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ON SUBDIRECT IRREDUCIBILITY AND ITS VARIANTS

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The concept of subdirect irreducibility was introduced for algebras by G. Birkhoff in [1]. For the general concrete categories, his definition can be extended, roughly speaking, as follows: a subdirectly irreducible object is one that cannot be constructed from simpler objects by using products and subobjects.

In [8], G. Sabidussi solved a problem of a kind of subdirect irreducibility for symmetric graphs without loops. In [7], A. Pultr and the author of this paper gave a characterization of subdirectly irreducibles for finite objects of regular categories (which include most of the "everyday-life" concrete categories). It was also proved in [7] that for finite objects the equivalence between the subdirect irreducibility of A and the productivity of the subcategory generated by all the objects which do not contain A as a subobject, and by the terminal object, holds. In general, for infinite objects, the situation is considerably more complicated and calls for discussing some variants of the definition of subdirect irreducibility. In this paper, a characterization of subdirect irreducibility and its variants for the infinite case is presented.

I am indebted to A. Pultr for valuable advice.

Conventions. The category of all the sets and mappings is denoted by Set. From category theory, the reader is just supposed to be acquainted with the most basic notions (as product, monomorphism, pullback) and with the notion of concrete category.

1. SEMIREGULAR CATEGORIES

First, we recall some definitions.

1.1. A *subobject* in a concrete category $(\mathfrak{K}, U)$ is a monomorphism $\mu: A \to B$ such that for every $f: UC \to UA$ for which there is a $\psi : C \to B$ with $U\psi = U\mu \circ f$, there exists a $\varphi: C \to A$ with $U\varphi = f$.

1.2. Remark. Subobjects have the following properties (see [7]):

1. Composition of subobjects is a subobject.
2. If $\alpha\beta$ is a subobject then β is a subobject.

3. If U preserves monomorphisms, $\mu = \mu'\varepsilon$ is a subobject and $U\varepsilon$ is onto then ε is an isomorphism.
4. If U preserves products and $\mu_i \colon A_i \to B_i$ $(i \in I)$ are subobjects then

$$\mu = \prod_I \mu_i : \prod_I A_i \to \prod_I B_i$$

is a subobject.

1.3. Let $(\mathfrak{K}, U)$ be a concrete category. For a set X define a preordered class

$$\mathfrak{K}UX = (\{A \mid UA = X\}, \prec)$$

putting $A \prec B$ iff there exists an $\alpha \colon A \to B$ with $U\alpha = 1_X$ (in this case we sometimes write $\alpha \colon A \prec B$). If A is an object of $\mathfrak{K}$, we write $\mathfrak{K}UA$ for $\mathfrak{K}U(UA)$.

1.4. Definition. A concrete category $(\mathfrak{K}, U)$ is said to be *semiregular* if it has the following properties:

(S 1) U preserves limits.

(S 2) If X is a set and $f \colon X \to UA$ an invertible mapping then there is an isomorphism φ with $U\varphi = f$.

(S 3) If α is an isomorphism and $U\alpha = 1_{UA}$ then $\alpha = 1_A$.

(S 4) Every $\mathfrak{K}UX$ is a set.

(S 5) For every morphism φ there is a subobject decomposition $\varphi = \mu\varepsilon$ with μ a subobject and $U\varepsilon$ onto.

1.5. Remarks. 1. (S 1) implies, particularly, that the underlying mapping of a monomorphism in a semiregular category is one-to-one.

2. Notice that in most everyday-life concrete categories the conditions (S 1 – 5) are satisfied.

3. The conditions of semiregularity are weaker that those of regularity (see [7]). We do not require the finiteness of $\mathfrak{K}UX$ for finite X and the reflectivity of subcategories closed with respect to products and subobjects. (E.g., the category of [0, 1]-labelled graphs – see 7.1 – is semiregular but not regular.)

1.6. We recall some properties of regular categories (see [7]) which are satisfied in semiregular categories as well. For proving them, it suffices to use only the conditions (S 1 – 5).

1. Every $\mathfrak{K}UX$ is a partially ordered set and every invertible $f \colon X \to Y$ naturally induces an isomorphism $\mathfrak{K}UX \cong \mathfrak{K}UY$.

2. For every $\mu \colon A \to B$ with $U\mu$ one-to-one there exists a unique subobject decomposition $\mu = \mu'\varepsilon$ with $U\varepsilon = 1_{UA}$.

3. Let X, I be sets, $A \in \mathfrak{K}UX$, $A = \bigwedge_I A_i$, $B = \prod_I A_i$; then there exists a subobject $\mu \colon A \to B$.

4. Let X, I be sets, $A_i \in \mathfrak{K}UX$ for each $i \in I$, $\prod_I A_i \to^{p_i} A_i$ a product with projections.

If there is a subobject $\mu\colon A \to \prod_I A_i$ such that $p_i\mu\colon A \prec A_i$ then $A = \bigwedge_I A_i$.

5. Let X be a finite set, $A, B \in \mathfrak{K}UX$, $A \prec B$, If there is a subobject $\mu\colon A \to B$ then $A = B$.

1.7. Lemma. *Let $(\mathfrak{K}, U)$ be a semiregular category, T a terminal object of $\mathfrak{K}$, $\mu\colon A \to T$ a subobject. Then every $\nu\colon A \to B$ is a subobject.*

Proof. By (S 1), UT is a one-point set.

(a) Let card $UA = 1$. Then $U\mu$ is onto and according to 1.2.3 μ is an isomorphism. Let $\psi\colon C \to B$, $f\colon UC \to UA$ with $U\psi = U\nu \circ f$. Then $U(A \to T \xrightarrow{\mu^{-1}} A) = f$.

(b) Let card $UA = 0$. Let f, ψ have the same properties as in (a). Then $UC = UA = \emptyset$, $U\mu \circ f = U(C \to T)$ and there exists a φ such that $U\varphi = f$.

2. SUBDIRECT IRREDUCIBILITY AND ITS VARIANTS

2.1. An object A of a concrete category $(\mathfrak{K}, U)$ is said to be (finitely) *subdirectly irreducible* (abbreviated : FSI, SI; cf. [1], [7], [8]) if for every subobject

$$\mu\colon A \to \prod_I A_i \quad (I \text{ finite, respectively})$$

such that all $U(p_i\mu)$ are onto, at least one $p_i\mu$ is an isomorphism.

2.2. For an object A of $(\mathfrak{K}, U)$ denote by $A \urcorner (\mathfrak{K}, U)$ (shortly, $A \urcorner \mathfrak{K}$) the full concrete subcategory of $(\mathfrak{K}, U)$ with the class of objects obj $(A \urcorner \mathfrak{K}) = \{B \in \text{obj}\, \mathfrak{K} \mid \exists \mu\colon A \to B \text{ a subobject} \Rightarrow B \text{ is the terminal object of } \mathfrak{K}\}$.

It was proved in [7] that a finite object of a regular categoty $\mathfrak{K}$ is SI (FSI) iff $A \urcorner \mathfrak{K}$ is closed under (finite, respectively) products. This assertion holds for semiregular categories as well. For infinite objects, however, one can prove only that for every A (finitely) subdirectly irreducible, $A \urcorner \mathfrak{K}$ is closed under (finite, respectively) products.

2.3. Definition. An object A of a concrete category $(\mathfrak{K}, U)$ is said to be *weakly* (finitely) *subdirectly irreducible* (WSI, WFSI, respectively) if $A \urcorner \mathfrak{K}$ is closed under (finite) products.

2.4. The following scheme of implications obviously holds for semiregular categories:

$$\begin{array}{ccc} \boxed{SI} & \Rightarrow & \boxed{WSI} \\ \Downarrow & & \Downarrow \\ \boxed{FSI} & \Rightarrow & \boxed{WFSI} \end{array}$$

We shall show later (see 4.4, 4.7, 5.5) that in general none of these implications is an equivalence.

3.1. Definition. (a) An object A of a semiregular category $(\mathfrak{K}, U)$ is said to be *maximal* if it is a maximal element of the poset $\mathfrak{K}UA$. It is said to be *weakly maximal* if for every $B \in \mathfrak{K}UA$ such that $B \succ A$ there exists a subobject $\mu: A \to B$.

(b) An object A is said to be (finitely) *meet irreducible* if $A = \bigwedge_I A_i$ (I finite, respectively) implies that there is an $i_0 \in I$ such that $A_{i_0} = A$.

A is said to be *weakly* (finitely) *meet irreducible* if $A = \bigwedge_I A_i$ (I finite, respectively) implies that there is an $i_0 \in I$ and a subobject $\mu: A \to A_{i_0}$.

3.2. A *monomorphic system* is a system $(\mu_i: A \to B_i)_{i \in I}$ of morphisms such that if $\mu_i \alpha = \mu_i \beta$ for all $i \in I$ then $\alpha = \beta$.

3.3. Analogously as in [7] one can prove the following assertions:

1. Let A be a maximal object of a semiregular (finitely) productive category $(\mathfrak{K}, U)$. Then A is (finitely) subdirectly irreducible iff for any (finite) monomorphic system $(\mu_i: A \to B_i)_{i \in I}$ at least one μ_i is a monomorphism.

2. Let A be a weakly maximal object of a semiregular (finitely) productive $(\mathfrak{K}, U)$. Then A is WSI (WFSI) iff for every (finite) monomorphic system $(\mu_i: A \to B_i)_{i \in I}$ there exists a subobject $v: A \to B_{i_0}$ for some $i_0 \in I$.

3. Let A be a non-maximal (finitely) meet irreducible object of a semiregular (finitely) productive $(\mathfrak{K}, U)$. For every $\varphi: A \to B$ with $U\varphi$ not one-to-one let there exist an $\iota: A \prec C$, $A \neq C$ and a $\bar{\varphi}: C \to B$ such that $\bar{\varphi}\iota = \varphi$. Then A is (finitely) subdirectly irreducible.

4. Let A be a weakly (finitely) meet irreducible object of a semiregular (finitely) productive $(\mathfrak{K}, U)$. For every $\varphi: A \to B$ with $B \in A \urcorner \mathfrak{K}$ let there exist an $\iota: A \prec C$, $A \neq C$ with $C \in A \urcorner \mathfrak{K}$ and a $\bar{\varphi}: C \to B$ such that $\bar{\varphi}\iota = \varphi$. Then A is weakly (finitely) subdirectly irreducible.

3.4. Lemma. *Let A be a non-maximal object of a semiregular finitely productive $(\mathfrak{K}, U)$. If A is finitely subdirectly irreducible then for every morphism $\varphi: A \to B$ such that $U\varphi$ is not one-to-one there exists an $\iota: A \prec C$, $A \neq C$ and a $\bar{\varphi}: C \to B$ such that $\bar{\varphi}\iota = \varphi$.*

Proof. Let $\varepsilon: A \prec D$, $A \neq C$. Suppose that $\varphi: A \to B$ such that $U\varphi$ is not one-to-one, cannnot be extended to a stronger structure. Define $\mu: A \to B \times D$ by $p_B\mu = \varphi$, $p_D\mu = \varepsilon(p_B, p_D$ are the projections). According to 1.6.2 there is a subobject decomposition $\mu = \mu'\varepsilon'$ with $\varepsilon': A \prec A'$. By the assumption of the non-existence of a non-trivial extension of φ, $A = A'$, $\mu = \mu'$ and A is not FSI.

3.5. Lemma. *Let A be an object of a semiregular finitely productive $(\mathfrak{K}, U)$ which is not weakly maximal. If A is* WFSI *then for every $\varphi: A \to B$ with $B \in A \urcorner \mathfrak{K}$ there exists an $\iota: A \prec C$, $A \neq C$, $C \in A \urcorner \mathfrak{K}$, and a $\bar{\varphi}: C \to B$ such that $\bar{\varphi}\iota = \varphi$.*

Proof. Suppose that $\varphi: A \to B$ with $B \in A \urcorner \mathfrak{K}$ cannot be extended to a stronger structure $C \in A \urcorner \mathfrak{K}$; let $\varepsilon: A \prec D$, $A \neq D$, $D \in A \urcorner \mathfrak{K}$. Similarly as in the proof of the previous lemma define $\mu: A \to B \times D$ by $p_B\mu = \varphi$, $p_D\mu = \varepsilon$. According to 1.6.2 there is a subobject decomposition $\mu = \mu'\varepsilon'$ with $\varepsilon': A \prec A'$. Then there exists a subobject $v: A \to A'$ and $\mu' v$ is a subobject.

(a) If $B \times D$ is not a terminal object then $B \times D \notin A \urcorner \mathfrak{K}$ and A is not WFSI.

(b) If $B \times D$ is a terminal object then, by 1.7, $p_D\mu' v$ is a subobject and D is a terminal object. According to (S 1) card UD = card $UA = 1$ and, by 1.6.5, $A = D$ which is a contradiction.

3.6. Theorem. *An object A of a semiregular $(\mathfrak{K}, U)$ with (finite) products is (finitely) subdirectly irreducible iff either A is maximal and for any (finite) monomorphic system $(\mu_i: A \to B_i)_I$ there exists an $i_0 \in I$ such that μ_{i_0} is a monomorphism, or A is not maximal, it is (finitely) meet irreducible, and for every $\varphi: A \to B$ such that $U\varphi$ is not one-to-one there exists an $\iota: A \prec C$, $A \neq C$ and a $\bar{\varphi}: C \to B$ such that $\bar{\varphi}\iota = \varphi$.*

Proof follows from 1.6.3, 2.4, 3.3.1, 3.3.3 and 3.4.

3.7. Theorem. *An object A of a semiregular $(\mathfrak{K}, U)$ with (finite) products is weakly (finitely) subdirectly irreducible iff either A is weakly maximal and for any (finite) monomorphic system $(\mu_i: A \to B_i)_I$ there exists an $i_0 \in I$ and a subobject $v: A \to B_{i_0}$, or A is not weakly maximal, it is weakly (finitely) meet irreducible and for every $\varphi: A \to B$ with $B \in A \urcorner \mathfrak{K}$ there exists an $\iota: A \prec C$, $A \neq C$ with $C \in A \urcorner \mathfrak{K}$ and a $\bar{\varphi}: C \to B$ such that $\bar{\varphi}\iota = \varphi$.*

Proof follows from 1.6.3, 2.4, 3.3.2, 3.3.4 and 3.5.

4. CATEGORIES $S(F)$

4.1. Let F be a covariant or a contravariant functor of Set into itself. The category $S(F)$ (cf. e.g. [4], [5]) is defined as follows: The objects are couples (X, r) with $r \subset FX$, the morphisms $(X, r) \to (Y, s)$ are triples $((X, r), f, (Y, s))$ with $f: X \to Y$ such that $F(f)(r) \subset s(F(f)(s) \subset r$ in the contravariant case). The composition and the forgetful functor are the obvious ones. One easily sees that $S(F)$ is semiregular.

4.2. Proposition. (a) *If F is covariant, the* SI *objects of $S(F)$ are the (X, FX) with* card $X \leqq 2$ *and the $(X, FX \setminus \{u\})$ where for every $f: X \to Y$ which is not one-to-one there is a $v \neq u$ with $F(f)(u) = F(f)(v)$.*

(b) *If F is contravariant, the* SI *objects of $S(F)$ are the $(X, \emptyset)$ with* card $X \leqq 2$ *and the $(X, \{u\})$ with $u \in FX \setminus \bigcup\{F(f)(FY) \mid f: X \to Y$ not one-to-one$\}$.*

Proof. Since $(2, F2)$ $((2, \emptyset))$ is the cogenerator in $S(F)$, one can easily see by 3.6 that the maximal SI (X, FX) $((X, \emptyset)$, respectively) have to have card $X \leqq 2$. On the other hand, a maximal object with card $X \leqq 2$ is SI.

The non-maximal meet irreducibles are $(X, FX \setminus \{u\})$ (or $(X, \{u\})$). The condition in 3.6 gives for every $f: X \to Y$ not one-to-one: $F(f)(FX \setminus \{u\}) = F(f)(FX)$ ($F(f)(s) \subset \{u\} \Rightarrow F(f)(s) = \emptyset$, respectively) which is equivalent to the condition of Proposition.

4.3. Proposition. (a) *If F is covariant, the* WSI *objects of $S(F)$ are the (X, FX) with* card $X \leqq 2$ *and the $(X, FX \setminus \{u\})$ where for every $((X, FX \setminus \{u\}), f, (Y, r))$ with $(Y, r) \in (X, FX \setminus \{u\}) \urcorner S(F)$ there is $F(f)(u) \in r$.*

(b) *If F is contravariant, the* WSI *of $S(F)$ are the $(X, \emptyset)$ with* card $X \leqq 2$ *and the $(X, \{u\})$ with*

$$\{\varphi: (X, \{u\}) \to (Y, r) \mid (Y, r) \in (X, \{u\}) \urcorner S(F),\ r \neq \emptyset\} = \emptyset\,.$$

Proof. One can easily see that every weakly maximal object of $S(F)$ is maximal. If (X, s) is not maximal then

$$(X, s) = \bigwedge_{u \in FX \setminus s} (X, FX \setminus \{u\}) \text{ in the covariant case}$$

$$((X, s) = \bigwedge_{u \in s} (X, \{u\}) \text{ in the contravariant case})\,.$$

Therefore, non-maximal weakly meet irreducible are the $(X, FX \setminus \{u\})$ ($(X, \{u\})$, respectively).

Now, one can easily complete the proof using 3.7.

4.4. Remark. The following example will show that there exist WSI objects which are not SI in $S(F)$: Define $FX = \{A \subset X \mid \text{card } A = \omega_0\} \cup \{0_X\}$; for $f: X \to Y$, define $F(f)(A) = f(A)$ if card $f(A) = \omega_0$, $F(f)(A) = 0_Y$ otherwise, $F(f)(0_X) = 0_Y$.

The SI objects of $S(F)$ are the $(X, \emptyset)$ with card $X \leqq 1$, the (X, FX) with card $X \leqq 2$ and the $(X, FX \setminus \{A\})$ with $A \in FX$, card $(X \setminus A) \leqq 1$. The WSI objects are, moreover, the $(X, FX \setminus \{0_X\})$ with card $X \geqq \omega_0$.

Proof. By 4.2 and 4.3, maximal objects are SI and WSI iff their cardinality is less or equal to 2. Therefore, it remains to consider the objects $(X, FX \setminus \{A\})$ and $(X, FX \setminus \{0_X\})$.

(a) If $A \subset X$, card $A = \omega_0$, card $(X \setminus A) \geqq 2$, define $Y = A \cup \{X \setminus A\}$ and $f: X \to Y$ as a factor mapping. Then $(Y, FY \setminus \{A\}) \in (X, FX \setminus \{A\}) \urcorner S(F)$ and $F(f)(A) \notin FY \setminus \{A\}$. By 4.3, $(X, FX \setminus \{A\})$ is not WSI.

(b) If $A \subset X$, card $A = \omega_0$, card $(X \setminus A) \leqq 1$ then for every $f: X \to Y$ not one-to-one there exists a $B \subset X$, card $B = \omega_0$, $B \neq A$, such that $F(f)(A) = F(f)(B)$. By 4.2, $(X, FX \setminus \{A\})$ is SI.

(c) If X is finite then $FX = \{0_X\}$. Consider $((X, \emptyset), f, (1, \emptyset))$ with a constant f. By 4.2 and 4.3, $(X, \emptyset)$ is SI (WSI, respectively) iff card $X \leqq 1$.

(d) Consider the $(X, FX \setminus \{0_X\})$ with card $X \geqq \omega_0$. Let $((X, FX \setminus \{0_X\}), f, (Y, r))$ be a morphism with $(Y, r) \in (X, FX \setminus \{0_X\}) \urcorner S(F)$. Then either card $f(X) =$ card X and $0_Y \in r$ because $(Y, r) \in (X, FX \setminus \{0_X\}) \urcorner S(F)$, or card $f(X) <$ card X and there exists an $A \subset X$ with $\operatorname{card} f(A) < \omega_0 = \operatorname{card} A$; therefore, $F(f)(A) = 0_Y \in r$ as well. By 4.3, $(X, FX \setminus \{0_X\})$ is WSI.

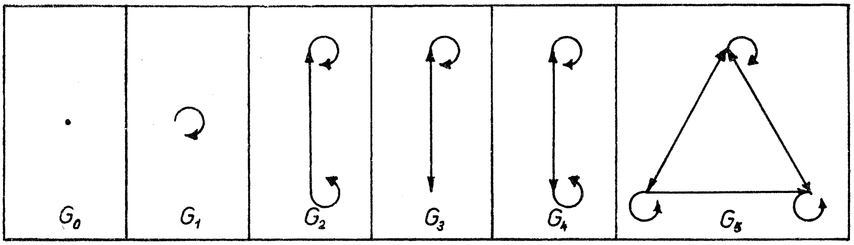
On the other hand, take $x, y \in X$, $x \neq y$, and define $Y = X \setminus \{x\}$, $r = FY \setminus \{0_Y\}$, $f: X \to Y$ by $f(x) = y$, $f(z) = z$ otherwise. Then f is not one-to-one, $F(f)(0_X) \neq F(f)(A)$ for any $A \in FX \setminus \{0_X\}$ and, by 4.2, $(X, FX \setminus \{0_X\})$ is not SI (moreover, it is not FSI).

4.5. Relational systems. The category $\mathrm{Rel}(\Delta)$ of sets with relational systems of the type $\Delta = (A_i)_{i \in I}$ can be represented as $S(F)$ with

$$FX = \bigcup_{i \in I} X^{A_i} \times \{i\}, \quad F(f)(\alpha, i) = (f\alpha, i).$$

By 4.2 and 4.3 we see that the SI objects in $\mathrm{Rel}(\Delta)$ are the (X, FX) with $\operatorname{card} X \leqq 2$ and the $(X, FX \setminus \{(\alpha, i)\})$ such that $\operatorname{card}(X \setminus \alpha(A_i)) \leqq 1$. There are no other WSI objects in $\mathrm{Rel}(\Delta)$. (Thus, the list of all the SI and WSI objects in $\mathrm{Rel}(\Delta)$ is given by the same formula as in the finite case – see [7].)

4.6. Binary relations (directed graphs). Put $n = 1$, $A_1 = 2$. Then $\mathrm{Rel}(\Delta) = \mathrm{Graph}$ is the category of binary relations (i.e. directed graphs). By 4.5 we obtain that the WSI objects coincide with the SI ones. The list goes as follows (cf. [7]):



4.7. Remark. For finite graphs, all the implications from 2.4 are equivalences. The situation is different for infinite directed graphs. For example $K_{\omega_0} = (\omega_0, \omega_0 \times \omega_0)$ and $L_{\omega_0} = (\omega_0, \{(j, k) \mid j \leqq k\})$ are WFSI (but according to 4.6 not WSI).

Proof. (a) Let $\mu: K_{\omega_0} \to \prod_I A_i$ be a subobject, I finite. Then there is an $i_0 \in I$ such that $\operatorname{card} UA_{i_0} \geqq \omega_0$. Thus, A_{i_0} contains K_{ω_0} as a subobject.

(b) Suppose that $L_{\omega_0} \urcorner$ Graph is not finitely productive. Then there exist objects $A, B \in L_{\omega_0} \urcorner$ Graph such that there exists a subobject $\mu: L_{\omega_0} \to A \times B$. We can suppose without loss of generality that Up_A, Up_B are bijections. Define a symmetric binary relation R on ω_0 putting $(n, q) \in R$ iff the restriction of edges of A to the set $\{Up_A(n), Up_A(q)\}$ is a complete binary relation.

By the Ramsey theorem, there exists either a complete countable subgraph of

(ω_0, R), or a countable discrete subgraph of (ω_0, R). In the former case, $B \notin L_{\omega_0} \urcorner$ $\urcorner$ Graph, in the latter case $A \notin L_{\omega_0} \urcorner$ Graph which both contradict the assumption.

4.8. Hypergraphs. Define a functor P^+ by

$$P^+(X) = \{M \mid M \subset X\}, \quad P^+(f)(M) = f(M).$$

Then $S(P^+)$ is the category of (generalized) hypergraphs. By 4.2 and 4.3, the SI objects in $S(P^+)$ are the (X, FX) with card $X \leqq 2$ and the $(X, FX \setminus \{M\})$ with $M \subset X$, card $(X \setminus M) \leqq 1$. There are no WSI objects which are not SI. Thus, there are arbitrarily large subdirectly irreducibles in this category.

5. UNARY ALGEBRAS

5.1. Consider the category of unary algebras and homomorphisms. One easily sees that every unary algebra is maximal. Thus, it suffices to proceed by the first points of 3.6 and 3.7. One sees that a WSI unary algebra (X, R) cannot have more than two components, and if it has two components, one of them has to have just one point and the other has just one point or no point x with $(x, x) \in R$.

5.2. Lemma. *Let* (X, R) *be a* WSI *unary algebra. Then*: $a \neq b \neq c \,\&\, (a, b) \in R \,\&\, (c, b) \in R \Rightarrow a = c$.

Proof. Suppose the contrary.

(1) Let there exist three distinct points $a_0, x, y \in A$ such that $(x, a_0) \in R$, $(y, a_0) \in R$, $(a_0, a_0) \in R$. Then define

$$M = \{a \in X \mid (a, a_0) \in R\}$$

and for every $a \in M \setminus \{a_0\}$ define

$$B_a = \{z \in X \mid \exists z_0 = z, \ldots, z_n = a,\ (z_i, z_{i+1}) \in R \text{ for } i = 0, \ldots, n-1\}$$

and a morphism $\mu_a: (X, R) \to (B_a, R \cap (B_a \times B_a))$ by $U\mu_a(z) = z$ if $z \in B_a$, $U\mu_a(z) = a_0$ otherwise. Then $(B_a, R \cap (B_a \times B_a)) \in (X, R) \urcorner \mathfrak{K}$, $(\mu_a)_{a \in M}$ is a monomorphic system and by 3.7, (X, R) is not WSI which is a contradiction.

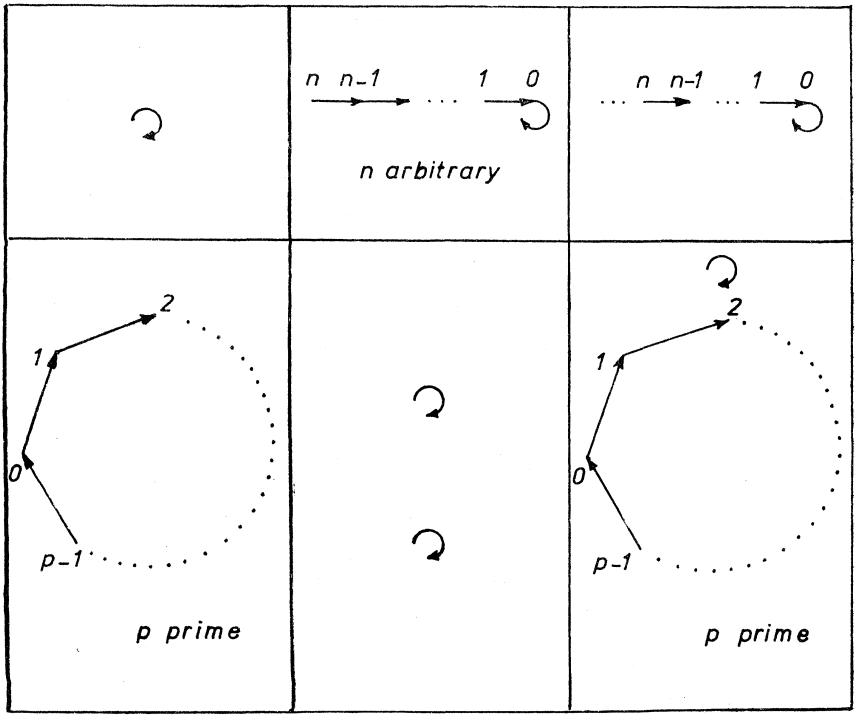
(2) Let the condition (1) fail to be satisfied. Then define $P = \{a \in X \mid \text{card}\{u \mid (u, a) \in R\} \geqq 2\}$ and for every $a \in P$ put $D_a = \{z \in X \mid \urcorner \exists z_0 = a, z_1, \ldots, z_n = z\}$, define $f_a : X \to D_a$ by $f_a(z) = z$ if $z \in D_a$ and $f_a(z) = a$ otherwise and put $R_a = (f_a \times f_a)(R)$.

Define an equivalence $\approx$ on X putting $x \approx y$ iff there exist sequences $x_0 = x$, $x_1, \ldots, x_n = z$, $y_0 = y, y_1, \ldots, y_n = z$ such that $(x_i, x_{i+1}) \in R$, $(y_i, y_{i+1}) \in R$. Let $\varphi: (X, R) \to (X, R)/\approx$ be the factor morphism, φ_a the morphism with $U\varphi_a = f_a$. Then $(\varphi, (\varphi_a)_{a \in P})$ is a monomorphic system and by 3.7, (X, R) is not WSI which is a contradiction.

5.3. Lemma. *If a cycle is* WSI *then it is finite and its length is a prime.*

Proof follows directly from 3.7.

5.4. Proposition. *A non-void unary algebra is* SI *iff it is contained in the following list*:



There are no WSI which are not SI.

Proof. By 5.1 – 5.3, any WSI object must be contained in the list presented. On the other hand, one can easily see that all these objects are SI.

5.5. Remark. *Let* Z *be the set of integers. Define a unary algebra* (Z, R) *putting* $R(z) = z + 1$ *for every* $z \in Z$. *Then* (Z, R) *is* FSI *(but it is not* SI *according to* 5.4).

Proof. Suppose there is a subobject

$$\mu: (Z, R) \to \prod_{i \in F} (A_i, R_i) \text{ where } F \text{ is a finite set },$$

$U(p_i\mu)$ are onto but no $p_i\mu$ is isomorphic. Then for every $i \in F$ there exists an integer $n(i)$ and a positive integer $q(i)$ such that $R_i^{q(i)}(p_i\, \mu(x)) = p_i\, \mu(x)$ whenever $x \geqq n(i)$.

Let q be the smallest common multiple of $q(i)$, $n = \max \{n(i) \mid i \in F\}$. Then $R^q(x) = x$ whenever $x \geqq n$ which contradicts the assumption.

5.6. If we consider the category of *partial unary algebras* we can see that with the exception of $(\{x\}, \emptyset)$ every weakly meet irreducible partial unary algebra is maximal, i.e. it is a unary algebra. Thus, a partial unary algebra is SI iff it is either $(\{x\}, \emptyset)$, or an SI unary algebra (see 5.4).

6. TOPOLOGICAL SPACES

6.1. Denote Top (Top_1, CR_1) the category of all the topological (T_1-topological, completely regular T_1-, respectively) spaces and all the continuous mappings between them. We shall use the notation $(A, \mathcal{T})$ for a topological space with A as the underlying set and $\mathcal{T}$ as the system of all the open subsets of A.

6.2. Proposition. SI *topological spaces are the* $(A, \{\emptyset, A\})$ *with* card $A \leqq 2$ *and* $(2, \{\emptyset, 1, 2\})$. *There are no* WSI *topological spaces which are not* SI.

Proof. One can easily see that a topological space $(A, \mathcal{T})$ is weakly maximal iff $\mathcal{T}$ is indiscrete. All the indiscrete spaces are maximal. Using 3.6, 3.7 and the fact that $(2, \{\emptyset, 2\})$ is the cogenerator of Top, we can prove that an indiscrete topological space is SI iff the cardinality of its underlying set is less or equal to 2.

Every non-indiscrete topological space $(A, \mathcal{T})$ satisfies the identity $(A, \mathcal{T}) = \bigwedge_{M \in \mathcal{T} \setminus \{\emptyset, A\}} (A, \{\emptyset, M, A\})$. Thus, all the meet irreducibles (moreover, all the weakly meet irreducibles) are either of the form $\{\emptyset, M, A\}$, or maximal. Using the quotient map $(A, \{\emptyset, M, A\}) \to (2, \{\emptyset, 1, 2\})$ for card $A \geqq 3$ one can easily see, by 3.6 and 3.7, that the only non-indiscrete WSI topological space is $(2, \{\emptyset, 1, 2\})$ which is also SI.

6.3. Proposition. *A* T_1*-space* $(A, \mathcal{T})$ *is* SI *in* Top_1 *iff* card $A \leqq 2$.

Proof. 1. Obviously, every T_1-space $(A, \mathcal{T})$ with card $A \leqq 2$ is SI.

2. If $(A, \mathcal{T}_{\max})$ is a topological space with the maximal T_1-topology and card $A > 2$, there exist three distinct points $a, b, c \in A$. Define $m_1 \colon A \to A/\{a, b\}$ ($m_2 \colon A \to A/\{a, c\}$) as a quotient mapping glueing a with b (a with c, respectively). Then (m_1, m_2) is a monomorphic system with no monomorphism. Therefore, by 3.6, $(A, \mathcal{T}_{\max})$ is not SI.

3. One can prove that non-maximal meet irreducibles are of the form $(A, \mathcal{T}_M)$ where $M \subset A$, card $M = 1$, card $A \geqq \omega_0$, $\mathcal{T}_M = \{X \subset A \mid \mathrm{card}\,(A \setminus X) < \omega_0\} \cup \{\emptyset, M\}$.

Using the quotient mapping sending $A \setminus M$ to a single point one can prove, by 3.6, that $(A, \mathcal{T}_M)$ is not SI.

6.4. Proposition. *A* T_1*-space* $(A, \mathscr{T})$ *is* WSI *in* Top_1 *iff either* card $A \leqq 2$, *or* A *is infinite and* $\mathscr{T}$ *is a maximal* T_1*-topology.*

Proof. 1. By [7], the equivalence SI $\Leftrightarrow$ WSI holds for finite spaces.

2. Let $(A, \mathscr{T})$ be a T_1-space with A infinite. Then it is weakly maximal iff $\mathscr{T}$ is a maximal T_1-topology, i.e.

$$\mathscr{T} = \{X \subset A \mid \operatorname{card}(A \setminus X) < \omega_0\} \cup \{\emptyset\} .$$

Suppose there is a subobject $\mu: (A, \mathscr{T}) \to \prod_{i \in I} (A_i, \mathscr{T}_i)$, $(A_i, \mathscr{T}_i) \in (A, \mathscr{T}) \urcorner \mathrm{Top}_1$. By (S 5), we can suppose that all $p_i\mu$ are onto (p_i: $\prod_{i \in I} (A_i, \mathscr{T}_i) \to (A_i, \mathscr{T}_i)$ are projections). Thus, card $A_i <$ card A for every $i \in I$. If there exists an $i_0 \in I$ such that card $A_{i_0} \geqq 2$ then there exists an $x \in A_{i_0}$ such that $B = (p_{i_0})^{-1}(x)$ is infinite. Then, for $f = p_{i_0}\mu$, $\overline{\{x\}} = \overline{f(B)} \supset f(\bar{B}) = f(A) = A_{i_0}$which contradicts the assumption $(A_{i_0}, \mathscr{T}_{i_0}) \in \mathrm{Top}_1$. Hence, card $A_i \leqq 1$ for any $i \in I$ and card $A \leqq 1$ which is a contradiction. Therefore, $(A, \mathscr{T})$ is WSI.

3. One can prove that an infinite space $(A, \mathscr{T})$ with a nonmaximal T_1-topology is weakly meet irreducible iff there exists $M \subset A$ with card $M = 1$ or card $M \geqq \omega_0$ such that $\operatorname{card}(A \setminus M) \geqq \omega_0$ and $\mathscr{T} = \mathscr{T}_M = \{X \subset A \mid \operatorname{card}(A \setminus X) < \omega_0\} \cup \cup \{X \subset M \mid \operatorname{card}(M \setminus X) < \omega_0\} \cup \{\emptyset\}$. Using 3.7 and the quotient mapping sending $A \setminus M$ to a single point one can see that $(A, \mathscr{T}_M)$ is not WSI.

6.5. Remarks. 1. Top_1 is an example of a category in which concepts of the meet irreducibility and the weak meet irreducibility are not equivalent. ($(A, \mathscr{T}_M) = (A, \mathscr{T}_{M \setminus \{x\}}) \wedge (A, \mathscr{T}_{M \setminus \{y\}})$ for M infinite, $x, y \in M$, $x \neq y$, is weakly meet irreducible but not meet irreducible.)

2. Top_1 is an example of a category in which cardinalities of all the SI objects are finite while cardinalities of the WSI objects are not bounded at all.

6.6. Proposition. *A completely regular* T_1*-space* $(A, \mathscr{T})$ *is* WSI *in* CR_1 *iff either* card $A \leqq 2$, *or* A *is homeomorphic to a one-dimensional subspace of the real unit interval.*

Proof. By 3.7, a completely regular $(A, \mathscr{T})$ with A finite is WSI iff card $A \leqq 2$. According to the Tychonoff theorem every $(A, \mathscr{T})$ can be embedded as a subspace into a power of the real unit interval I. Every 0-dimensional subspace of I can be embedded as a subspace into the Cantor discontinuum. Thus, every infinite WSI has to be a one-dimensional subspace of I.

On the other hand, let A be a one-dimensional subspace of I, $\mu: A \to \prod_J A_j$ a subspace. Then A contains a copy I' of I as a subject. There exists a $j_0 \in J$ such that card $p_{j_0}\,\mu(I') > 1$. Hence, $p_{j_0}\,\mu(I')$ is a locally connected metrizable continuum and it contains a copy of I as a subobject (see e.g. [2]). Thus, $(A, \mathscr{T})$ is WSI.

6.7. Proposition. *A completely regular* T_1*-space* $(A, \mathscr{T})$ *is* SI *in* CR_1 *iff* card $A \leqq 2$.

Proof. If A is finite then (by 3.6) $(A, \mathscr{T})$ is SI iff card $A \leqq 2$. If A is infinite then, by 2.4 and 6.6, it suffices to show that no one-dimensional subspace of I is SI. (It can be done analogously as in 6.3.)

7. NOTES ON REPRESENTABILITY

By the Birkhoff theorem, every algebra of a finite type can be embedded into a product of subdirectly irreducibles. G. Sabidussi proved in [8] that every finite graph has a subdirect representation (in his sense) as well. In [7], it was proved that every finite object of a regular category can be embedded as a subobject into a product of SI objects. We shall show that in general, for semiregular categories, an analogue of the Birkhoff theorem does not hold.

7.1. Example. Define a category $\mathrm{Graph}_{[0,1]}$ of $[0, 1]$-labelled graphs as follows: the objects of $\mathrm{Graph}_{[0,1]}$ are couples (A, v) where $v\colon A \times A \to [0, 1]$ ($[0, 1]$ is the closed unit interval); the morphisms $(A, v) \to (B, w)$ are mappings $f : A \to B$ such that $w(f(a), f(b)) \geqq v(a, b)$ for any $a, b \in A$.

1. Weakly maximal objects are the (A, v) where $v(x, y) = 1$ for any $x, y \in A$. One can easily see, by 3.6 and 3.7, that a weakly maximal (A, v) is WSI iff card $A \leqq 2$ and that all these WSI objects are SI.

2. Let $M = \{(a, b) \in A \times A \mid v(a, b) < 1\} \neq \emptyset$. Then define for any $(a, b) \in M$ and for any r such that $v(a, b) < r < 1$:

$$v_{a,b,r}(x, y) = \max\left(v(x, y), r\right).$$

$(A, v_{a,b,r}) \in (A, v) \urcorner \mathrm{Graph}_{[0,1]}$ and $(A, v) = \bigwedge_{(a,b)\in M} \bigwedge_{v(a,b)<r<1} (A, v_{a,b,r})$. Thus, (A, v) is not WSI.

3. One can easily see that every subobject of a product of WSI objects is maximal. Hence, non-maximal objects have not a subdirect representation in $\mathrm{Graph}_{[0,1]}$.

7.2. Example. *In the category* $S(F)$ *with* F *defined in* 4.4, *every object has a representation with* WSI *objects but* $(\omega_0, F\omega_0 \setminus \{0_{\omega_0}\})$ *has not a representation with* SI *objects.*

Proof. (1) Every maximal object (X, FX) can be embedded as a subobject into a power of $(2, F2)$ because $(2, F2)$ is a cogenerator.

(2) For $(X, FX \setminus \{A\})$ define $\varphi_1\colon (X, FX \setminus \{A\}) \prec (X, FX)$, $\varphi_2\colon (X, FX \setminus \{A\}) \to (A \cup \{X \setminus A\}, F(A \cup \{X \setminus A\}) \setminus \{A\})$ as a factor morphism onto the WSI object. According to (1) every (X, FX) has a subdirect representation and therefore every $(X, FX \setminus \{A\})$ has a subdirect representation as well.

(3) Every $(X, \emptyset)$ with a finite X can be embedded as a subobject into the product $(X, \{0_X\}) \times (1, \emptyset)$ and therefore it has a subdirect representation.

(4) Every $(X, FX \setminus \{0_X\})$ with X infinite is WSI.

(5) Every (X, R) which is not meet irreducible can be embedded into a product of meet irreducibles and by (1)–(4) it has a representation with WSI objects.

(6) On the other hand, suppose that there exists a subobject $\mu: (\omega_0, F\omega_0 \setminus \{0_{\omega_0}\}) \rightarrow \rightarrow \prod_{i \in I} (A_i, R_i)$ where (A_i, R_i) are SI. Then either card $A_i \geqq \omega_0$ and by 4.4, $0_{A_i} \in R_i$, or card $A_i < \omega_0$, hence card $A_i \leqq 2$ and according to the definition of F we have $0_{A_i} \in R_i$ as well. Thus, $0_{\omega_0} \in F\omega_0 \setminus \{0_{\omega_0}\}$ which is a contradiction.

7.3. Remarks. 1. It can be proved (see [9]) that every object has an SI representation e.g. in the following categories: relational systems, hypergraphs, symmetric graphs, topological spaces, preordered sets etc.

2. In CR_1, every object can be embedded as a subobject into a product of objects from $A = \{(\emptyset, \emptyset), (1, \{\emptyset, 1\}), (2, P(2)), I\}$ (I is the real unit interval with the usual topology). The system of SI objects (see 6.7) is too small for generating all the category in the sense of SI while the system of WSI is greater than the minimal system A.

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