ such that* $\text{var}(a) = \text{var}(b)$. *Then exactly one of the following five cases takes place:*
(1) *(a, b) is nice;*
(2) *either $a \in V$ or $b \in V$;*
(3) *$\Delta = \Delta_0 \cup \Delta_1 \cup \{F\}$ for some symbol F with $n_F = 2$ and either $a = F(x, c)$, $b = F(d, y)$ or $a = F(d, y)$, $b = F(x, c)$ for some variables x, y and terms $c, d \notin V$ such that $x \notin \text{var}(c)$ and $y \notin \text{var}(d)$;*
(4) *$\Delta = \Delta_0 \cup \Delta_1 \cup \{F\}$ for some symbol F with $n_F \geqq 2$, $a \notin V$, $b \notin V$ and either a or b equals $F(y_1, \ldots, y_{n_F})$ for some pairwise different variables $y_1, \ldots, y_{n_F}$;*
(5) *$\Delta = \Delta_0 \cup \Delta_1 \cup \{F, G\}$ for some different symbols F, G with $n_F = n_G \geqq 2$, $a = F(y_1, \ldots, y_n)$ for some pairwise different variables $y_1, \ldots, y_n$ (where $n = n_F$) and $b = G(z_1, \ldots, z_n)$ for some $z_1, \ldots, z_n$ with $\{y_1, \ldots, y_n\} = \{z_1, \ldots, z_n\}$.*
Moreover, in the last case the equation (a, b) is parallel.

Proof. It is easy.

5. STRICTLY LARGE TYPES, 1-SPECIAL EQUATIONS

An equation (a, b) is said to be 1-special if $\text{var}(a) = \text{var}(b)$, $a \neq b$ and either $a \in V$ or $b \in V$.

5.1. Proposition. *Let Δ be strictly large and let (a, b) be a 1-special equation. Then $\text{Cn}(a, b)$ is just the least element T of $\mathcal{L}_\Delta$ with the following two properties:*
(1) *the least* EDZ-*theory containing T equals $W_\Delta \times W_\Delta$;*
(2) *if (u, v) is a nice equation then $(u, v) \in T$ iff $(u, v) \in \text{Cn}(a, b)$.*

Proof. Evidently, the theory $\text{Cn}(a, b)$ has both these properties. Let T be any element of $\mathcal{L}_\Delta$ with these two properties. It is enough to consider the case when $a \in V$; put $x = a$. By (1) there exists a term c such that $\text{var}(c) = \{x\}$, $c \neq x$ and $(x, c) \in T$. The equation $(c, \sigma_b^x(c))$ is nice and belongs to C, so that $(c, \sigma_b^x(c)) \in T$ by (2). Further, we have $(\sigma_b^x(x), \sigma_b^x(c)) \in T$, i.e. $(b, \sigma_b^x(c)) \in T$ and so $(b, c) \in T$. Hence $(x, b) \in T$ and so $\text{Cn}(a, b) \subseteq T$.

6. STRICTLY LARGE TYPES, 2-SPECIAL EQUATIONS

An equation (a, b) of type Δ is said to be 2-special if $\Delta = \Delta_0 \cup \Delta_1 \cup \{F\}$ for some binary symbol F (we shall write uv instead of $F(u, v)$) and there are terms a_0, b_0 and variables x, y such that $a = xa_0$, $b = b_0y$, $a_0 \notin V$, $a < b$, $x \notin \text{var}(a_0)$, $y \notin \text{var}(b_0)$, $\text{var}(a) = \text{var}(b)$.

A 2-special equation (a, b) of type Δ such that Δ_1 is non-empty is said to be 21-special.

A 2-special equation $(a, b) = (xa_0, b_0y)$ such that $x = y$ is said to be 22-special.

6.1. Proposition. *Let (a, b) be a 21-special equation. Put $C = \text{Cn}(a, b)$. Then C is just the greatest element T of $\mathcal{L}_\Delta$ with the following two properties:*
(1) *$T \subseteq E_\Delta$;*
(2) *if (u, v) is a nice equation then $(u, v) \in T$ iff $(u, v) \in C$.*

Proof. Evidently, the theory C has these properties. Let T be any element of $\mathcal{L}_\Delta$ with these properties and let $(c, d) \in T$, $c \neq d$; we must prove $(c, d) \in C$. There exists a unary symbol $G \in \Delta$. Evidently, the equation (Gc, Gd) is nice and belongs to T, so that it belongs to C. But from $(Gc, Gd) \in C$ we get $(c, d) \in C$ quite easily.

6.2. Lemma. *Let $(a, b) = (xa_0, b_0y)$ be a 2-special equation and let Δ_1 be empty. Put $C = \text{Cn}(a, b)$. Let there exist a substitution I such that $(b, I(b)) \in C$ and the terms $b, I(b)$ are not similar. Let T be any element of $\mathcal{L}_\Delta$ such that whenever (u, v) is either a parallel or a nice equation then $(u, v) \in T$ iff $(u, v) \in C$. Then $(a, b) \in T$.*

Proof. Denote by M the free monoid over $\{1, 2\}$; the unit of M will be denoted by $\emptyset$. If $e = a_1 \dots a_n \in M$ where $a_i \in \{1, 2\}$ then n is called the length of e. If $e, f \in M$ and $f = eg$ ($f = ge$, resp.) for some $g \in M$, then e is said to be a beginning (an end, resp.) of f. If $f = g_1 e g_2$ for some $g_1, g_2 \in M$ then e is said to be a connected part of f. By an irreducible element of M we mean any element $e \in M$ such that $e \neq \emptyset$ and whenever $e = f^n$ for some $f \in M$ and $n \geqq 1$ then $n = 1$. The following three assertions can be proved easily:

(A1) Let $e, f \in M \setminus \{\emptyset\}$ and $ef = fe$. Then $e = g^n$ and $f = g^m$ for some $g \in M$ and some $n, m \geqq 1$.

(A2) Every element of $M \setminus \{\emptyset\}$ can be uniquely expressed in the form e^n for some irreducible element e of M and some $n \geqq 1$.

(A3) Let e be an irreducible element of M and $f, g \in M$ be such that $ee = feg$. Then either $f = \emptyset$ or $g = \emptyset$.

If $e = r_1 \ldots r_n \in M$ (where $r_i \in \{1, 2\}$) and if $t, t_1, \ldots, t_n$ are terms, then we define a term $[t, r_1, t_1, \ldots, r_n, t_n]$ as follows: if $n = 0$, this term equals t; if $r_n = 1$, it equals $t_n[t, r_1, t_1, \ldots, r_{n-1}, t_{n-1}]$; if $r_n = 2$, it equals $[t, r_1, t_1, \ldots, r_{n-1}, t_{n-1}]\, t_n$. For every $i \in \{0, \ldots, n\}$ put $[t, r_1, t_1, \ldots, r_n, t_n]_i = [t, r_1, t_1, \ldots, r_i, t_i]$.

The depth $\partial(t)$ of a term t is defined as follows: if $t \in V \cup \Delta_0$ then $\partial(t) = 0$; if $t = t_1 t_2$ then $\partial(t) = 1 + \operatorname{Max}(\partial(t_1), \partial(t_2))$.

Since $(b, I(b)) \in C$, we have $\operatorname{var}(I^k(a)) = \operatorname{var}(a)$ for all positive integers k; since $b, I(b)$ are not similar, there is a $z \in \operatorname{var}(a)$ with $\lambda(I(z)) \geqq 2$. Hence $\lambda(a) < \lambda(I(a)) < \lambda(I^2(a)) < \ldots$; for some $k \geqq 1$ we get $\lambda(I^k(a)) > \lambda(b)$ and so $I^k(a) \nleqq b$. We shall fix one positive integer k with this property and put $a' = I^k(a)$. Evidently $(a, a') \in C$ and $a' \nleqq b$.

By a 1-term (2-term, resp.) we shall mean a term of the form zt (of the form tz, resp.) where t is a term and $z \in V \setminus \operatorname{var}(t)$. If there exists a term t such that $(a, t) \in C$ and t is neither a 1-term nor a 2-term then both (a, t) and (t, b) are nice, so that $(a, t) \in T$ and $(t, b) \in T$, so that $(a, b) \in T$ and we are through. Hence it is enough to assume that there is no such term t. Especially, a' is a 1-term.

There exists a substitution H such that $H(a)$ is a subterm of b; we can write $b = [H(a), r_1, b_1, \ldots, r_n, b_n]$ for some $e = r_1 \ldots r_n \in M \setminus \{\emptyset\}$ and some terms $b_1, \ldots, b_n$; we have $r_n = 2$.

If (a', b) is parallel then $(a', b) \in T$, $(a, a') \in T$, $(a, b) \in T$ and we are through. It remains to consider the case $b \leqq a'$. Since a' is a 1-term, there is a substitution K_0 such that $K_0(b)$ is a proper subterm of a'. We can write $a' = [K_0(b), \bar{s}_1, \bar{c}_1, \ldots, \bar{s}_{m_0}, \bar{c}_{m_0}]$ for some $f = \bar{s}_1 \ldots \bar{s}_{m_0} \in M \setminus \{\emptyset\}$ and some terms $\bar{c}_1, \ldots, \bar{c}_{m_0}$; we have $\bar{s}_{m_0} = 1$. Put $K = K_0 H$, $m = n + m_0$, $s_1 \ldots s_m = ef$, $c_i = K_0(b_i)$ for $i \in \{1, \ldots, n\}$ and $c_{n+i} = \bar{c}_i$ for $i \in \{1, \ldots, m_0\}$. We have evidently $a' = [K(a), s_1, c_1, \ldots, s_m, c_m]$.

Now we shall prove two easy assertions.

(A4) If $l \geqq 1$ then efe is not a connected part of e^l. Suppose, on the contrary, that $e^l = g_1 efeg_2$ for some g_1, g_2. By (A2) we have $e = h^p$ for some irreducible $h \in M$ and some $p \geqq 1$. By (A3) it follows easily from $h^{pl} = g_1 h^p feg_2$ that $g_1 = h^q$ for some $q \geqq 0$. Similarly, g_2 is a power of h. From this it follows that $f = h^j$ for some j. But e ends with 2 and f ends with 1, a contradiction.

(A5) If $l \geqq 1$ and if $j \geqq 1$ is such that e^j is longer than $efefe$, then e^j is not a connected part of $(ef)^l$. Indeed, if e^j was a connected part of $(ef)^l$, then, since it is longer than $efefe$, evidently efe would be a connected part of e^j, a contradiction by (A4).

For every $p \geqq 1$ and $i \in \{1, \ldots, pn\}$ put

$$B^{(p,i)} = [H^p(a), r_1, H^{p-1}(b_1), \ldots, r_n, H^{p-1}(b_n), r_1, H^{p-2}(b_1), \ldots \\ \ldots, r_n, H^{p-2}(b_n), \ldots, r_1, b_1, \ldots, r_n, b_n]_i .$$

For every $p \geqq 1$ and $i \in \{1, \ldots, pm\}$ put

$$C^{(p,i)} = [K^p(a), s_1, K^{p-1}(c_1), \ldots, s_m, K^{p-1}(c_m), s_1, K^{p-2}(c_1), \ldots \\ \ldots, s_m, K^{p-2}(c_m), \ldots, s_1, c_1, \ldots, s_m, c_m]_i .$$

Put $B^{(p)} = B^{(p,pn)}$ and $C^{(p)} = C^{(p,pm)}$. Evidently $B^{(p)}$ are 2-terms and $C^{(p)}$ are 1-terms; we have $(a, B^{(p)}) \in C$ and $(a, C^{(p)}) \in C$.

Let us fix an integer L such that

$$L > m + \operatorname{Max}\left(\partial(H(z)), \partial(K(z))\right) \quad \text{for all} \quad z \in \operatorname{var}(a) .$$

It is easy to see that for any $p \geqq 1$,

$$pn \leqq \partial B^{(p)} \leqq \partial B^{(p+1)} < \partial B^{(p)} + L,$$

$$pm \leqq \partial C^{(p)} \leqq \partial C^{(p+1)} < \partial C^{(p)} + L.$$

It follows that there are positive integers p, q such that

(B1) $pn > L + \lambda$ where λ is the length of $eefefe$,

(B2) $qm > L + \lambda$,

(B3) $\partial B^{(p)} > \operatorname{Max}(\partial c_1, \ldots, \partial c_m, \partial K c_1, \ldots, \partial K c_m, \ldots, \partial K^{L+\lambda} c_1, \ldots, \partial K^{L+\lambda} c_m)$,

(B4) $\partial C^{(q)} > \operatorname{Max}(\partial b_1, \ldots, \partial b_n, \partial H b_1, \ldots, \partial H b_n, \ldots, \partial H^{L+\lambda} b_1, \ldots, \partial H^{L+\lambda} b_n)$,

(B5) $|\partial B^{(p)} - \partial C^{(q)}| < L$.

Let us fix such a pair p, q.

Suppose $B^{(p)} \leqq C^{(q)}$, so that $N(B^{(p)})$ is a subterm of $C^{(q)}$ for some substitution N. Since $\partial C^{(q)} < \partial B^{(p)} + L \leqq \partial N(B^{(p)}) + L$, $N(B^{(p)})$ can not be a subterm of $C^{(q,qm-L)}$. Hence one of the following two cases takes place.

Case 1. $N(B^{(p)})$ is a subterm of one of the terms $c_1, \ldots, c_m, Kc_1, \ldots, Kc_m, \ldots \ldots, K^{L-1}c_1, \ldots, K^{L-1}c_m$. This is a contradiction with (B3).

Case 2. $N(B^{(p)}) = C^{(q,i)}$ for some $i \in \{qm - L + 1, \ldots, qm\}$. Put $e^p = r_{1,1} \ldots \ldots r_{1,pn}$ and $(ef)^q = r_{2,1} \ldots r_{2,qm}$. Denote by g the greatest common end of e^p and $r_{2,1} \ldots r_{2,i}$; denote by j the length of g. If it were $j \geqq \lambda$, then evidently a power of e longer than $efefe$ would be a connected part of $(ef)^q$, a contradiction with (A5). Hence $j < \lambda$. Evidently $N(B^{(p,pn-j)}) = C^{(q,i-j)}$ and $N(B^{(p,pn-j-1)}) = J$ where $C^{(q,i-j)} = [C^{(q,i-j-1)}, s_{i-j}, J]$; but $J \in \{c_1, \ldots, c_m, Kc_1, \ldots, Kc_m, \ldots, K^{L+\lambda}c_1, \ldots \ldots, K^{L+\lambda}c_m\}$ and we get a contradiction by (B3).

Similarly, we can not have $C^{(q)} \leqq B^{(p)}$. Thus $(B^{(p)}, C^{(q)})$ is a parallel equation; since it belongs to C, it follows that it belongs to T. We have evidently $(b, B^{(p)}) \in T$ and $(a, C^{(q)}) \in T$; hence $(a, b) \in T$.

6.3. Proposition. *Let $(a, b) = (xa_0, b_0x)$ be a 22-special equation and let Δ_1 be empty. Put $C = \mathrm{Cn}(a, b)$. Then C is just the least element T of $\mathscr{L}_\Delta$ with the following three properties:*
(1) $T \subseteq E_\Delta$;
(2) *there is a term c such that $a < c$ and $(a, c) \in T$;*
(3) *if (u, v) is either parallel or nice then $(u, v) \in T$ iff $(u, v) \in C$.*

Proof. Evidently, C has these properties. Let T be any element of $\mathscr{L}_\Delta$ with these three properties. We must prove $(a, b) \in T$. Let z be any element of $\Delta_0 \cup \mathrm{var}(a_0)$. The equation $(za_0, \sigma_z^x(c))$ is nice and belongs to T, so it belongs to C. Let $u_0, \ldots, u_k$ be an (a, b)-proof from za_0 to $\sigma_z^x(c)$. It is easy to prove by induction on $i \in \{0, \ldots, k\}$ that there exists a term t_i such that $x \notin \mathrm{var}(t_i)$ and either $(a, xt_i) \in C$, $u_i = zt_i$ or $(a, t_ix) \in C$, $u_i = t_iz$. Especially, for $i = k$ we get: there is a term t such that $x \notin$ $\notin \mathrm{var}(t)$ and either $(a, xt) \in C$, $\sigma_z^x(c) = zt$ or $(a, tx) \in C$, $\sigma_z^x(c) = tz$. From this it follows that there are only two possibilities: either (a, c) is nice or $c = xt$ in the case $\sigma_z^x(c) = zt$ and $c = tx$ in the case $\sigma_z^x(c) = tz$. In any case we get $(a, c) \in C$. If there is a substitution I such that $(b, I(b)) \in C$ and the terms $b, I(b)$ are not similar, then $(a, b) \in T$ follows from 6.2. Now let there be no such I. Then it is easy to see that $c = ux$ for some term u, the equation (b, c) is nice and belongs to C, hence $(b, c) \in T$ and so $(a, b) \in T$.

6.4. Proposition. *Let $(a, b) = (xa_0, b_0y)$ be a 2-special equation and let Δ_1 be empty. Put $C = \mathrm{Cn}(a, b)$. Then C is just the least element T of $\mathscr{L}_\Delta$ with the following three properties:*
(1) $T \subseteq E_\Delta$;
(2) *there is a term c such that $a < c$ and $(a, c) \in T$;*
(3) *if (u, v) is either parallel or nice or 22-special then $(u, v) \in T$ iff $(u, v) \in C$.*

Proof. Evidently, C has these properties. Let T be any element of $\mathscr{L}_\Delta$ with these three properties. We must prove $(a, b) \in T$. If $x = y$, then (a, b) is 22-special and everything is clear. Let $x \neq y$. If there is a substitution I such that $(b, I(b)) \in C$ and the terms $b, I(b)$ are not similar, then $(a, b) \in T$ follows from 6.2. Now let there be no such I. Evidently, the equation (b, c) is nice. The equation $(\sigma_x^y(a), \sigma_x^y(c))$ is nice and belongs to T, so it belongs to C.

Let us prove that if $(a, t) \in C$ then either $a \sim t$ or $t = t_1y$ for some term t_1 with $y \notin \mathrm{var}(t_1)$. There exists a minimal (a, b)-proof $v_0, \ldots, v_l$ from a to t. If there is no $i \in \{1, \ldots, l\}$ such that $v_{i-1} = h(b)$ and $v_i = h(a)$ for some substitution h, then the assertion is evident. Now let there be such an i and denote by i the least positive integer with this property; since there is no substitution I as above, we have $b \sim h(b)$ and we can suppose that h is an automorphism of W_Δ. Suppose $i < l$. Then evidently

$v_{i+1} = h(b) = t_{i-1}$, a contradiction with the minimality of $v_0, \ldots, v_l$. Hence $i = l$, $t = h(a)$ and we get $a \sim t$.

Let $u_0, \ldots, u_k$ be an (a, b)-proof from $\sigma_x^y(a)$ to $\sigma_x^y(c)$. Let us prove by induction on $i \in \{0, \ldots, k\}$ that $u_i = \sigma_x^y(t)$ for some term t with $(a, t) \in C$. For $i = 0$ it is evident. Let $i < k$, $u_i = \sigma_x^y(t)$, $(a, t) \in C$. If $t \sim a$, then $t = p(a)$ for some automorphism p of W_Δ; we have $u_{i+1} = \sigma_x^y\, p(b)$ and $(a, p(b)) \in C$. As we have proved above, it remains to consider the case $t = t_1 y$ for some term t_1 with $y \notin \mathrm{var}(t_1)$. We have $u_i = t_1 x$. There is a substitution g such that either $g(a)$ or $g(b)$ is a subterm of $t_1 x$ and u_{i+1} is obtained from $t_1 x$ if this subterm is replaced by $g(b)$ in the first case and by $g(a)$ in the second case. If the subterm of $t_1 x$ is contained in t_1, then everything is evident. It remains the case when $t_1 x = g(b)$ and $u_{i+1} = g(a)$. Define a substitution h as follows: $h(y) = y$; $h(z) = g(z)$ for all $z \in V \setminus \{y\}$. Then $u_{i+1} = \sigma_x^y(h(a))$ and $t = h(b)$, so that $(a, h(a)) \in C$.

Especially, $\sigma_x^y(c) = \sigma_x^y(t)$ for some term t with $(a, t) \in C$. It follows that one of the following three cases takes place (recall from the proof of 6.2 the notions of 1-term and 2-term and notice that evidently c is not a 1-term):

(i) c is not a 2-term;

(ii) $c = t$;

(iii) c is obtained from t by the transposition $x \mapsto y \mapsto x$.

In the first two cases we get $(a, b) \in T$ immediately. In case (iii) the equation (a, c) is 22-special, so that $(a, c) \in C$ by (3); now we get $(a, b) \in T$ easily.

7. STRICTLY LARGE TYPES, 3-SPECIAL EQUATIONS

Recall that $V = \{x_1, x_2, x_3, \ldots\}$ is the set of variables. Throughout this section let x be a fixed variable (e.g. put $x = x_1$).

Let a type Δ be given. We denote by E the free monoid over $\Delta^{(1)}$; the elements of E can be identified with finite sequences of ele ments of $\Delta^{(1)}$ and the empty sequence is the unit of E. If $e, f \in E$ and $f = eg$ for some $g \in E$, then e is said to be a beginning of f. Let t be any term. In Section 6 of [2] we have defined a finite subset $E(t)$ of E; for every $e \in E(t)$ we have defined a subterm $t\langle e\rangle$ of t. For every $e \in E(t)$ and every term s define a term $\sigma_{e:s}(t)$ as follows:

(i) if e is empty then $\sigma_{e:s}(t) = s$;

(ii) if $e = (F, i) f$ for some $(F, i) \in \Delta^{(1)}$ and $f \in E$ then $t = F(t_1, \ldots, t_{n_F})$ for some terms $t_1, \ldots, t_{n_F}$ and we put $\sigma_{e:s}(t) = F(t_1, \ldots, t_{i-1}, \sigma_{f:s}(t_i), t_{i+1}, \ldots, t_{n_F})$.

Thus $\sigma_{e:s}(t)$ is the term obtained from t if "the e-th subterm is replaced by s".

For every $F \in \Delta$ we denote by $E_F(t)$ the set of all $e \in E(t)$ such that $t\langle e\rangle = F(t_1, \ldots, t_{n_F})$ for some terms $t_1, \ldots, t_{n_F}$.

A finite sequence $e_1, \ldots, e_k$ $(k \geqq 0)$ of elements of $E(t)$ is said to be independent if the following is true: whenever $i, j \in \{1, \ldots, k\}$ and e_i is a beginning of e_j then $i = j$.

An equation (a, b) of type Δ is said to be 3-special if $\Delta = \Delta_0 \cup \Delta_1 \cup \{F\}$ for some symbol F with $n_F \geqq 2$, $a = F(x_1, \ldots, x_{n_F})$, $\mathrm{var}(a) = \mathrm{var}(b)$ and $a \leqq b$.

An equation (a, b) of type Δ is said to be 31-special if it is 3-special and $b = \sigma_a^x(w)$ for some term w such that $\mathrm{var}(w) = \{x\}$. An equation (a, b) is said to be 32-special if it is 3-special and satisfies some 31-special equation (a, b') with $a \neq b'$. An equation (a, b) is said to be 33-special if it is 3-special but not 32-special.

7.1. Proposition. *Let (a, b) be 31-special. Put $C = \mathrm{Cn}(a, b)$. Then C is just the greatest element T of $\mathscr{L}_\Delta$ with the following two properties:*

(1) $T \subseteq E_\Delta$;

(2) *if (u, v) is either nice or 2-special then $(u, v) \in T$ iff $(u, v) \in C$.*

Proof. Evidently, the theory C has both these properties. Let T be any element of $\mathscr{L}_\Delta$ with these two properties. Let F and w be as above; put $n = n_F$. It is enough to prove that if $(a, c) \in T$ for some term c then $(a, c) \in C$. Let $y_2, \ldots, y_n$ be pairwise different variables different from $x_1, \ldots, x_n$. The equation $(F(a, y_2, \ldots, y_n), F(c, y_2, \ldots \ldots, y_n))$ is nice and belongs to T, so that it belongs to C. Let $u_0, \ldots, u_k$ be an (a, b)-proof from $F(a, y_2, \ldots, y_n)$ to $F(c, y_2, \ldots, y_n)$.

Let us prove by induction on $i \in \{0, \ldots, k\}$ that if t is a term such that $\mathrm{var}(t) = \{x_1, \ldots, x_n\}$ and $F(t, y_2, \ldots, y_n)$ is a subterm of u_i then $(a, t) \in C$. For $i = 0$ it is evident. Let $i \geqq 1$. There is a substitution h and an $e \in E(u_{i-1})$ such that either $u_{i-1}\langle e\rangle = h(a)$ and $u_i = \sigma_{e:h(b)}(u_{i-1})$ or $u_{i-1}\langle e\rangle = h(b)$ and $u_i = \sigma_{e:h(a)}(u_{i-1})$. There is an $f \in E(u_i)$ such that $u_i\langle f\rangle = F(t, y_2, \ldots, y_n)$. If either e, f are independent or f is a proper beginning of e or $F(t, y_2, \ldots, y_n)$ is a subterm of one of the terms $h(x_1), \ldots, h(x_n)$, everything is evident from the induction assumption. If $f = e$ and $u_i\langle e\rangle = h(a)$, then it is evident, too, since $h(a)$ is a subterm of $h(b)$; indeed, (a, b) is 31-special and so a is a subterm of b. It remains to consider the case when $u_{i-1}\langle e\rangle = h(a)$, $u_i\langle e\rangle = h(b)$ and e is a beginning of f and $F(t, y_2, \ldots, y_n) = h(v)$ for some subterm $v \notin V$ of b. We have $v = F(v_1, \ldots, v_n)$ for some terms $v_1, \ldots, v_n$. Since $h(v_2) = y_2, \ldots, h(v_n) = y_n$, $v_2, \ldots, v_n$ are pairwise different variables. From this and from $b = \sigma_a^x(w)$ it follows easily that $v = F(x_1, \ldots, x_n) = a$. Now $h(a) = F(t, y_2, \ldots, y_n)$ and we can use the induction assumption.

Especially, for $i = k$ we get $(a, c) \in C$.

7.2. Proposition. *Let (a, b) be 32-special. Put $C = \mathrm{Cn}(a, b)$. Then C is just the least element T of $\mathscr{L}_\Delta$ with the following two properties:*

(1) $T \subseteq E_\Delta$;

(2) *if (u, v) is either nice or 31-special then $(u, v) \in T$ iff $(u, v) \in C$.*

Proof. It is easy.

The rest of this section is devoted to the proof of the following proposition.

7.3. Proposition. *Let (a, b) be a 33-special equation. Put $C = \mathrm{Cn}(a, b)$. Then C is just the greatest element T of $\mathscr{L}_\Delta$ with the following two properties:*

(1) $T \subseteq E_\Delta$;

(2) *if* (u, v) *is either nice or* 31-*special then* $(u, v) \in T$ *iff* $(u, v) \in C$.

Proof. Evidently, the theory C has both these properties. Let T be any element of $\mathscr{L}_\Delta$ with these two properties. Let $(c, d) \in T$, $c \neq d$. We must prove $(c, d) \in C$. Let F be as above and put $n = n_F$. If t is a term and $e_1, \ldots, e_k$ is an independent sequence of elements of $E_F(t)$, we put $S_{e_1,\ldots,e_k}(t) = \sigma_{e_1:h_1(b)} \ldots \sigma_{e_k:h_k(b)}(t)$ where $h_1, \ldots, h_k$ are substitutions such that $t\langle e_1\rangle = h_1(a), \ldots, t\langle e_k\rangle = h_k(a)$.

7.4. Lemma. *Let* t *be a term,* $e \in E_F(t)$ *and let* $e_1, \ldots, e_k$ *be an independent sequence of elements of* $E_F(t)$. *Put* $u = S_e(t)$ *and* $v = S_{e_1,\ldots,e_k}(t)$. *Then there exist an independent sequence* $f_1, \ldots, f_p$ *of elements of* $E_F(u)$ *and an independent sequence* $g_1, \ldots, g_q$ *of elements of* $E_F(v)$ *such that* $S_{f_1,\ldots,f_p}(u) = S_{g_1,\ldots,g_q}(v)$.

Proof. Consider first the case $e = e_i(F, j) f$ for some $i \in \{1, \ldots, k\}$, some $j \in \{1, \ldots, n\}$ and some f. Denote by $h_1, \ldots, h_m$ all the (pairwise different) elements $h \in E(b)$ such that $b\langle h\rangle = x_j$. We can put $(f_1, \ldots, f_p) = (e_1, \ldots, e_k)$ and $(g_1, \ldots, g_q) = (e_i h_1 f, \ldots, e_i h_m f)$.

Now consider the case $e = e_i$ for some $i \in \{1, \ldots, k\}$. Then we can put $(f_1, \ldots, f_p) = (e_1, \ldots, e_{i-1}, e_{i+1}, \ldots, e_k)$ and $(g_1, \ldots, g_q) = \emptyset$.

Finally, consider the remaining case. Then denote by $f_1, \ldots, f_p$ all the (pairwise different) elements $f \in E$ such that one of the following two cases takes place:

(i) $f \in \{e_1, \ldots, e_k\}$ and e is not a beginning of f;

(ii) there exist an $i \in \{1, \ldots, n\}$, an $l \in E$ with $e(F, i)\, l \in \{e_1, \ldots, e_k\}$ and an $h \in E(b)$ with $b\langle h\rangle = x_i$ such that $f = ehl$.

Further, put $(g_1, \ldots, g_q) = (e)$. It is easy to verify $S_{f_1,\ldots,f_p}(u) = S_{g_1,\ldots,g_q}(v)$.

By a direct (a, b)-proof we mean an (a, b)-proof $u_0, \ldots, u_k$ such that for every $i \in \{1, \ldots, k\}$, (u_{i-1}, u_i) is an immediate consequence of (a, b); i.e. for every $i \in \{1, \ldots, k\}$ there exists an $e \in E_F(u_{i-1})$ such that $u_i = S_e(u_{i-1})$.

7.5. Lemma. *Let* $u_0, \ldots, u_m$ *be a direct* (a, b)-*proof; let* u *be a term such that* $u = S_{e_1,\ldots,e_k}(u_0)$ *for some independent sequence* $e_1, \ldots, e_k$ *of elements of* $E_F(u_0)$. *Then there exists an independent sequence* $f_1, \ldots, f_p$ *of elements of* $E_F(u_m)$ *such that there is a direct* (a, b)-*proof from* u *to* $S_{f_1,\ldots,f_p}(u_m)$.

Proof. It follows from 7.4 by induction on m.

7.6. Lemma. *Let* $(u, v) \in C$. *Then there exist a term* w_0, *a direct* (a, b)-*proof from* u *to* w_0 *and a direct* (a, b)-*proof from* v *to* w_0.

Proof. It follows from 7.5 by induction on the length of an (a, b)-proof from u to v.

Let us fix pairwise different variables $y_1, \ldots, y_n$ different from $x_1, \ldots, x_n$ and not belonging to $\mathrm{var}(c)$.

For every term t define a set $\mathrm{SU}(t)$ of terms as follows: if $t = x_1$ then $u \in \mathrm{SU}(t)$ iff $(F(y_1, \ldots, y_n), u) \in C$; if $t \in V \setminus \{x_1\}$ then $\mathrm{SU}(t) = \{t\}$; if $t = G(t_1, \ldots, t_{n_G})$ for some $G \in \Delta$ and some terms $t_1, \ldots, t_{n_G}$ then $\mathrm{SU}(t) = \{G(u_1, \ldots, u_{n_G}); u_1 \in \mathrm{SU}(t_1), \ldots \ldots, u_{n_G} \in \mathrm{SU}(t_{n_G})\}$.

7.7. Lemma. *Let $u_0, \ldots, u_k$ be a direct (a, b)-proof and let there exist a term u such that $x_1 \in \mathrm{var}(u)$, $y_1, \ldots, y_n \notin \mathrm{var}(u)$ and $u_0 \in \mathrm{SU}(u)$. Then there exists a term v such that $(u, v) \in C$ and $u_k \in \mathrm{SU}(v)$.*

Proof. It is enough to consider the case $k = 1$, since the general case follows from this one by induction. There is an $e \in E_F(u_0)$ such that $u_1 = S_e(u_0)$. If there is an $f \in E(u)$ such that $u\langle f \rangle = x_1$ and f is a beginning of e, we can evidently put $v = u$. Consider the opposite case. Then $e \in E_F(u)$ and it is not much difficult to see that we can put $v = S_e(u)$.

Evidently, it is enough to assume that $x \in \mathrm{var}(c)$ and the equation $(\sigma^x_{F(y_1,\ldots,y_n)}(c), \sigma^x_{F(y_1,\ldots,y_n)}(d))$ is nice. This equation belongs to T and so it belongs to C. By 7.6 there exist a term w_0, a direct (a, b)-proof from $\sigma^x_{F(y_1,\ldots,y_n)}(c)$ to w_0 and a direct (a, b)-proof from $\sigma^x_{F(y_1,\ldots,y_n)}(d)$ to w_0. By 7.7, there are terms w_1, w_2 such that $(c, w_1) \in C$, $w_0 \in \mathrm{SU}(w_1)$, $(d, w_2) \in C$, $w_0 \in \mathrm{SU}(w_2)$.

7.8. Lemma. *Let t, u be terms such that* $\mathrm{var}(t) = \mathrm{var}(u)$, $y_1, \ldots, y_n \notin \mathrm{var}(t)$ *and* $\mathrm{SU}(t) \cap \mathrm{SU}(u)$ *is non-empty. Then $t = u$.*

Proof. By induction on $\lambda(t) + \lambda(u)$. First of all, let $t = x = x_1$ and $u \neq x$. There is a term $p \in \mathrm{SU}(t) \cap \mathrm{SU}(u)$. Since $p \in \mathrm{SU}(u)$, we have $(p, \sigma^x_{F(y_1,\ldots,y_n)}(u)) \in C$. Since $p \in \mathrm{SU}(t)$, we have $(p, F(y_1, \ldots, y_n)) \in C$. Hence $(\sigma^x_{F(y_1,\ldots,y_n)}(u), F(y_1, \ldots, y_n)) \in C$. From this $(a, \sigma^x_a(u)) \in C$; but $\mathrm{var}(u) = \{x\}$ and $u \neq x$, a contradiction with the fact that (a, b) is 33-special. If $u = x$, the proof is quite analogous. If either t or u belongs to $(V \setminus \{x\}) \cup \Delta_0$, everything is evident. Now let $t = G(t_1, \ldots, t_{n_G})$ and $u = H(u_1, \ldots, u_{n_H})$ for some $G, H \in \Delta$ and some terms $t_1, \ldots, t_{n_G}, u_1, \ldots, u_{n_H}$. Since $S(t) \cap S(u)$ is non-empty, we have $G = H$. Some term p belongs to $S(t) \cap S(u)$. We can write $p = G(p_1, \ldots, p_{n_G})$ for some $p_1, \ldots, p_{n_G}$; we have $p_1 \in \mathrm{SU}(t_1) \cap \mathrm{SU}(u_1), \ldots, p_{n_G} \in \mathrm{SU}(t_{n_G}) \cap \mathrm{SU}(u_{n_G})$. By induction, $t_1 = u_1, \ldots, t_{n_G} = u_{n_G}$ and so $t = u$.

By 7.8 we get $w_1 = w_2$. From this $(c, d) \in C$ follows immediately. This ends the proof of 7.3.

8. STRICTLY LARGE TYPES, THE FORMULAS

If $f(X, \ldots)$ is a formula and X is its free variable, then we define two new formulas ${}^X[f(X, \ldots)]$ and ${}_X[f(X, \ldots)]$ as follows:

$$ {}^X[f(X, \ldots)] \equiv f(X, \ldots) \;\&\; \forall X'(f(X', \ldots) \rightarrow X' \leqq X), $$

$$ {}_X[f(X, \ldots)] \equiv f(X, \ldots) \;\&\; \forall X'(f(X', \ldots) \rightarrow X \leqq X'). $$

Definition. $\varphi_{76}(X, Y, A, B) \equiv \varphi_{75}(X, Y, Y) \;\&\; \exists P, Q(P \prec Q \;\&\; \varphi_{60}(X, Y, P, A) \;\&\; \&\; \varphi_{60}(X, Y, Q, B))$.

8.1. Lemma. *Let Δ be a strictly large type. Then:*

(i) $\varphi_{75}(X, Y, Y)$ *in* $\mathscr{F}_\Delta$ *iff there are* $(F, i) \in \Delta^{(2)}$ *and terms* a, b *such that* $X = (F, i)^*$ *and* $Y = H_{F,i}(a, b)$.

(ii) $\varphi_{76}(X, Y, A, B)$ *in* $\mathscr{F}_\Delta$ *iff there are* $(F, i) \in \Delta^{(2)}$ *and terms* a, b *such that* $X = (F, i)^*$, $Y = H_{F,i}(a, b)$, $A = a^*$, $B = b^*$.

Definition. (i) $\varphi_{77}(X, Y) \equiv \exists A, B(\varphi_{76}(X, Y, A, B) \,\&\, \forall U, C(\varphi_{31}(Y, U) \rightarrow ((\varphi_{76}(X, U, A, C) \rightarrow B = C) \,\&\, (\varphi_{76}(X, U, C, B) \rightarrow A = C))))$.

(ii) $\varphi_{78}(X, Y) \equiv \varphi_{77}(X, Y) \,\&\, \exists A, B(\varphi_{76}(X, Y, A, B) \,\&\, \neg A \leqq B \,\&\, \neg B \leqq A)$.

(iii) $\varphi_{79}(X, Y) \equiv \varphi_{77}(X, Y) \,\&\, \exists A, B(\varphi_{76}(X, Y, A, B) \,\&\, (\omega_1(A) \text{ VEL } \omega_1(B)))$.

(iv) $\varphi_{80}(X, Y) \equiv \varphi_{77}(X, Y) \,\&\, \exists A, B, C, D, U, X_1, X_2(\alpha_2(U) \,\&\, \forall P(\alpha(P) \rightarrow (\alpha_0(P) \text{ VEL } \alpha_1(P) \text{ VEL } U = P)) \,\&\, U \prec X_1 \,\&\, U \prec X_2 \,\&\, X_1 \neq X_2 \,\&\, \varphi_{29}(X_1, C, A) \,\&\, \varphi_{29}(X_2, D, B) \,\&\, \neg\omega_1(C) \,\&\, \neg\omega_1(D) \,\&\, \varphi_{76}(X, Y, A, B))$.

(v) $\varphi_{81}(X, Y) \equiv \varphi_{77}(X, Y) \,\&\, \exists A, B, U(\varphi_{76}(X, Y, A, B) \,\&\, \neg\omega_1(A) \,\&\, \neg\omega_1(B) \,\&\, \bar{\alpha}_2(U) \,\&\, \forall P(\alpha(P) \rightarrow (\alpha_0(P) \text{ VEL } \alpha_1(P) \text{ VEL } U = P)) \,\&\, (A = U \text{ VEL } B = U))$.

(vi) $\varphi_{82}(X, Y) \equiv \varphi_{77}(X, Y) \,\&\, \exists A, B(\varphi_{76}(X, Y, A, B) \,\&\, \bar{\alpha}_2(A) \,\&\, \bar{\alpha}_2(B) \,\&\, A \neq B \,\&\, \forall U(\alpha(U) \rightarrow (\alpha_0(U) \text{ VEL } \alpha_1(U) \text{ VEL } U = A \text{ VEL } U = B)))$.

(vii) $\varphi_{83}(X, Y) \equiv \varphi_{77}(X, Y) \,\&\, \neg\varphi_{79}(X, Y) \,\&\, \neg\varphi_{80}(X, Y) \,\&\, \neg\varphi_{81}(X, Y) \,\&\, \neg\varphi_{82}(X, Y)$.

(viii) $\varphi_{84}(X, Y) \equiv \varphi_{80}(X, Y) \,\&\, \forall Z, A, B((\varphi_{76}(X, Z, A, B) \,\&\, \varphi_{31}(Y, Z)) \rightarrow (\exists C\, \varphi_{17}(C, A) \leftrightarrow \exists D\, \varphi_{17}(D, B)))$.

(ix) $\varphi_{85}(X, Y) \equiv \varphi_{81}(X, Y) \,\&\, \forall A, A_1, A_2, A_3, B_1, B_2, B_3, C((\varphi_{65}(X, A, A_1, A_2) \,\&\, \varphi_{65}(X, B, B_1, B_2) \,\&\, \varphi_{68}(X, Y, A_2, A_3, C) \,\&\, A_2 \neq A_3 \,\&\, \omega_1(C) \,\&\, \varphi_{68}(X, Y, B_2, B_3, C) \,\&\, B_2 \neq B_3) \rightarrow \exists U\, \varphi_{68}(X, Y, A, B, U))$.

(x) $\varphi_{86}(X, Y) \equiv \varphi_{81}(X, Y) \,\&\, \exists Z(\varphi_{85}(X, Z) \,\&\, \varphi_{75}(X, Y, Z) \,\&\, \neg\varphi_{72}(X, Z))$.

8.2. Lemma. *Let Δ be a strictly large type. Let* $j \in \{77, 78, \ldots, 86\}$. *Then* $\varphi_j(X, Y)$ *in* $\mathscr{F}_\Delta$ *iff there are* $(F, i) \in \Delta^{(2)}$ *and terms* a, b *such that* $X = (F, i)^*$, $Y = H_{F,i}(a, b)$ *and:*

(i) *if* $j = 77$ *then* $\text{var}(a) = \text{var}(b)$;

(ii) *if* $j = 78$ *then* (a, b) *is a parallel equation;*

(iii) *if* $j = 79$ *then* $\text{var}(a) = \text{var}(b)$ *and* (a, b) *is as in* 4.2(2);

(iv) *if* $j = 80$ *then* $\text{var}(a) = \text{var}(b)$ *and* (a, b) *is as in* 4.2(3);

(v) *if* $j = 81$ *then* $\text{var}(a) = \text{var}(b)$ *and* (a, b) *is as in* 4.2(4);

(vi) *if* $j = 82$ *then* $\text{var}(a) = \text{var}(b)$ *and* (a, b) *is as in* 4.2(5);

(vii) *if* $j = 83$ *then* (a, b) *is a nice equation;*

(viii) *if* $j = 84$ *then* $\text{var}(a) = \text{var}(b)$ *and* (a, b) *is as in* 4.2(3) *with* $x = y$;

(ix) *if* $j = 85$ *then* $\text{var}(a) = \text{var}(b)$, (a, b) *is as in* 4.2(4) *and there are a term* w *and a variable* x *such that* $\text{var}(w) = \{x\}$ *and either* $b = \sigma_a^x(w)$ *or* $a = \sigma_b^x(w)$;

(x) *if* $j = 86$ *then* $\mathrm{var}(a) = \mathrm{var}(b)$, (a, b) *is as in* 4.2(4) *and* (a, b) *has a non-trivial consequence as in the last case.*

Definition. (i) $\psi_{18}(X, Y, T) \equiv \exists A, B, C, D, E(\varphi^{\varepsilon}_{78}(X, Y) \& \varphi^{\varepsilon}_{76}(X, Y, A, B) \& C = A \vee B \& \psi_{13}(D) \& E = D \wedge C \& T \leqq E \& \forall U(\varepsilon(U) \rightarrow (T \leqq U \leftrightarrow C \leqq U)) \& \forall P \exists Q(P < T \rightarrow (\varepsilon(Q) \& P \leqq Q \& \neg A \leqq Q \& \neg B \leqq Q)))$.

(ii) $\psi_{19}(X, Y, T) \equiv \psi_{18}(X, Y, T) \& \forall Z \exists U((\varphi^{\varepsilon}_{75}(X, Y, Z) \& \varphi^{\varepsilon}_{78}(X, Z)) \rightarrow (\psi_{18}(X, Z, U) \& U \leqq T))$.

(iii) $\psi_{20}(X, Y, T) \equiv \varphi^{\varepsilon}_{83}(X, Y) \& {}^{T}[\bar{\psi}_{13}(T) \& \forall A, B(\psi_{19}(X, A, B) \rightarrow (B \leqq T \leftrightarrow \varphi^{\varepsilon}_{75}(X, Y, A)))]$.

(iv) $\psi_{21}(X, Y, T) \equiv \varphi^{\varepsilon}_{79}(X, Y) \& \neg\varphi^{\varepsilon}_{72}(X, Y) \& {}_{T}[\forall U((\varepsilon(U) \& T \leqq U) \rightarrow \omega_1(U)) \& \forall A, B(\psi_{20}(X, A, B) \rightarrow (B \leqq T \leftrightarrow \varphi^{\varepsilon}_{75}(X, Y, A)))]$.

(v) $\psi_{22}(X, Y, T) \equiv \varphi^{\varepsilon}_{80}(X, Y) \& \neg\varphi^{\varepsilon}_{78}(X, Y) \& \exists Q\, \alpha^{\varepsilon}_{1}(Q) \& {}^{T}[\bar{\psi}_{13}(T) \& \forall A, B(\psi_{20}(X, A, B) \rightarrow (B \leqq T \leftrightarrow \varphi^{\varepsilon}_{75}(X, Y, A)))]$.

(vi) $\psi_{23}(X, Y, T) \equiv \varphi^{\varepsilon}_{84}(X, Y) \& \neg\varphi^{\varepsilon}_{78}(X, Y) \& \neg\exists Q\, \alpha^{\varepsilon}_{1}(Q) \& {}_{T}[\bar{\psi}_{13}(T) \& \forall A, B((\psi_{19}(X, A, B)\ \mathsf{VEL}\ \psi_{20}(X, A, B)) \rightarrow (B \leqq T \leftrightarrow \varphi^{\varepsilon}_{75}(X, Y, A))) \& \exists C, D(\varphi^{\varepsilon}_{76}(X, Y, C, D) \& \forall E((\psi_3(E) \& T \leqq E) \rightarrow (C \leqq E \& D \leqq E)))]$.

(vii) $\psi_{24}(X, Y, T) \equiv \varphi^{\varepsilon}_{80}(X, Y) \& \neg\varphi^{\varepsilon}_{78}(X, Y) \& \neg\exists Q \alpha^{\varepsilon}_{1}(Q) \& {}_{T}[\bar{\psi}_{13}(T) \& \forall A, B((\psi_{19}(X, A, B)\ \mathsf{VEL}\ \psi_{20}(X, A, B)\ \mathsf{VEL}\ \psi_{23}(X, A, B)) \rightarrow (B \leqq T \leftrightarrow \varphi^{\varepsilon}_{75}(X, Y, A)) \& \exists C, D(\varphi^{\varepsilon}_{76}(X, Y, C, D) \& \forall E((\psi_3(E) \& T \leqq E) \rightarrow (C \leqq E \& D \leqq E)))]$.

(viii) $\psi_{25}(X, Y, T) \equiv \varphi^{\varepsilon}_{85}(X, Y) \& {}^{T}[\bar{\psi}_{13}(T) \& \forall A, B((\psi_{20}(X, A, B)\ \mathsf{VEL}\ \psi_{22}(X, A, B)\ \mathsf{VEL}\ \psi_{24}(X, A, B)) \rightarrow (B \leqq T \leftrightarrow \varphi^{\varepsilon}_{75}(X, Y, A)))]$.

(ix) $\psi_{26}(X, Y, T) \equiv \varphi^{\varepsilon}_{86}(X, Y) \& {}_{T}[\bar{\psi}_{13}(T) \& \forall A, B((\psi_{20}(X, A, B)\ \mathsf{VEL}\ \psi_{25}(X, A, B)) \rightarrow (B \leqq T \leftrightarrow \varphi^{\varepsilon}_{75}(X, Y, A)))]$.

(x) $\psi_{27}(X, Y, T) \equiv \varphi^{\varepsilon}_{81}(X, Y) \& \neg\varphi^{\varepsilon}_{86}(X, Y) \& {}^{T}[\bar{\psi}_{13}(T) \& \forall A, B((\psi_{20}(X, A, B)\ \mathsf{VEL}\ \psi_{25}(X, A, B)) \rightarrow (B \leqq T \leftrightarrow \varphi^{\varepsilon}_{75}(X, Y, A)))]$.

(xi) $\psi_{28}(X, Y, T) \equiv \psi_{19}(X, Y, T)\ \mathsf{VEL}\ \psi_{20}(X, Y, T)\ \mathsf{VEL}\ \psi_{21}(X, Y, T)\ \mathsf{VEL}\ \psi_{22}(X, Y, T)\ \mathsf{VEL}\ \psi_{24}(X, Y, T)\ \mathsf{VEL}\ \psi_{26}(X, Y, T)\ \mathsf{VEL}\ \psi_{27}(X, Y, T)\ \mathsf{VEL}\ (\varphi^{\varepsilon}_{72}(X, Y) \& \omega_0(T))$.

8.3. Lemma. *Let* Δ *be a strictly large type. Then* $\psi_{28}(X, Y, T)$ *in* $\mathscr{L}_{\Delta}$ *iff there are* $(F, i) \in \Delta^{(2)}$ *and two terms* a, b *such that* $\mathrm{var}(a) = \mathrm{var}(b)$, $X = \mathrm{Z}((F, i)^*)$, $Y = \mathrm{Z}(H_{F,i}(a, b))$ *and* $T = \mathrm{Cn}(a, b)$.

Proof. It is a formalization of Sections 3, 4, 5, 6 and 7.

& **Definition.** (i) $\psi_{29}(X, Y, T) \equiv \varphi^{\varepsilon}_{75}(X, Y, Y) \& \neg\varphi^{\varepsilon}_{77}(X, Y) \& \neg\bar{\psi}_{13}(T) \& \forall A, B(\psi_{28}(X, A, B) \rightarrow (B \leqq T \leftrightarrow \varphi^{\varepsilon}_{75}(X, Y, A)))$.

(ii) $\psi_{30}(X, Y, T) \equiv \psi_{28}(X, Y, T)\ \mathsf{VEL}\ \psi_{29}(X, Y, T)$.

(iii) $\psi_{31}(X) \equiv \exists A, B\, \psi_{30}(A, B, X)$.

(iv) $\psi_{32}(X) \equiv \exists A, B(\varphi^{\varepsilon}_{73}(A, B) \& \forall C(\varphi^{\varepsilon}_{74}(A, B, C) \leftrightarrow \exists D(\psi_{30}(A, C, D) \& D \leqq X)))$.

8.4. Lemma. *Let Δ be a strictly large type. Then*:

(i) $\psi_{29}(X, Y, T)$ *in* $\mathscr{L}_\Delta$ *iff there are* $(F, i) \in \Delta^{(2)}$ *and two terms* a, b *such that* $\mathrm{var}(a) \neq \mathrm{var}(b)$, $X = \mathrm{Z}((F, i)^*)$, $Y = \mathrm{Z}(H_{F,i}(a, b))$ *and* $T = \mathrm{Cn}(a, b)$.

(ii) $\psi_{30}(X, Y, T)$ *in* $\mathscr{L}_\Delta$ *iff there are* $(F, i) \in \Delta^{(2)}$ *and two terms* a, b *such that* $X = \mathrm{Z}((F, i)^*)$, $Y = \mathrm{Z}(H_{F,i}(a, b))$ *and* $T = \mathrm{Cn}(a, b)$.

(iii) $\psi_{31}(X)$ *in* $\mathscr{L}_\Delta$ *iff* X *is one-based.*

(iv) $\psi_{32}(X)$ *in* $\mathscr{L}_\Delta$ *iff* X *is finitely based.*

Proof. It is a well known and easy fact that if Δ is large then every equational theory which is not contained in E_Δ is uniquely determined by its intersection with E_Δ together with the fact that it is not contained in E_Δ. From this the assertion (i) follows. The rest is obvious.

8.5. Lemma. *Let Δ be strictly large and let h be an automorphism of* $\mathscr{L}_\Delta$. *Then* $h = Q_{c,f}$ *for some* $(c, f) \in H_\Delta$.

Proof. By 2.8, a restriction of h is an automorphism of the lattice of EDZ-theories of type Δ. Hence by Theorem 7.7 of [2] there exists a pair $(c, f) \in G_\Delta = H_\Delta$ such that $h(\mathrm{Z}(A)) = Q_{c,f}(\mathrm{Z}(A))$ for all $A \in \mathscr{F}_\Delta$. Using the formula ψ_{30} we see that $h(T) = Q_{c,f}(T)$ for every one-based equational theory T. Since any equational theory is the join of one-based theories, we get $h = Q_{c,f}$.

Let Δ be a strictly large type and let $(a_1, b_1), \ldots, (a_n, b_n)$ be a non-empty finite sequence of equations such that if $i \neq j$ then the sets $\mathrm{var}(a_i) \cup \mathrm{var}(b_i)$, $\mathrm{var}(a_j) \cup \mathrm{var}(b_j)$ are disjoint. Then we define a formula $\Theta_{\Delta,(a_1,b_1),\ldots,(a_n,b_n)}(X)$ as follows. Let us fix a pair $(F, i) \in \Delta^{(2)}$ and pairwise different variables $x, y_1, \ldots, y_n$ not belonging to $\mathrm{var}(a_1) \cup \mathrm{var}(b_1) \cup \ldots \cup \mathrm{var}(a_n) \cup \mathrm{var}(b_n)$. Put $t_1 = H_{F,i,y_1}(a_1, b_1), \ldots, t_n = H_{F,i,y_n}(a_n, b_n)$, $t = H_{F,i,x}(t_1, \ldots, t_n)$. Put

$$\Theta_{\Delta,(a_1,b_1),\ldots,(a_n,b_n)}(X) \equiv \exists Y(\vartheta_{\Delta,t}(Y)\ \& $$

$$\&\ \forall A, B, C(\psi_{30}(A, B, C) \rightarrow (C \leqq X \leftrightarrow \varphi_{74}^{\varepsilon}(A, Y, B)))) .$$

Now we get evidently

8.6. Lemma. *Let Δ be strictly large and let $(a_1, b_1), \ldots, (a_n, b_n)$ be as above. Then* $\Theta_{\Delta,(a_1,b_1),\ldots,(a_n,b_n)}(X)$ *in* $\mathscr{L}_\Delta$ *iff* $X = h(\mathrm{Cn}((a_1, b_1), \ldots, (a_n, b_n)))$ *for some automorphism h of* $\mathscr{L}_\Delta$.

9. LARGE BUT NOT STRICTLY LARGE TYPES

An equation (a, b) of type Δ is called 1-nice if Δ is large but not strictly large, $a < b$ and either a, b contain no variables or there are a variable x, a symbol $F \in \Delta_1$ and two integers n, m with $0 < n < m$ such that $a = F^n x$, $b = F^m x$.

9.1. Proposition. *Let* (a, b) *be a* 1-*nice equation. Put* $C = \mathrm{Cn}(a, b)$. *Then* C *is just the greatest element* T *of* $\mathscr{L}_\Delta$ *with the following three properties*:
(1) $T \subseteq E_\Delta$;
(2) *the least* EDZ-*theory containing* T *equals* $\mathrm{Z}(a)$;
(3) *if* (u, v) *is a parallel equation then* $(u, v) \in T$ *iff* $(u, v) \in C$.

Proof. Evidently, the theory C has all these three properties. Let T be any element with these three properties. Let $(c, d) \in T$. We must prove $(c, d) \in C$; it is enough to consider the case $c < d$. There are two sequences $p, q \in \Delta^{(-)}$ and an element $z \in V \cup \Delta_0$ such that $c = pz$ and $d = qz$.

Consider first the case when a, b contain no variables. By (2) we have $a \leqq c$ and so $z \in \Delta_0$, $(a, b) = (sz, tz)$ for some sequences $s, t \in \Delta^{(-)}$ and s is a beginning of p. There is a symbol $F \in \Delta_1$ such that t ends with F and a symbol $G \in \Delta_1 \setminus \{F\}$. Let $n > \mathrm{Max}\,(\lambda(b), \lambda(d))$. The equation $(FG^n pz, FG^n qz)$ is parallel and belongs to T, so that it belongs to C. Let $u_0, \ldots, u_k$ be an (a, b)-proof from $FG^n pz$ to $FG^n qz$. It is easy to verify by induction on $i \in \{0, \ldots, k\}$ that $u_i = FG^n rz$ for some r with $(pz, rz) \in C$. Especially, $(pz, qz) \in C$.

Now consider the case $(a, b) = (F^n x, F^m x)$. Let us fix a symbol $G \in \Delta_1 \setminus \{F\}$ and an integer $n > \mathrm{Max}\,(\lambda(b), \lambda(d))$.

Let $z \in \Delta_0$. If the equation $(G^n pz, G^n qz)$ is parallel then by (3) it belongs to C and so evidently $(pz, qz) \in C$. Suppose that it is not parallel, so that $q = G^k p$ for some $k \geqq 1$. Then the equation (Fpz, Fqz) is parallel, so that it belongs to C; but then the number of the G's in Fp equals the number of the G's in Fq, evidently a contradiction.

Let $z \in V$. The equation $(FG^n pG^n Fz, FG^n qG^n Fz)$ is parallel and belongs to T, so that it belongs to C; hence the number of the G's in $FG^n pG^n F$ equals the number of the G's in $FG^n qG^n F$; hence the number of the G's in p equals the number of the G's in q. Hence q consists not only of the G's. The equation (pqz, qpz) is either parallel or trivial and belongs to T, so that it belongs to C. From this it follows that $p = G^k p_0 G^l$ and $q = G^k q_0 G^l$ for some $k, l \geqq 0$ and sequences p_0, q_0 that neither begin nor end with G. Now it is clear that $(G^n pG^n z, G^n qG^n z)$ is a parallel equation, so that it belongs to C; but then evidently $(pz, qz) \in C$.

An equation (a, b) of type Δ is called 2-nice if Δ is large but not strictly large and there are a variable x, two non-empty sequences $s, t \in \Delta^{(-)}$ and two symbols $F, G \in \Delta_1$ such that $a = sx$, $b = tx$, $a < b$, neither s nor t starts with F and neither s not t ends with G.

9.2. Proposition. *Let* $(a, b) = (sx, tx)$ *be* 2-*nice. Put* $C = \mathrm{Cn}(a, b)$. *Then* C *is just the greatest element* T *of* $\mathscr{L}_\Delta$ *with the following two properties*:
(1) $T \subseteq E_\Delta$;
(2) *if* (u, v) *is either parallel or* 1-*nice then* $(u, v) \in T$ *iff* $(u, v) \in C$.

Proof. Evidently, C has these properties. Let $T \in \mathscr{L}_\Delta$ have these properties and let $(c, d) \in T$. We must prove $(c, d) \in C$. It is enough to consider the case when

$c < d$ and $c = px$, $d = qx$ for some $p, q \in \Delta^{(-)}$. Let $n > \text{Max}\,(\lambda(b), \lambda(d))$. Let F, G be as above. The equation (pqx, qpx) is either parallel or trivial, so that it belongs to C. Denote by k (resp. l) the greatest non-negative integer such that F^k is a beginning of pq (resp. F^l is a beginning of qp). Since neither s nor t starts with F, evidently $k = l$. We can assume that (px, qx) is not 1-nice. From this it follows that the largest beginning of the form F^r in p is the same as in q. Analogously, the largest end of the form G^r in p is the same as in q. From this it follows that $(F^n pG^n x, F^n qG^n x)$ is a parallel equation, so that it belongs to C. But then evidently $(px, qx) \in C$.

An equation (a, b) of type Δ is called 3-nice if Δ is large but not strictly large, (a, b) is not 2-nice and there are a variable x, two non-empty sequences $s, t \in \Delta^{(-)}$ and a symbol $F \in \Delta_1$ such that $a = sx$, $b = tx$, $a < b$ and neither s nor t starts with F.

An equation (a, b) of type Δ is called 4-nice if Δ is large but not strictly large, (a, b) is not 2-nice and there are a variable x, two non-empty sequences $s, t \in \Delta^{(-)}$ and a symbol $F \in \Delta_1$ such that $a = sx$, $b = tx$, $a < b$ and neither s nor t ends with F.

An equation (a, b) of type Δ is called 5-nice if Δ is large but not strictly large, (a, b) is neither 2-nice nor 3-nice nor 4-nice and there are a variable x and two non-empty sequences $s, t \in \Delta^{(-)}$ such that $a = sx$, $b = tx$, $a < b$.

9.3. Proposition. *Let $(a, b) = (sx, tx)$ be 5-nice. Put $C = \text{Cn}(a, b)$. Then C is just the least element T of $\mathscr{L}_\Delta$ with the following three properties:*
(1) $T \subseteq E_\Delta$;
(2) *the least* EDZ-*theory containing T equals* $\text{Z}(a)$;
(3) *if (u, v) is either parallel or 2-nice or 3-nice or 4-nice then $(u, v) \in T$ iff $(u, v) \in C$.*

Proof. Evidently, C has these three properties. Let T have these three properties. We must prove $(a, b) \in T$. By (2) there exists a sequence $u \in \Delta^{(-)}$ such that $(sx, ux) \in$ $\in T$ and $sx < ux$.

Denote by F the first symbol in s and by G the first symbol in t. We have $F \neq G$ and $\Delta_1 = \{F, G\}$. Denote by n the positive integer such that $G^n F$ is a beginning of t. Evidently, whenever $(sx, t'x) \in C$ then $t' = G^{kn} Ft''$ for some $k \geqq 0$ and some t''. Consider the following cases.

Case 1. There is a sequence $t_0 \in \Delta^{(-)}$ such that $(sx, t_0 x) \in C$, $sx < t_0 x$ and t_0 begins with F. Put $t = t_1 s t_2$, $t_0 = t_{01} s t_{02}$, let k be the length of $t_1 t_2$ and let l be the length of $t_{01} t_{02}$. We have $(sx, t_1^l s t_2^l x) \in C$, $(sx, t_{01}^k s t_{02}^k x) \in C$; since the sequences $t_1^l s t_2^l$ and $t_{01}^k s t_{02}^k$ are of the same length, the equation $(t_1^l s t_2^l x, t_{01}^k s t_{02}^k x)$ is parallel and so belongs to T. The equation $(tx, t_1^l s t_2^l x)$ is either 2-nice or 3-nice and belongs to C, so that it belongs to T. The equation $(sx, t_{01}^k s t_{02}^k x)$ is either 2-nice or 3-nice and belongs to C, so that it belongs to T. Hence $(sx, tx) \in T$.

Case 2. $t \neq Gs$ and there is no sequence t_0 as in Case 1. We have $(Gsx, Gux) \in T$ and this equation is either parallel or 2-nice or 3-nice, so it belongs to C. Let $u_0, \ldots, u_k$ be an (a, b)-proof from Gsx to Gux. Let us prove by induction on $i \in \{0, \ldots, k\}$

that $u_i = Gpx$ for some p with $(sx, px) \in C$. For $i = 0$ it is evident. Let $u_i = Gpx$, $(sx, px) \in C$, $i < k$. The term u_{i+1} is obtained from u_i if either a connected part s in Gp is replaced by t or a connected part t in Gp is replaced by s. If the connected part is a part of p, everything is evident. If $p = s$ and the connected part is not a part of p, then $t = Gs$, a contradiction. It remains to consider the case when $p \neq s$, so that p starts with G and u_{i+1} is obtained from u_i if a beginning t of u_i is replaced by s. Since t is a beginning of Gp and Gp starts with GG, we have $n \geqq 2$. We have $p = G^{ln}Fq$ for some q and some $l \geqq 1$. But then t must begin with $GG^{ln}F$, a contradiction. The induction is finished. Especially, $(sx, ux) \in C$. Hence $(ux, tx) \in C$; since $(sx, ux) \in C$ and we are not in Case 1, either (ux, tx) or (tx, ux) is either parallel or 2-nice or 3-nice or trivial and so $(ux, tx) \in T$. Hence $(sx, tx) \in T$.

Case 3. $t = Gs$. Then (sx, tx) is either 2-nice or 4-nice, a contradiction. This case is impossible.

9.4. Proposition. *Let $(a, b) = (sx, tx)$ be either 3-nice or 4-nice. Put $C = \mathrm{Cn}(a, b)$. Then C is just the least element T of $\mathscr{L}_\Delta$ with the following three properties:*
(1) $T \subseteq E_\Delta$;
(2) *the least* EDZ-*theory containing T equals* $\mathrm{Z}(a)$;
(3) *if (u, v) is either parallel or 2-nice then $(u, v) \in T$ iff $(u, v) \in C$.*

Proof. Evidently, C has these three properties. Let T have these three properties. We must prove $(a, b) \in T$. We shall consider only the case when (a, b) is 4-nice; the 3-nice case is quite similar. Since (a, b) is not 2-nice, Δ contains precisely two unary symbols. Let H be the last symbol in s, so that H is the last symbol in t, too.

By (2) there exists a sequence u such that $(sx, ux) \in T$ and $sx < ux$. Suppose that u does not end with H. Then (sux, usx) is a parallel equation belonging to T, so that it belongs to C; but s, t both end with the same symbol and so any consequence of (sx, tx) must evidently have the same property; we get a contradiction. This proves that u ends with H, too.

Denote by F the first symbol in s and by G the first symbol in t. Since (a, b) is 4-nice, we have $F \neq G$ and $\Delta_1 = \{F, G\}$. Now we can define the positive integer n and distinguish Cases 1, 2, 3 as in the proof of 9.3. Cases 1 and 2 can be solved similarly as in the proof of 9.3. The case $t = Gs$ remains. Evidently, $(sx, px) \in C$ iff $p = G^k s$ for some $k \geqq 0$. We have $(Gsx, Gux) \in T$; this equation is either parallel or 2-nice and so belongs to C. Hence $(sx, Gux) \in C$ and so $u = G^k s$ for some $k \geqq 1$. We get $(sx, ux) \in C$. Now $(sx, tx) \in T$ follows easily.

An equation (a, b) of type Δ is called 6-nice if Δ is a large but not strictly large type and there are a variable x and a non-empty sequence $s \in \Delta^{(-)}$ such that $a = x$ and $b = sx$.

9.5. Proposition. *Let $(a, b) = (x, sx)$ be a 6-nice equation. Put $C = \mathrm{Cn}(a, b)$. Then C is just the least element T of $\mathscr{L}_\Delta$ with the following three properties:*
(1) $T \subseteq E_\Delta$;

(2) *the least* EDZ-*theory containing* T *equals* $W_\Delta \times W_\Delta$;
(3) *if* (u, v) *is either* 2-*nice or* 3-*nice or* 4-*nice or* 5-*nice then* $(u, v) \in T$ *iff* $(u, v) \in C$.

Proof. Evidently, C has (1), (2), (3). Let T have (1), (2), (3). There is a non-empty sequence $u \in \Delta^{(-)}$ such that $(x, ux) \in T$. The equation (ux, sux) belongs to C and is either 2-nice or 3-nice or 4-nice or 5-nice, so that it belongs to T. Evidently $(sx, sux) \in T$ and so $(ux, sx) \in T$; we get $(x, sx) \in T$.

10. LARGE BUT NOT STRICTLY LARGE TYPES, THE FORMULAS

10.1. Lemma. *Let* Δ *be a large unary type, let* $F \in \Delta_1$, *let* x, y *be two different variables and let* $T \in \mathscr{L}_\Delta$. *Then* $(Fx, Fy) \in T \subseteq Z(F^*)$ *iff the following three conditions are satisfied*:
(1) $T \nsubseteq E_\Delta$;
(2) *the least* EDZ-*theory containing* T *equals* $Z(F^*)$;
(3) *whenever* A *is an* EDZ-*theory such that* $A \subseteq Z(F^*)$ *and* $A \neq 1_{W_\Delta}$ *then* $Z(F^*) = A \vee T$.

Proof. First assume that $(Fx, Fy) \in T \subseteq Z(F^*)$. Then (1) and (2) are evident. Let A be an EDZ-theory, $1_{W_\Delta} \neq A \subseteq Z(F^*)$. We have $A = Z(U)$ for some non-empty full set U. Let $a, b \in F^*$. We must prove $(a, b) \in A \vee T$. There are sequences $s_1, s_2, t_1, t_2, u \in \Delta^{(-)}$ and variables z_1, z_2 such that $a = s_1Fs_2z_1$, $b = t_1Ft_2z_2$ and $ux \in U$. We have $(s_1Fs_2z_1, s_1Fux) \in T$, $(s_1Fux, t_1Fux) \in A$, $(t_1Ft_2z_2, t_1Fux) \in T$ and so $(a, b) \in A \vee T$.

Now assume that (1), (2), (3) are satisfied. By (2) we have $T \subseteq Z(F^*)$ and so it remains to prove that $(Fx, Fy) \in T$. By (1) there is a sequence $s \in \Delta^{(-)}$ such that $(sx, sy) \in T$; by (2), F is contained in s. Put $A = Z(sx)$. By (3) we have $(Fx, Fy) \in A \vee T$. Suppose $(Fx, Fy) \notin T$. Then there are terms a, b such that $(Fx, a) \in T$, $(a, b) \in A$ and $a \neq b$. Since $(a, b) \in A$, we have $a = tz$ for some sequence $t \in \Delta^{(-)}$ and a variable z such that $sx \leqq tz$. Since $(sx, sy) \in T$ and $sx \leqq tz$, we have $(tz, ty) \in T$. We get $(Fx, ty) \in T$ and so $(Fx, Fy) \in T$.

Definition. $\psi_{33}(X, Y) \equiv \alpha_1^\varepsilon(X) \mathbin{\&} {}_Y[\neg\bar{\psi}_{13}(Y) \mathbin{\&} \forall A(\varepsilon(A) \to (Y \leqq A \leftrightarrow X \leqq A)) \mathbin{\&} \forall A((\varepsilon(A) \mathbin{\&} A \leqq X \mathbin{\&} \neg\omega_0(A)) \to X = A \vee Y)]$.

10.2. Lemma. *Let* Δ *be a large unary type. Then* $\psi_{33}(X, Y)$ *in* $\mathscr{L}_\Delta$ *iff there are an* $F \in \Delta_1$ *and two different variables* x, y *such that* $X = Z(F^*)$ *and* $Y = \mathrm{Cn}(Fx, Fy)$.

Definition. $\psi_{34}(A, B, T) \equiv \tau^\varepsilon(A) \mathbin{\&} \tau^\varepsilon(B) \mathbin{\&} \neg A \leqq B \mathbin{\&} \neg B \leqq A \mathbin{\&} (\exists P(\alpha_0^\varepsilon(P) \mathbin{\&} A \leqq P) \leftrightarrow \exists Q(\alpha_0^\varepsilon(Q) \mathbin{\&} B \leqq Q)) \mathbin{\&} \exists C, D, E({}_C[\varepsilon(C) \mathbin{\&} A \leqq C \mathbin{\&} B \leqq C] \mathbin{\&} \psi_{13}(D) \mathbin{\&} E = D \wedge C \mathbin{\&} T \leqq E \mathbin{\&} \forall U(\varepsilon(U) \to (T \leqq U \leftrightarrow C \leqq U))) \mathbin{\&} \forall P\, \exists Q(P < T \to (\varepsilon(Q) \mathbin{\&} P \leqq Q \mathbin{\&} \neg A \leqq Q \mathbin{\&} \neg B \leqq Q))$.

10.3. Lemma. *Let Δ be a large but not strictly large type. Then* $\psi_{34}(A, B, T)$ *in* $\mathscr{L}_{\Delta}$ *iff there is a parallel equation* (a, b) *such that* $A = \mathrm{Z}(a)$, $B = \mathrm{Z}(b)$ *and* $T = \mathrm{Cn}(a, b)$.

Definition. $\psi_{35}(X_1, X_2, Y) \equiv \varphi^{\varepsilon}_{33}(X_1, X_2, Y) \,\&\, ((\neg \exists C \alpha^{\varepsilon}_0(C)) \rightarrow$ $\rightarrow \exists A, B, T(\varphi^{\varepsilon}_3(X_2, A) \,\&\, \psi_{33}(X_2, B) \,\&\, \psi_{34}(A, Y, T) \,\&\, T \leqq B))$.

10.4. Lemma. *Let Δ be a large but not strictly large type. Then* $\psi_{35}(X_1, X_2, Y)$ *in* $\mathscr{L}_{\Delta}$ *iff there are two different symbols* $F, G \in \Delta_1$ *and a variable* x *such that* $X_1 = \mathrm{Z}(F^*)$, $X_2 = \mathrm{Z}(G^*)$ *and* $Y = \mathrm{Z}((GFx)^*)$.

Let Δ be a large but not strictly large type, let $F \in \Delta_1$ and let t be a term; let (A, U) be the fine F-code of t. Then $(\mathrm{Z}(A), \mathrm{Z}(U))$ is said to be the fine F-code of t in $\mathscr{L}_{\Delta}$. Similarly we can introduce the notions of an (F, G, w, x)-code and of a fine (F, G, w, x)-code of a non-empty finite sequence of terms in $\mathscr{L}_{\Delta}$ (see Section 5 of [2]).

Definition. $\psi_{36}(X, A, U_1, B, U_2, C, U_3, D, U_4) \equiv \exists X_2, Y(\psi_{35}(X, X_2, Y) \,\&$ $\&\, \varphi^{\varepsilon}_{50}(X, X_2, Y, A, U_1, B, U_2, C, U_3, D, U_4))$.

10.5. Lemma. *Let Δ be a large but not strictly large type. Then* $\psi_{36}(X, A, U_1, B, U_2, C, U_3, D, U_4)$ *in* $\mathscr{L}_{\Delta}$ *iff there are an* $F \in \Delta_1$ *and terms* a, b, c, d *such that* $X = \mathrm{Z}(F^*)$, $(A, U_1), (B, U_2), (C, U_3), (D, U_4)$ *are the fine* F*-codes of* a, b, c, d *(respectively) and* (c, d) *is a consequence of* (a, b).

Definition. (i) $\varphi_{87}(X, A, U_1, B, U_2) \equiv \varphi_{43}(X, A, U_1) \,\&\, \varphi_{43}(X, B, U_2) \,\&$ $\&\, (U_1 = U_2 \;\mathsf{VEL}\; (\alpha_0(U_1) \,\&\, \alpha_0(U_2)))$.

(ii) $\varphi_{88}(X, A, U_1, B, U_2) \equiv \varphi_{87}(X, A, U_1, B, U_2) \,\&\, \neg A \leqq B \,\&\, \neg B \leqq A$.

(iii) $\varphi_{89}(X, A, U_1, B, U_2) \equiv \varphi_{87}(X, A, U_1, B, U_2) \,\&\, A \leqslant B \,\&\, A \neq B \,\&\, ((\alpha_0(U_1) \,\&$ $\&\, \alpha_0(U_2)) \;\mathsf{VEL}\; \exists Y(\alpha_1(Y) \,\&\, \varphi_2(Y, B) \,\&\, \neg \omega_1(A)))$.

(iv) $\varphi_{90}(X, A, U_1, B, U_2) \equiv \varphi_{87}(X, A, U_1, B, U_2) \,\&\, A \leqslant B \,\&\, A \neq B \,\&\, \neg \omega_1(A) \,\&$ $\&\, \neg \alpha_0(U_1) \,\&\, U_1 = U_2 \,\&\, \exists X_2, Y, P, Q(\varphi_{33}(X, X_2, Y) \,\&\, \alpha_1(P) \,\&\, \alpha_1(Q) \,\&$ $\&\, \neg \varphi_{36}(X, X_2, Y, P, A) \,\&\, \neg \varphi_{36}(X, X_2, Y, P, B) \,\&\, \neg \varphi_{46}(X, X_2, Y, Q, A) \,\&$ $\&\, \neg \varphi_{46}(X, X_2, Y, Q, B))$.

(v) $\varphi_{91}(X, A, U_1, B, U_2) \equiv \varphi_{87}(X, A, U_1, B, U_2) \,\&\, A \leqslant B \,\&\, A \neq B \,\&\, \neg \omega_1(A) \,\&$ $\&\, \neg \alpha_0(U_1) \,\&\, U_1 = U_2 \,\&\, \neg \varphi_{90}(X, A, U_1, B, U_2) \,\&\, \exists X_2, Y, P(\varphi_{33}(X, X_2, Y) \,\&$ $\&\, \alpha_1(P) \,\&\, ((\neg \varphi_{36}(X, X_2, Y, P, A) \,\&\, \neg \varphi_{36}(X, X_2, Y, P, B)) \;\mathsf{VEL}\; (\neg \varphi_{46}(X, X_2, Y, P,$ $A) \,\&\, \neg \varphi_{46}(X, X_2, Y, P, B))))$.

(vi) $\varphi_{92}(X, A, U_1, B, U_2) \equiv \varphi_{87}(X, A, U_1, B, U_2) \,\&\, A \leqslant B \,\&\, A \neq B \,\&\, \neg \omega_1(A) \,\&$ $\&\, \neg \alpha_0(U_1) \,\&\, U_1 = U_2 \,\&\, \neg \varphi_{90}(X, A, U_1, B, U_2) \,\&\, \neg \varphi_{91}(X, A, U_1, B, U_2)$.

(vii) $\varphi_{93}(X, A, U_1, B, U_2) \equiv \varphi_{87}(X, A, U_1, B, U_2) \,\&\, \omega_1(A) \,\&\, \neg \omega_1(B)$.

10.6. Lemma. *Let Δ be a large but not strictly large type. Let* $i \in \{87, 88, \ldots, 93\}$. *Then* $\varphi_i(X, A, U_1, B, U_2)$ *in* $\mathscr{F}_{\Delta}$ *iff there are an* $F \in \Delta_1$ *and an equation* (a, b) *such that* $X = F^*$, (A, U_1) *is the fine* F*-code of* a, (B, U_2) *is the fine* F*-code of* b *and*:

(i) *if* $i = 87$ *then* $\mathrm{var}(a) = \mathrm{var}(b)$;
(ii) *if* $i = 88$ *then* (a, b) *is parallel*;
(iii) *if* $i = 89$ *then* (a, b) *is* 1-*nice*;
(iv) *if* $i = 90$ *then* (a, b) *is* 2-*nice*;
(v) *if* $i = 91$ *then* (a, b) *is either* 3-*nice or* 4-*nice*;
(vi) *if* $i = 92$ *then* (a, b) *is* 5-*nice*;
(vii) *if* $i = 93$ *then* (a, b) *is* 6-*nice*.

Definition. (i) $\psi_{37}(X, A, U_1, B, U_2, T) \equiv \varphi^{\varepsilon}_{88}(X, A, U_1, B, U_2) \,\&\, \psi_{34}(A, B, T)$.

(ii) $\psi_{38}(X, A, U_1, B, U_2, T) \equiv \varphi^{\varepsilon}_{89}(X, A, U_1, B, U_2) \,\&\, {}^{T}[\bar{\psi}_{13}(T) \,\&\, \forall P(\varepsilon(P) \to \to (T \leqq P \leftrightarrow A \leqq P)) \,\&\, \forall C, V_1, D, V_2, U(\psi_{37}(X, C, V_1, D, V_2, U) \to \to (U \leqq T \leftrightarrow \psi_{36}(X, A, U_1, B, U_2, C, V_1, D, V_2)))]$.

(iii) $\psi_{39}(X, A, U_1, B, U_2, T) \equiv \varphi^{\varepsilon}_{90}(X, A, U_1, B, U_2) \,\&\, {}^{T}[\bar{\psi}_{13}(T) \,\&\, \forall C, V_1, D, V_2, U((\psi_{37}(X, C, V_1, D, V_2, U) \,\mathsf{VEL}\, \psi_{38}(X, C, V_1, D, V_2, U)) \to \to (U \leqq T \leftrightarrow \psi_{36}(X, A, U_1, B, U_2, C, V_1, D, V_2)))]$.

(iv) $\psi_{40}(X, A, U_1, B, U_2, T) \equiv \varphi^{\varepsilon}_{91}(X, A, U_1, B, U_2) \,\&\, {}_{T}[\bar{\psi}_{13}(T) \,\&\, \forall P(\varepsilon(P) \to \to (T \leqq P \leftrightarrow A \leqq P)) \,\&\, \forall C, V_1, D, V_2, U((\psi_{37}(X, C, V_1, D, V_2, U) \,\mathsf{VEL}\, \psi_{39}(X, C, V_1, D, V_2, U)) \to (U \leqq T \leftrightarrow \psi_{36}(X, A, U_1, B, U_2, C, V_1, D, V_2)))]$.

(v) $\psi_{41}(X, A, U_1, B, U_2, T) \equiv \varphi^{\varepsilon}_{92}(X, A, U_1, B, U_2) \,\&\, {}_{T}[\bar{\psi}_{13}(T) \,\&\, \forall P(\varepsilon(P) \to \to (T \leqq P \leftrightarrow A \leqq P)) \,\&\, \forall C, V_1, D, V_2, U((\psi_{37}(X, C, V_1, D, V_2, U) \,\mathsf{VEL}\, \psi_{39}(X, C, V_1, D, V_2, U) \,\mathsf{VEL}\, \psi_{40}(X, C, V_1, D, V_2, U)) \to (U \leqq T \leftrightarrow \psi_{36}(X, A, U_1, B, U_2, C, V_1, D, V_2)))]$.

(vi) $\psi_{42}(X, A, U_1, B, U_2, T) \equiv \varphi^{\varepsilon}_{93}(X, A, U_1, B, U_2) \,\&\, {}_{T}[\bar{\psi}_{13}(T) \,\&\, \forall P((\varepsilon(P) \,\&\, \& \, T \leqq P) \to \omega_1(P)) \,\&\, \forall C, V_1, D, V_2, U((\psi_{39}(X, C, V_1, D, V_2, U) \,\mathsf{VEL}\, \psi_{40}(X, C, V_1, D, V_2, U) \,\mathsf{VEL}\, \psi_{41}(X, C, V_1, D, V_2, U)) \to (U \leqq T \leftrightarrow \psi_{36}(X, A, U_1, B, U_2, C, V_1, D, V_2)))]$.

(vii) $\psi_{43}(X, A, U_1, B, U_2, T) \equiv \psi_{37}(X, A, U_1, B, U_2, T) \,\mathsf{VEL}\, \psi_{38}(X, A, U_1, B, U_2, T) \,\mathsf{VEL}\, \psi_{38}(X, B, U_2, A, U_1, T) \,\mathsf{VEL}\, \psi_{39}(X, A, U_1, B, U_2, T) \,\mathsf{VEL}\, \psi_{39}(X, B, U_2, A, U_1, T) \,\mathsf{VEL} \ldots \mathsf{VEL}\, \psi_{42}(X, A, U_1, B, U_2, T) \,\mathsf{VEL}\, \psi_{42}(X, B, U_2, A, U_1, T) \,\mathsf{VEL}\, (\varphi^{\varepsilon}_{87}(X, A, U_1, B, U_2) \,\&\, A = B \,\&\, U_1 = U_2 \,\&\, \omega_0(T))$.

10.7. Lemma. *Let* Δ *be a large but not strictly large type. Then* $\psi_{43}(X, A, U_1, B, U_2, T)$ *in* $\mathscr{L}_\Delta$ *iff there are an* $F \in \Delta_1$ *and an equation* (a, b) *such that* $\mathrm{var}(a) = \mathrm{var}(b)$, $X = \mathrm{Z}(F^*)$, (A, U_1) *is the fine F-code of a in* $\mathscr{L}_\Delta$, (B, U_2) *is the fine F-code of b in* $\mathscr{L}_\Delta$ *and* $T = \mathrm{Cn}(a, b)$.

Proof. It is a formalization of Section 9.

Definition. (i) $\psi_{44}(X, A, U_1, B, U_2, T) \equiv \varphi^{\varepsilon}_{43}(X, A, U_1) \,\&\, \varphi^{\varepsilon}_{43}(X, B, U_2) \,\&\, \& \neg\varphi^{\varepsilon}_{87}(X, A, U_1, B, U_2) \,\&\, \neg\bar{\psi}_{13}(T) \,\&\, \forall C, V_1, D, V_2, U(\psi_{43}(X, C, V_1, D, V_2, U) \to \to (U \leqq T \leftrightarrow \psi_{36}(X, A, U_1, B, U_2, C, V_1, D, V_2)))$.

(ii) $\psi_{45}(X, A, U_1, B, U_2, T) \equiv \psi_{43}(X, A, U_1, B, U_2, T) \,\mathsf{VEL}\, \psi_{44}(X, A, U_1, B, U_2, T)$.

10.8. Lemma. *Let Δ be a large but not strictly large type. Then* $\psi_{45}(X, A, U_1, B, U_2, T)$ *in* $\mathscr{L}_\Delta$ *iff there are an* $F \in \Delta_1$ *and an equation* (a, b) *such that* $X = Z(F^*)$, (A, U_1) *is the fine F-code of a in* $\mathscr{L}_\Delta$, (B, U_2) *is the fine F-code of b in* $\mathscr{L}_\Delta$ *and* $T = \mathrm{Cn}(a, b)$.

Definition. (i) $\psi_{46}(X) \equiv \exists Y, A, U_1, B, U_2\, \psi_{45}(Y, A, U_1, B, U_2, X)$.

(ii) $\psi_{47}(X_1, X_2, Y, A_1, B_1, A_2, B_2, D, T) \equiv \psi_{35}(X_1, X_2, Y)\, \&\, \varphi^{\varepsilon}_{44}(X_1, X_2, Y, A_1, B_1, D)\, \&\, \varphi^{\varepsilon}_{44}(X_1, X_2, Y, A_2, B_2, D)\, \&\, \forall C, V_1, E, V_2, U(\psi_{45}(X_1, C, V_1, E, V_2, U) \rightarrow (U \leqq T \leftrightarrow \varphi^{\varepsilon}_{49}(X_1, X_2, Y, A_1, B_1, A_2, B_2, D, C, V_1, E, V_2)))$.

(iii) $\psi_{48}(X) \equiv \exists X_1, X_2, Y, A_1, B_1, A_2, B_2, D\, \psi_{47}(X_1, X_2, Y, A_1, B_1, A_2, B_2, D, X)$.

10.9. Lemma. *Let Δ be a large but not strictly large type. Then:*

(i) $\psi_{46}(X)$ *in* $\mathscr{L}_\Delta$ *iff X is one-based.*

(ii) $\psi_{47}(X_1, X_2, Y, A_1, B_1, A_2, B_2, D, T)$ *in* $\mathscr{L}_\Delta$ *iff there are two different symbols* $F, G \in \Delta_1$, *a variable x, an integer* $n \geqq 1$ *and equations* $(a_1, b_1), \ldots, (a_n, b_n)$ *such that* $X_1 = Z(F^*)$, $X_2 = Z(G^*)$, $Y = Z((GFx)^*)$, (A_1, B_1, D) *is a fine* (F, G, GF, x)-*code of* $a_1, \ldots, a_n$, (A_2, B_2, D) *is a fine* (F, G, GF, x)-*code of* $b_1, \ldots, b_n$ *and* $T = \mathrm{Cn}((a_1, b_1), \ldots, (a_n, b_n))$.

(iii) $\psi_{48}(X)$ *in* $\mathscr{L}_\Delta$ *iff X is finitely based.*

10.10. Lemma. *Let Δ be large but not strictly large and let h be an automorphism of* $\mathscr{L}_\Delta$. *Then* $h = Q_{c,f}$ *for some* $(c, f) \in H_\Delta$.

Proof. By 2.8, a restriction of h is an automorphism of the lattice of EDZ-theories of type Δ. Hence by Theorem 7.7 of [2] there exists a pair $(c, f) \in G_\Delta$ such that $h(Z(A)) = Z(\bar{P}_{c,f}(A))$ for all $A \in \mathscr{F}_\Delta$.

If Δ is not unary then $H_\Delta = G_\Delta$; if Δ is unary then H_Δ can be identified with the subgroup of G_Δ formed by the pairs (d, g) such that $d = 1$. We shall now prove $(c, f) \in H_\Delta$. Suppose, on the contrary, that Δ is unary and $c = 2$. There are two different symbols $F, G \in \Delta$. We have $\psi_{35}(Z(F^*), Z(G^*), Z((GFx)^*))$ and so $\psi_{35}(h(Z(F^*)), h(Z(G^*)), h(Z((GFx)^*)))$, i.e. $\psi_{35}(Z(H^*), Z(K^*), Z((HKx)^*))$ for some $H, K \in \Delta$, a contradiction.

Now evidently $h(Z(A)) = Q_{c,f}(Z(A))$ for all $A \in \mathscr{F}_\Delta$. Using the formula ψ_{45} we see that $h(T) = Q_{c,f}(T)$ for every one-based equational theory T. Since every element of $\mathscr{L}_\Delta$ is the join of one-based equational theories, we get $h = Q_{c,f}$.

Let Δ be a large but not strictly large type and let $(a_1, b_1), \ldots, (a_n, b_n)$ be a non-empty finite sequence of equations of type Δ. Then we define a formula $\Theta_{\Delta,(a_1,b_1),\ldots,(a_n,b_n)}(X)$ as follows. Let us fix a symbol $F \in \Delta_1$. For every $i \in \{1, \ldots, n\}$ denote by (a_i^*, u_i^*) the fine F-code of a_i and by (b_i^*, v_i^*) the fine F-code of b_i. Put $(t_1, \ldots, t_{4n}) = (a_1, u_1, b_1, v_1, \ldots, a_n, u_n, b_n, v_n)$. For every $i \in \{1, \ldots, 4n\}$, the term t_i can be uniquely expressed in the form $t_i = F_{i,k_i} \ldots F_{i,1} y_i$ where $y_i \in V \cup \Delta_0$ and $F_{i,1}, \ldots, F_{i,k_i} \in \Delta_1$. In Section 7 of [2] we have introduced the formula $\mu_{t_1,\ldots,t_{4n}}$. Put

$\Theta_{\Delta,(a_1,b_1),\ldots,(a_n,b_n)}(X) \equiv \exists P_1, P_2, Q, X_1, \ldots, X_{4n}, Y_1, \ldots, Y_{4n}, Z_{1,1}, \ldots, Z_{1,k_1}, \ldots, Z_{4n,1}, \ldots, Z_{4n,k_{4n}}, T_1, \ldots, T_n(\psi_{35}(P_1, P_2, Q)\, \&\, \mu^{\varepsilon}_{t_1,\ldots,t_{4n}}(P_1, P_2, Q, X_1, \ldots, X_{4n},$

$Y_1, \ldots, Y_{4n}, Z_{1,1}, \ldots, Z_{4n,k_{4n}}) \& \psi_{45}(P_1, X_1, X_2, X_3, X_4, T_1) \& \ldots \& \psi_{45}(P_1, X_{4n-3}, X_{4n-2}, X_{4n-1}, X_{4n}, T_n) \& X = T_1 \vee \ldots \vee T_n)$.

10.11. Lemma. *Let Δ be large but not strictly large and let $(a_1, b_1), \ldots, (a_n, b_n)$ be a non-empty finite sequence of equations. Then $\Theta_{\Delta,(a_1,b_1),\ldots,(a_n,b_n)}(X)$ in $\mathscr{L}_\Delta$ iff $X = h(\mathrm{Cn}((a_1, b_1), \ldots, (a_n, b_n)))$ for some automorphism h of $\mathscr{L}_\Delta$.*

11. TYPES CONTAINING ONLY NULLARY SYMBOLS

In this section we shall investigate the types Δ such that $\Delta = \Delta_0$. In this case the lattice $\mathscr{L}_\Delta$ is isomorphic to the equivalence lattice of Δ, with one greatest element added.

Definition. (i) $\psi_{49}(X) \equiv \exists A(\psi_2(A) \& X < A \& \neg\exists B(X < B < A))$.

(ii) $\psi_{50}(X) \equiv \exists A, B(\omega_0(A) \& \psi_{49}(B) \& A = X \wedge B) \& \neg\, \omega_1(X)$.

(iii) $\psi_{51}(X, Y) \equiv \neg\omega_1(X) \& \neg\omega_1(Y) \& \psi_3(X) \& \psi_3(Y) \& \exists A(\omega_0(A) \& A = X \wedge Y) \& (\omega_0(X)\ \mathsf{VEL}\ \omega_0(Y)\ \mathsf{VEL}\ \exists B(B = X \vee Y \& \neg\psi_3(B)))$.

(iv) $\psi_{52}(X, Y) \equiv \psi_{51}(X, Y) \& \exists A, B, P(\omega_0(A) \& B = X \vee Y \& \psi_{50}(P) \& A = X \wedge P \& A = Y \wedge P \& \forall U((\psi_3(U) \& P \leqq U) \rightarrow X \leqq U) \& \forall U((\psi_3(U) \& B \leqq U) \rightarrow P \leqq U))$.

(v) $\psi_{53}(X) \equiv \neg\omega_1(X) \& \psi_3(X) \& \neg\exists A_1, A_2, B(A_1 < A_2 \& A_2 \leqq X \& \psi_{52}(A_1, B) \& \psi_{52}(B, A_1) \& \psi_{52}(A_2, B) \& \psi_{52}(B, A_2))$.

(vi) $\psi_{54}(X) \equiv \omega_1(X)\ \mathsf{VEL}\ \exists Y(\psi_{53}(Y) \& X \leqq Y)$.

(vii) $\psi_{55}(X) \equiv \exists A, B(\omega_0(A) \& \omega_1(B) \& (X = A\ \mathsf{VEL}\ X = B\ \mathsf{VEL}\ \neg\exists Y(A < Y < X)))$.

(viii) $\psi_{56}(X) \equiv \psi_{49}(X) \& \psi_3(X)$.

(ix) $\psi_{57}(X, Y, A) \equiv \psi_{56}(X) \& \psi_{56}(Y) \& X \neq Y \& \psi_{55}(A) \& \neg A \leqq X \& \neg A \leqq Y \& \neg\omega_1(A)$.

11.1. Lemma. *Let $\Delta = \Delta_0$. Then:*

(i) *$\psi_{49}(X)$ in $\mathscr{L}_\Delta$ iff $X = (C \times C) \cup (D \times D) \cup 1_{W_\Delta}$ for two non-empty disjoint subsets C, D of Δ with $C \cup D = \Delta$.*

(ii) *$\psi_{50}(X)$ in $\mathscr{L}_\Delta$ iff* $\mathrm{Card}(\Delta) \geqq 2$ *and there is a set M of pairwise disjoint two-element subsets of Δ such that $(u, v) \in X$ iff either $u = v$ or $\{u, v\} \in M$.*

(iii) *$\psi_{51}(X, Y)$ in $\mathscr{L}_\Delta$ iff there are two disjoint subsets C, D of Δ such that $X = (C \times C) \cup 1_{W_\Delta}$ and $Y = (D \times D) \cup 1_{W_\Delta}$.*

(iv) *$\psi_{52}(X, Y)$ in $\mathscr{L}_\Delta$ iff* $\mathrm{Card}(\Delta) \geqq 2$ *and there are two disjoint subsets C, D of Δ such that* $\mathrm{Card}(C) \leqq \mathrm{Card}(D)$, *$X = (C \times C) \cup 1_{W_\Delta}$ and $Y = (D \times D) \cup 1_{W_\Delta}$.*

(v) *$\psi_{53}(X)$ in $\mathscr{L}_\Delta$ iff $X = (C \times C) \cup 1_{W_\Delta}$ for some finite $C \subseteq \Delta$.*

(vi) *$\psi_{54}(X)$ in $\mathscr{L}_\Delta$ iff X is finitely based.*

(vii) *$\psi_{55}(X)$ in $\mathscr{L}_\Delta$ iff X is one-based.*

(viii) *$\psi_{56}(X)$ in $\mathscr{L}_\Delta$ iff* $\mathrm{Card}(\Delta) \geqq 2$ *and $X = ((\Delta \setminus \{c\}) \times (\Delta \setminus \{c\})) \cup 1_{W_\Delta}$ for some $c \in \Delta$.*

(ix) $\psi_{57}(X, Y, A)$ *in* $\mathscr{L}_\Delta$ *iff* $\mathrm{Card}(\Delta) \geqq 3$ *and there are two different symbols* $c, d \in \Delta$ *such that* $X = ((\Delta \setminus \{c\}) \times (\Delta \setminus \{c\})) \cup 1_{W_\Delta}$, $Y = ((\Delta \setminus \{d\}) \times (\Delta \setminus \{d\})) \cup \cup 1_{W_\Delta}$ *and* $A = \mathrm{Cn}(c, d)$.

11.2. Lemma. *Let* $\Delta = \Delta_0$ *and let h be an automorphism of* $\mathscr{L}_\Delta$. *Then* $h = Q_{c,f}$ *for some* $(c, f) \in H_\Delta$.

Proof. If $\mathrm{Card}(\Delta) \leqq 2$, it is clear, since then $\mathscr{L}_\Delta$ has only the identical automorphism. Let $\mathrm{Card}(\Delta) \geqq 3$. For every $o \in \Delta$ put $g(o) = ((\Delta \setminus \{o\}) \times (\Delta \setminus \{o\})) \cup 1_{W_\Delta}$, so that $o \mapsto g(o)$ is an injective mapping of Δ into $\mathscr{L}_\Delta$. It follows from 11.1(viii) that if $o \in \Delta$ then $h(g(o)) = g(c(o))$ for some $c(o) \in \Delta$; evidently, c is a permutation of Δ. Denote by f the identical permutation of the empty set, so that $(c, f) \in H_\Delta$. It is easy to see that $h = Q_{c,f}$.

Let $\Delta = \Delta_0$ and let $(a_1, b_1), \ldots, (a_n, b_n)$ be a finite non-empty sequence of non-trivial equations. Then we define a formula $\Theta_{\Delta,(a_1,b_1),\ldots,(a_n,b_n)}(X)$ as follows. Put $(t_1, \ldots, t_{2n}) = (a_1, b_1, \ldots, a_n, b_n)$. If $t_i \in V$ for some i, then put $\Theta_{\Delta,(a_1,b_1),\ldots,(a_n,b_n)}(X) \equiv \equiv \omega_1(X)$. If $t_i \in \Delta$ for all i, put

$$\Theta_{\Delta,(a_1,b_1),\ldots,(a_n,b_n)}(X) \equiv \exists A_1, \ldots, A_{2n}, B_1, \ldots, B_n$$

$(\psi_{57}(A_1, A_2, B_1) \,\&\, \ldots \,\&\, \psi_{57}(A_{2n-1}, A_{2n}, B_n) \,\&\, X = B_1 \vee \ldots \vee B_n \,\&\, g_1 \,\&\, g_2)$ where g_1 is the conjunction of the formulas $A_i = A_j$ $(i, j \in \{1, \ldots, 2n\},\ t_i = t_j)$ and g_2 is the conjunction of the formulas $A_i \neq A_j$ $(i, j \in \{1, \ldots, 2n\},\ t_i \neq t_j)$.

11.3. Lemma. *Let* $\Delta = \Delta_0$ *and* $\mathrm{Card}(\Delta) \geqq 3$. *Let* $(a_1, b_1), \ldots, (a_n, b_n)$ *be a finite non-empty sequence of non-trivial equations. Then* $\Theta_{\Delta,(a_1,b_1),\ldots,(a_n,b_n)}(X)$ *in* $\mathscr{L}_\Delta$ *iff* $X = h(\mathrm{Cn}((a_1, b_1), \ldots, (a_n, b_n)))$ *for some automorphism h of* $\mathscr{L}_\Delta$.

12. SMALL TYPES CONTAINING A UNARY SYMBOL

12.1. Lemma. *Let* $\Delta = \Delta_0 \cup \{F\}$ *for some unary symbol F. Then* $\psi_{34}(A, B, T)$ *in* $\mathscr{L}_\Delta$ *iff there are two different symbols* $c, d \in \Delta_0$ *and two integers* $n, m \geqq 0$ *such that* $A = \mathrm{Z}(F^n c)$, $B = \mathrm{Z}(F^m d)$ *and* $T = \mathrm{Cn}(F^n c, F^m d)$.

Definition. (i) $\psi_{58}(X, Y, T) \equiv \exists A, T_1, T_2, B(\psi_{34}(X, A, T_1) \,\&\, \psi_{34}(Y, A, T_2) \,\&\, \&\, Y < X \,\&\, B = T \vee T_1 \,\&\, {}_T[\forall U(\varepsilon(U) \rightarrow (T \leqq U \leftrightarrow X \leqq U)) \,\&\, T_2 \leqq B])$.

(ii) $\psi_{59}(X, Y, T) \equiv \varphi_5^\varepsilon(X) \,\&\, \varphi_5^\varepsilon(Y) \,\&\, Y < X \,\&\, {}^T[\forall U(\varepsilon(U) \rightarrow (T \leqq U \leftrightarrow X \leqq U)) \,\&\, \&\, \forall A, B, T_1(\psi_{34}(A, B, T_1) \rightarrow \neg T_1 \leqq T) \,\&\, \forall A, B, U, T_1((\psi_{58}(A, B, T_1) \,\&\, \varphi_8^\varepsilon(A, X) \,\&\, \&\, \varphi_8^\varepsilon(B, U) \,\&\, T_1 \leqq T) \rightarrow U \leqq Y) \,\&\, \forall A, B, T_1((\psi_{58}(A, B, T_1) \,\&\, \varphi_8^\varepsilon(A, X) \,\&\, \varphi_8^\varepsilon(B, Y)) \rightarrow \rightarrow T_1 \leqq T)]$.

(iii) $\psi_{60}(X) \equiv \exists A, B(\psi_{34}(A, B, X)$ VEL $\psi_{58}(A, B, X)$ VEL $\psi_{59}(A, B, X))$ VEL $\varphi_5^\varepsilon(X)$ VEL $\exists A, B(\varphi_5^\varepsilon(A) \,\&\, \tau^\varepsilon(B) \,\&\, {}_X[\varepsilon(X) \,\&\, A \leqq X \,\&\, B \leqq X])$ VEL $\omega_0(X)$.

(iv) $\psi_{61}(X) \equiv \exists A, B(\psi_{59}(A, B, X) \,\&\, \omega_1(A) \,\&\, \alpha_1^\varepsilon(B))$.

(v) $\psi_{62}(X, Y) \equiv \varepsilon(Y) \& \forall U(\varphi_1^\varepsilon(U, Y) \leftrightarrow (\alpha_0^\varepsilon(U) \& (\exists A, B, T, A_1(\psi_{34}(A, B, T) \& \& A \leqq U \& \varphi_8^\varepsilon(A, A_1) \& \neg A_1 \leqq X) \text{ VEL } \exists A, B, T, A_1, B_1, T_1(\psi_{58}(A, B, T) \& A \leqq U \& \& \varphi_8^\varepsilon(A, A_1) \& \varphi_8^\varepsilon(B, B_1) \& \psi_{59}(A_1, B_1, T_1) \& \neg T_1 \leqq X))))$.

12.2. Lemma. *Let $\Delta = \Delta_0 \cup \{F\}$ where $n_F = 1$ and* $\text{Card}(\Delta_0) \geqq 2$. *Then:*

(i) $\psi_{58}(X, Y, T)$ *in* $\mathscr{L}_\Delta$ *iff there is a symbol $c \in \Delta_0$ and two integers n, m such that $0 \leqq n < m$, $X = \text{Z}(F^n c)$, $Y = \text{Z}(F^m c)$ and $T = \text{Cn}(F^n c, F^m c)$.*

(ii) $\psi_{59}(X, Y, T)$ *in* $\mathscr{L}_\Delta$ *iff there is a variable x and two integers n, m such that $0 \leqq n < m$, $X = \text{Z}(F^n x)$, $Y = \text{Z}(F^m x)$ and $T = \text{Cn}(F^n x, F^m x)$.*

(iii) $\psi_{60}(X)$ *in* $\mathscr{L}_\Delta$ *iff X is one-based.*

(iv) $\psi_{61}(X)$ *in* $\mathscr{L}_\Delta$ *iff $X = \text{Cn}(x, Fx)$ (where $x \in V$).*

(v) $\psi_{62}(X, Y)$ *in* $\mathscr{L}_\Delta$ *iff Y is the* EDZ-*theory determined by the nullary symbols that are contained in any base for X.*

Let f be a formula and L be a variable not contained in f. Then we define a formula $f^{(L)}$ as follows: if f is without quantifiers then $f^{(L)} \equiv f$; if $f \equiv \neg g$ then $f^{(L)} \equiv \neg g^{(L)}$; if $f \equiv g \& h$ then $f^{(L)} \equiv g^{(L)} \& h^{(L)}$; similarly for VEL, $\rightarrow$, $\leftrightarrow$; if $f \equiv \forall X\, g$ then $f^{(L)} \equiv \forall X(L \leqq X \rightarrow g^{(L)})$; if $f \equiv \exists X\, g$ then $f^{(L)} \equiv \exists X(L \leqq X \& g^{(L)})$.

Definition. $\psi_{63}(X) \equiv \exists Y, L, A(\psi_{62}(X, Y) \& \psi_{61}(L) \& A = L \vee Y \& \psi_{53}^{(L)}(A))$.

12.3. Lemma. *Let $\Delta = \Delta_0 \cup \{F\}$ where $n_F = 1$ and* $\text{Card}(\Delta_0) \geqq 2$. *Then $\psi_{63}(X)$ in $\mathscr{L}_\Delta$ iff X is finitely based.*

12.4. Lemma. *Let $\Delta = \Delta_0 \cup \{F\}$ where $n_F = 1$ and* $\text{Card}(\Delta_0) \geqq 2$. *Let h be an automorphism of $\mathscr{L}_\Delta$. Then $h = Q_{c,f}$ for some $(c, f) \in H_\Delta$.*

Let $\Delta = \Delta_0 \cup \{F\}$ where $n_F = 1$ and $\text{Card}(\Delta_0) \geqq 2$; let $(a_1, a_2), \ldots, (a_{2n-1}, a_{2n})$ be a finite non-empty sequence of equations; for every $i \in \{1, \ldots, 2n\}$ let $a_i = F^{k(i)} y_i$ where $y_i \in V \cup \Delta_0$; suppose that whenever $i \in \{1, \ldots, 2n\}$ is odd then either $k(i) \leqq k(i + 1)$, $y_i = y_{i+1}$ or $k(i) = k(i + 1)$, $y_i \in V$, $y_{i+1} \in V$, $y_i \neq y_{i+1}$ or $y_i \in \Delta_0$, $y_{i+1} \in \Delta_0$, $y_i \neq y_{i+1}$. Put

$$\Theta_{\Delta,(a_1,a_2),\ldots,(a_{2n-1},a_{2n})}(X) \equiv \exists A_{1,0}, \ldots, A_{1,k(1)}, \ldots, A_{2n,0}, \ldots, A_{2n,k(2n)}, B_1, \ldots$$
$$\ldots, B_n((A_{1,0} \prec A_{1,1} \prec \ldots \prec A_{1,k(1)})^\varepsilon \& \ldots \& (A_{2n,0} \prec A_{2n,1} \prec \ldots \prec A_{2n,k(2n)})^\varepsilon \&$$
$$\& g_1 \& g_2 \& g_3 \& g_4 \& g_5 \& g_6 \& g_7 \& g_8 \& g_9 \& X = B_1 \vee \ldots \vee B_n)$$

where

g_1 is the conjunction of the formulas $\omega_1(A_{i,0})$ $(i \in \{1, \ldots, 2n\},\ y_i \in V)$,

g_2 is the conjunction of the formulas $\alpha_0^\varepsilon(A_{i,0})$ $(i \in \{1, \ldots, 2n\},\ y_i \in \Delta_0)$,

g_3 is the conjunction of the formulas $A_{i,0} = A_{j,0}$ $(i, j \in \{1, \ldots, 2n\},\ y_i = y_j \in \Delta_0)$,

g_4 is the conjunction of the formulas $A_{i,0} \neq A_{j,0}$ $(i, j \in \{1, \ldots, 2n\},\ y_i \in \Delta_0,\ y_j \in \Delta_0,\ y_i \neq y_j)$,

g_5 is the conjunction of the formulas $\omega_0(B_i)$ $(i \in \{1, \ldots, n\},\ a_{2i-1} = a_{2i})$,

g_6 is the conjunction of the formulas $\psi_{34}(A_{2i-1,k(2i-1)}, A_{2i,k(2i)}, B_i)$ $(i \in \{1, \ldots, n\}$, $y_{2i-1} \in \Delta_0$, $y_{2i} \in \Delta_0$, $y_{2i-1} \neq y_{2i})$,

g_7 is the conjunction of the formulas $\psi_{58}(A_{2i-1,k(2i-1)}, A_{2i,k(2i)}, B_i)$ $(i \in \{1, \ldots, n\}$, $k(2i-1) < k(2i)$, $y_{2i-1} = y_{2i} \in \Delta_0)$,

g_8 is the conjunction of the formulas $\psi_{59}(A_{2i-1,k(2i-1)}, A_{2i,k(2i)}, B_i)$ $(i \in \{1, \ldots, n\}$, $k(2i-1) < k(2i)$, $y_{2i-1} = y_{2i} \in V)$,

g_9 is the conjunction of the formulas $B_i = A_{2i} \,\&\, \varphi_5^\varepsilon(B_i)$ $(i \in \{1, \ldots, n\}$, $y_{2i-1} \in V$, $y_{2i} \in V$, $y_{2i-1} \neq y_{2i})$.

12.5. Lemma. *Let* $\Delta = \Delta_0 \cup \{F\}$ *where* $n_F = 1$ *and* $\mathrm{Card}(\Delta_0) \geqq 2$; *let* $(a_1, a_2), \ldots, (a_{2n-1}, a_{2n})$ *be as above. Then* $\Theta_{\Delta,(a_1,a_2),\ldots,(a_{2n-1},a_{2n})}(X)$ *in* $\mathscr{L}_\Delta$ *iff* $X = h(\mathrm{Cn}((a_1, a_2), \ldots, (a_{2n-1}, a_{2n})))$ *for some automorphism* h *of* $\mathscr{L}_\Delta$.

Definition. $\psi_{64}(X) \equiv \neg\overline{\psi}_{13}(X)$ VEL $\exists A(\alpha_0^\varepsilon(A) \,\&\, X \leqq A)$ VEL $\neg\exists A, Y_1, Y_2(\alpha_0^\varepsilon(A) \,\&\, Y_1 \leqq A \,\&\, X = Y_1 \vee Y_2 \,\&\, X \neq Y_2)$.

12.6. Lemma. *Let* $\Delta = \Delta_0 \cup \{F\}$ *where* $n_F = 1$. *If* Δ_0 *is empty then every equational theory of type* Δ *is one-based. If* $\mathrm{Card}(\Delta_0) = 1$ *then every equational theory of type* Δ *is two-based; we have* $\psi_{64}(X)$ *in* $\mathscr{L}_\Delta$ *iff* X *is one-based.*

13. THE MAIN RESULTS

Definition. $\Phi(X) \equiv (\psi_{31}(X) \,\&\, \exists A\, \bar{\alpha}_2^\varepsilon(A))$ VEL $(\psi_{46}(X) \,\&\, \psi_5 \,\&\, \neg\exists A\, \bar{\alpha}_2^\varepsilon(A))$ VEL $(\psi_4 \,\&\, \psi_{55}(X))$ VEL $(\psi_{60}(X) \,\&\, \neg\psi_4 \,\&\, \neg\psi_5 \,\&\, \exists A, B(\alpha_0^\varepsilon(A) \,\&\, \alpha_0^\varepsilon(B) \,\&\, A \neq B))$ VEL $(\neg\psi_4 \,\&\, \neg\psi_5 \,\&\, \neg\exists A\, \alpha_0^\varepsilon(A))$ VEL $(\psi_{64}(X) \,\&\, \neg\psi_4 \,\&\, \neg\psi_5 \,\&\, \exists!!\, A\, \alpha_0^\varepsilon(A))$.

13.1. Theorem. *Let* Δ *be any type. Then* $\Phi(X)$ *in* $\mathscr{L}_\Delta$ *iff* X *is one-based. Consequently, the set of one-based equational theories of type* Δ *is definable in* $\mathscr{L}_\Delta$.

Definition. $\Psi(X) \equiv (\psi_{32}(X) \,\&\, \exists A\, \bar{\alpha}_2^\varepsilon(A))$ VEL $(\psi_{48}(X) \,\&\, \psi_5 \,\&\, \neg\exists A\, \bar{\alpha}_2^\varepsilon(A))$ VEL $(\psi_4 \,\&\, \psi_{54}(X))$ VEL $(\neg\psi_4 \,\&\, \neg\psi_5 \,\&\, ((\exists A, B(\alpha_0^\varepsilon(A) \,\&\, \alpha_0^\varepsilon(B) \,\&\, A \neq B)) \to \psi_{63}(X)))$.

13.2. Theorem. *Let* Δ *be any type. Then* $\Psi(X)$ *in* $\mathscr{L}_\Delta$ *iff* X *is finitely based. Consequently, the set of finitely based equational theories of type* Δ *is definable in* $\mathscr{L}_\Delta$.

Theorems 13.1 and 13.2 follow immediately from 8.4, 10.9, 11.1, 12.2, 12.3 and 12.6.

13.3. Theorem. *Let a type* Δ *be given.*

(i) *If* $\Delta \neq \{o_1, o_2\}$, $\Delta \neq \{F\}$ *and* $\Delta \neq \{F, o\}$ *for any unary symbol* F *and any nullary symbols* o, o_1, o_2 $(o_1 \neq o_2)$, *then the mapping* $(c, f) \mapsto Q_{c,f}$ *is an isomorphism of* H_Δ *onto the automorphism group of* $\mathscr{L}_\Delta$.

(ii) *If* $\Delta = \{o_1, o_2\}$ *where* o_1, o_2 *are two different nullary symbols, then* $\mathscr{L}_\Delta$ *has only the identical automorphism.*

(iii) *If either* $\Delta = \{F\}$ *or* $\Delta = \{F, o\}$ *where F is unary and o is nullary, then the automorphism group of* $\mathscr{L}_\Delta$ *is isomorphic to the group of permutations of an infinite countable set.*

Proof. For (i) see 8.5, 10.10, 11.2 and 12.4. The assertions (ii) and (iii) are easy and they are left to the reader.

13.4. Theorem. *For any type* Δ *and any finite sequence* $(a_1, b_1), \ldots, (a_n, b_n)$ *of equations of type* Δ *there exists a formula* $\Theta(X)$ *such that* $\Theta(X)$ *in* $\mathscr{L}_\Delta$ *iff* $X = h(\mathrm{Cn}((a_1, b_1), \ldots, (a_n, b_n)))$ *for some automorphism h of* $\mathscr{L}_\Delta$.

Proof. If $\Delta \subseteq \{o_1, o_2\}$ or $\Delta \subseteq \{F, o\}$ for some unary symbol F and some nullary symbols o_1, o_2, o, then the proof is not difficult and it is left to the reader. For the remaining types see Lemmas 8.6, 10.11, 11.3 and 12.5, where the corresponding formula $\Theta(X)$ is effectively constructed; notice that any finite sequence of equations is equivalent to a sequence for which the formula $\Theta(X)$ is constructed in these lemmas.

14. REMARKS AND OPEN PROBLEMS

Theorems 13.1, 13.2, 13.3 and 13.4 are the results of this treatment; Parts I and II are necessary, too. Let us mention briefly the idea of their proofs. First of all, it is necessary to find a formula defining in $\mathscr{L}_\Delta$ the et of EDZ-theories (equational theories with at most one block of cardinality $\geqq 2$); see Theorem 2.8. To do this, we need a characterization of modular elements in the lattice $\mathscr{L}_\Delta$, since it turns out that the set of EDZ-theories does not differ much from the set of modular elements of $\mathscr{L}_\Delta$; such a characterization is found in Part I. Then we must study definability and automorphisms in the lattice of EDZ-theories, or – which is the same – in the lattice of full sets of terms; this is done in Part II. If this is carried out, Propositions 3.1 and 3.2 enable us to characterize in the lattice $\mathscr{L}_\Delta$ equational theories generated by a parallel equation. Finally, we show that some special equations are determined by their parallel consequences, some less special equations are determined by their consequences that are either parallel or considered before, etc.; after finitely many steps all equations are exhausted. Here the case of a type consisting of a single binary symbol is the most difficult.

We see that the notion of EDZ-theory is fundamental for the investigation of definability in the lattice of equational theories. The varieties corresponding to EDZ-theories were studied in [5], [6] and [7]; they were called EDZ-varieties or varieties of algebras with equationally definable zeros there.

Let us formulate some open problems related to the contents of this paper.

The formulas $\Phi(X)$ and $\Psi(X)$, defining one-based and finitely based equational theories in the lattice $\mathscr{L}_\Delta$, are very long.

Problem 1. *Does there exist a short formula $f(X)$ such that for any type Δ, $f(X)$ in $\mathscr{L}_\Delta$ iff X is one-based?*

Problem 2. *Does there exist a short formula $f(X)$ such that for any type Δ, $f(X)$ in $\mathscr{L}_\Delta$ iff X is finitely based?*

The author's conjecture is that Problem 2 could have a positive solution.

Problem 3. *Let Δ be a large type. Is every equational theory T of type Δ such that $T \subseteq E_\Delta$ uniquely determined by its parallel equations and the* EDZ-*theory generated by T?*

A positive answer to Problem 3 would simplify the contents of this paper.

In [1] we have defined a quasiordering on the set W_Δ of Δ-terms. Put $\mathscr{T}_\Delta = W_\Delta/\sim$, so that $\mathscr{T}_\Delta$ is a poset.

Problem 4. *Investigate definability and automorphisms in the poset $\mathscr{T}_\Delta$.*

Problem 5. *Let Δ be an at most countable type. Is the set of recursively based equational theories of type Δ definable in $\mathscr{L}_\Delta$? Is every recursively based equational theory of type Δ definable up to automorphisms in $\mathscr{L}_\Delta$? Does there exist a definable element of $\mathscr{L}_\Delta$ which is not a recursively based equational theory?*

Problem 6. *Is the set of equational theories $T \in \mathscr{L}_\Delta$ such that the corresponding variety of Δ-algebras has the amalgamation property (or some other interesting property) definable in $\mathscr{L}_\Delta$?*

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