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ON THE EFFECT OF THE PERTURBATION OF A NONNEGATIVE MATRIX ON ITS PERRON EIGENVECTOR

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1. INTRODUCTION

The behaviour of the Perron (eigen-) root of an $n \times n$ nonnegative matrix A when some of its entries are positively perturbed is well known and well documented, e.g. [1], [5] and [6]. Rather less seems to be known about the behaviour of a corresponding Perron (eigen-) vector, or, to be more precise, the behaviour of its components when A is subjected to such a change. Since the Perron vector often has useful interpretations, for example in steady state distributions, priorization [4], in niche overlap [3], etc. it seems natural to investigate this problem.

Frequently in practical situation (as in certain instances of those mentioned above) the nonnegative matrix A is irreducible. From the Perron-Frobenius theory it is known that for such matrices the Perron root is positive and simple, and hence the Perron vectors, all of which are positive, are multiples by a positive scalar of each other. Section 2 will commence with a result (Theorem 2.1), on the topic of the title, for the case where A is nonnegative and irreducible. However, no assumptions concerning the adoption of a normalization strategy towards the (possible) Perron vectors before and after the perturbation will be made. With some re-arrangements of the inequalities obtained in Theorem 2.1, a corollary shows that the conclusions of the theorem hold when the condition that A be irreducible is replaced by the weaker conditions that: the Perron roots of A and its positive perturbation, say $\hat{A}$, are simple, but that the Perron roots of the nonnegative matrices sandwiched between A and $\hat{A}$ are not necessarily simple. Hence there is some justification in viewing the results of the corollary as being *global* (in nature).

Suppose that λ is a simple eigenvalue of an $n \times n$ (complex or real) matrix C. Then for any $n \times n$ matrix D, there exists a number $\varepsilon > 0$, possibly dependent on D, such that the matrix $C_\delta := C + \delta D$ has a simple eigenvalue λ_δ for each $\delta \in [-\varepsilon, \varepsilon]$. Moreover, if for each δ in this range the same normalization scheme is applied to an eigenvector corresponding to λ_δ, then both λ_δ and the components of the associated

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normalized eigenvector are analytic functions of δ in some open interval containing $[-\varepsilon, \varepsilon]$, e.g. Wilkinson [7, pp. 66–67]. In Section 3, under the assumption that A is an $n \times n$ nonnegative matrix whose Perron root is simple, we exploit this result to show (c.f. Theorem 3.1) that while (as mentioned previously) the Perron root of A is a *nondecreasing* function of a positive perturbation, in a (one sided) 'neighbourhood' of A whose size may depend on the particular perturbation, the entries of the corresponding normalized Perron vector are *nonincreasing* functions of a positive perturbation. We prove this *local* (in nature) result using an analytical approach (as, indeed, the result from [7] suggests would be the case). Subsequently, some of the intermediate relations derived in the course of proving Theorem 3.1, are used to study various specific cases of perturbations. These cases all fall within the framework of positive perturbations in one row of the matrix A, *which is the only type of modification considered in Sections* 2 *and* 3.

In the final section, 4, we present several observations concerning the perturbation in more than one row of a nonnegative matrix A whose Perron root is simple. Furthermore, mention of the difficulties in sharpening these observations is made.

Finally, the following notations will be required. The ith n-dimensional unit coordinate vector will be denoted by $e^{(i)}$. For an n-dimensional (column) vector x, x^{T} will denote the *transpose* of x, and for an $n \times n$ matrix E, $\varrho(E)$ denotes its *spectral radius*.

2. THE IRREDUCIBLE CASE AND OTHER INITIAL OBSERVATIONS

Theorem 2.1. *Let A be an $n \times n$ nonnegative irreducible matrix. Then for any nonnegative n-vector $v \neq 0$,*

$$\frac{y_i}{x_i} > \frac{y_k}{x_k}; \quad k \neq i, \quad 1 \leqq k \leqq n, \tag{2.1}$$

where $x = (x_1, \ldots, x_n)^{\mathrm{T}}$ and $y = (y_1, \ldots, y_n^{\mathrm{T}})$ are (positive) Perron vectors of A and $A + e^{(i)}v^{\mathrm{T}}$, respectively.

Proof. Suppose there exists an index $m \neq i$, $1 \leqq m \leqq n$, such that

$$\frac{y_m}{x_m} = \max_{1 \leqq k \leqq n} \frac{y_k}{x_k}.$$

Set $\varrho := \varrho(A)$ and $\tilde{\varrho} := \varrho(A + e^{(i)}v^{\mathrm{T}})$. Because A is irreducible and $e^{(i)}v^{\mathrm{T}} \geqq 0$, $e^{(i)}v^{\mathrm{T}} \neq 0$, $0 < \varrho < \tilde{\varrho}$. Hence

$$0 < \varrho \frac{y_m}{x_m} < \tilde{\varrho} \frac{y_m}{x_m} = \frac{1}{x_m} \sum_{k=1}^{n} a_{mk} \frac{y_k}{x_k} x_k \leqq \frac{y_m}{x_m} \sum_{k=1}^{n} a_{mk} \frac{x_k}{x_m} = \varrho \frac{y_m}{x_m},$$

which is not possible. Thus the inequalities in (2.1) are valid. □

Examples of dimension $n \geqq 4$ can be constructed to show that the largest *relative* change in the component of a Perron vector corresponding to the row in A which has been positively perturbed does not imply the largest *absolute* change for that component, that is, it does not follow from (2.1) that

$$y_i - y_k > x_i - x_k : k \neq i, \quad 1 \leqq k \leqq n .$$

We note, however, that if both Perron vectors x and y were to be normalized so that $x_i = y_i = 1$, then Theorem 2.1 implies that

$$x_k > y_k, \quad k \neq i, \quad k = 1, \ldots, n, \tag{2.2}$$

a manifestation which will be more deeply understood from the analytical approach of the next section.

If in Theorem 2.1 the assumption that A is irreducible is replaced by the assumption that the Perron roots of A and $A + e^{(i)}v^{\mathrm{T}}$ are simple, then the strict inequalities in (2.1) have to be weakened to accommodate the possibility that no greater relative change occurs in the ith component over all other components. Moreover, since some of the entries in the Perron vectors may well be zero, these inequalities have to be re-arranged to meet such an eventuality.

Corollary 2.2. *Let A be an $n \times n$ nonnegative matrix whose Perron root is simple and suppose that $v \neq 0$ is a nonnegative n-vector for which the Perron root of $A + e^{(i)}v^{\mathrm{T}}$ is simple. Let $x = (x_1, \ldots, x_n)^{\mathrm{T}}$ and $y = (y_1, \ldots, y_n)^{\mathrm{T}}$ be Perron roots of A and $A + e^{(i)}v^{\mathrm{T}}$, respectively. Then*

$$x_k y_i \geqq x_i y_k \tag{2.3}$$

for all $k = 1, 2, \ldots, n$.

Proof. Let J be the $n \times n$ matrix whose entries are all unity and for $\delta \in [0, \infty)$ set

$$A_\delta := A + \delta J \quad \text{and} \quad \tilde{A}_\delta := A_\delta + e^{(i)}v^{\mathrm{T}} .$$

Furthermore, for any δ in this range let $x(\delta) := (x_1(\delta), \ldots, x_n(\delta))^{\mathrm{T}}$ and $y(\delta) = (y_1(\delta), \ldots, y_n(\delta))^{\mathrm{T}}$ be Perron vectors of A_δ and $\tilde{A}_\delta$, respectively. Then for $\delta > 0$, A_δ is irreducible and so, by (2.1)

$$x_k(\delta)\, y_i(\delta) > x_i(\delta)\, y_k(\delta)$$

for all $k \neq i$, $1 \leqq k \leqq n$. Since the Perron root of A is simple, inequalities (2.3) follow by letting $\delta \to 0$. □

The following observations are now possible.

Proposition 2.3. *Under the assumption (and notations) of Corollary 2.2, if $y_i = 0$ then $y = \alpha x$ for some positive number α, whereas if $x_i = 0$ then inequalities (2.3) are trivially satisfied.*

Proof. Clearly the second observation follows from an inspection of (2.3). Suppose then that $y_i = 0$ (but that we have no prior knowledge concerning x_i). Without loss of generality (WLOG) we may assume that $i = 1$. Let $y^T = (0, \eta^T)$, where η is an $(n - 1)$-vector and partion the matrix A as follows:

$$A = \begin{bmatrix} a_{11} & a^T \\ b & B \end{bmatrix}, \tag{2.4}$$

where a, b and B are $(n - 1)$-vectors and an $(n - 1) \times (n - 1)$ matrix respectively. Since $y_1 = 0$, it readily follows that

$$B\eta = \varrho(A + e^{(1)}v^T)\,\eta\,.$$

But then, as η is necessarily a nonzero vector, $\varrho(A + e^{(1)}v^T)$ is an eigenvalue of B, showing that

$$\varrho(A + e^{(1)}v^T) \leqq \varrho(B)\,. \tag{2.5}$$

However, since B is a principal submatrix of A,

$$\varrho(B) \leqq \varrho(A) \leqq \varrho(A + e^{(1)}v^T)\,. \tag{2.6}$$

Hence, from (2.5) and (2.6) we have that

$$\varrho(A) = \varrho(A + e^{(1)}v^T)\,. \tag{2.7}$$

Set $\varrho := \varrho(A + e^{(1)}v^T)$ and consider the eigenvalue-eigenvector relationship

$$(A + e^{(1)}v^T)\,y = \varrho y\,. \tag{2.8}$$

Since $e^{(1)}v^T$ is a nonnegative matrix and since $\varrho y_1 = 0$, it follows that $(e^{(1)}v^T)\,y = 0$, and so (2.8) reduces to the equality

$$Ay = \varrho y\,. \tag{2.9}$$

Whence, because the Perron root of A is simple (and as $y \neq 0$ is a nonnegative vector), (2.7) and (2.9) show that there exists a positive constant α such that

$$y = \alpha x$$

and the proof is complete. □

Proposition 2.3 suggests that of more interest is the study of the effect of a positive perturbation on a Perron vector in the event that the entry of the eigenvector corresponding to the row of the yet unperturbed matrix is nonzero and hence positive. (Such is the case, of course, when A is irreducible.)

3. AN ANALYTICAL INVESTIGATION

As in the proof of Observation (2.3), in this section we shall continue to assume, WLOG, that the positive perturbation occurs in the first row of the matrix. More importantly, we shall suppose throughout, that *whenever the first component of* *any Perron vector is nonzero, then that eigenvector has already been normalized so that its first entry equals unity.* We shall refer to this normalized eigenvector as *the* Perron vector.

With Proposition 2.3 and the comment that followed in mind, we now proceed to prove one of our principal results.

Theorem 3.1. *Let A be an $n \times n$ nonnegative matrix whose Perron root is simple and such that the first entry of (any of) its Perron vectors is nonzero, and assume that $v \neq 0$ is a nonnegative vector. Then there exists a positive number ε_0, possibly dependent on v, such that*

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon} z_k(\varepsilon) \leqq 0 \tag{3.1}$$

for each $\varepsilon \in [0, \varepsilon_0]$ and for each $1 \leqq k \leqq n$, where for each $\varepsilon \in [0, \varepsilon_0]$, $z(\varepsilon) = (z_1(\varepsilon), \ldots, z_n(\varepsilon))^{\mathrm{T}}$ denotes the Perron vector of

$$A_\varepsilon := A + \varepsilon e^{(1)} v^{\mathrm{T}} .$$

Proof. We first note that for any $\varepsilon > 0$ the matrix $A + \varepsilon e^{(1)} v^{\mathrm{T}}$ cannot have a Perron vector whose first component is zero, otherwise, by Proposition 2.3, $z_1(0) = 0$, which contradicts our assumptions. Next, as indicated in the Introduction, because the Perron root of A is simple, there exists a sufficiently small ε_0 such that in $[0, \varepsilon_0]$ both $\bar{\varrho}(\varepsilon) := \varrho(A_\varepsilon)$ and $z(\varepsilon)$ (viewed as a simple eigenvalue and a corresponding normalized eigenvector) are differentiable with respect to ε.

For $\varepsilon \in (0, \varepsilon_0]$ consider the relationship

$$(A_\varepsilon - \bar{\varrho}(\varepsilon) I)\, z(\varepsilon) = 0 . \tag{3.2}$$

Then upon differentiating (3.2) with respect to ε in this range, one obtains

$$(A_\varepsilon - \bar{\varrho}(\varepsilon) I)\, z'(\varepsilon) + (e^{(1)} v^{\mathrm{T}} - \bar{\varrho}'(\varepsilon) I)\, z(\varepsilon) = 0 . \tag{3.3}$$

Let $l^{\mathrm{T}}(\varepsilon)$ be the left Perron vector of A_ε, $\varepsilon \in [0, \varepsilon_0]$. It is well known that $l^{\mathrm{T}}(\varepsilon)\, z(\varepsilon) \neq 0$ throughout $[0, \varepsilon_0]$, so that premultiplying (3.3) by $l^{\mathrm{T}}(\varepsilon)$, $\varepsilon \in (0, \varepsilon_0]$, and solving for $\bar{\varrho}'(\varepsilon)$ we obtain that

$$\bar{\varrho}'(\varepsilon) = \frac{l^{\mathrm{T}}(\varepsilon)\, e^{(1)} v^{\mathrm{T}}\, z(\varepsilon)}{l^{\mathrm{T}}(\varepsilon)\, z(\varepsilon)} . \tag{3.4}$$

Next, partition A as in (2.4) and write

$$(z(\varepsilon))^{\mathrm{T}} = (1, (w(\varepsilon))^{\mathrm{T}}) \quad \text{and} \quad (z'(\varepsilon))^{\mathrm{T}} = (0, (w'(\varepsilon))^{\mathrm{T}}) , \tag{3.5}$$

where $w(\varepsilon)$ is an $(n-1)$-vector. Then from (3.3), (2.4) and (3.5) we see that

$$(B - \bar{\varrho}(\varepsilon) I)\, w'(\varepsilon) = \bar{\varrho}'(\varepsilon)\, w(\varepsilon) , \tag{3.6}$$

where, here, I is the identity matrix of order $n - 1$. Clearly

$$\varrho(B) \leqq \varrho(A) = \bar{\varrho}(0) \leqq \bar{\varrho}(\varepsilon) \tag{3.7}$$

for every $\varepsilon \in [0, \varepsilon_0]$. *If* for all $\varepsilon \in (0, \varepsilon_0]$, $\bar{\varrho}(\varepsilon) > \varrho(B)$, then $(B - \bar{\varrho}(\varepsilon) I)^{-1}$ exists and is a nonpositive matrix. Thus as $w(\varepsilon) \geqq 0$ and since $\bar{\varrho}'(\varepsilon) \geqq 0$ (because $\bar{\varrho}(\varepsilon)$ is a nondecreasing function in the interval $[0, \varepsilon_0]$ and as also confirmed by inspection of (3.4)), we obtain at once from (3.6) that $w'(\varepsilon) \leqq 0$, so that, since $z_1(\varepsilon) = 1$ for all $\varepsilon \in (0, \varepsilon_0]$, (3.1) is readily seen to be satisfied.

Suppose, then, that $\bar{\varrho}(\varepsilon) \not> \varrho(B)$ for all $\varepsilon \in (0, \varepsilon_0]$ and let

$$\varepsilon_* = \sup \{\varepsilon \in (0, \varepsilon_0] \mid \bar{\varrho}(\varepsilon) = \varrho(B)\} .$$

If $\varepsilon_* < \varepsilon_0$, then exactly as in the preceding paragraph, we show that (3.1) is valid in $(\varepsilon_*, \varepsilon_0]$, while in $(0, \varepsilon_*]$, from (3.7) we have that

$$\varrho: = \bar{\varrho}(0) = \varrho(A) = \varrho(B) = \bar{\varrho}(\varepsilon) . \tag{3.8}$$

Thus $\bar{\varrho}'(\varepsilon) = 0$ in $(0, \varepsilon_*]$, in which case

$$w'(\varepsilon) \in \ker (B - \varrho I)$$

for each ε in this interval by (3.6). Now (3.8), (3.2), (2.4) and the first expression in (3.5) imply that

$$b + (B - \varrho I) w(0) = 0 . \tag{3.9}$$

But as ϱ is a simple eigenvalue of A, it follows that rank $(A - \varrho I) = n - 1$, so that upon taking account of (3.9) (and (2.4)), a simple rank 'analysis' shows that rank $(B - \varrho I) = n - 2$. Hence $\ker (B - \varrho I)$ is one dimensional and is therefore spanned by a Perron vector of B, call it $\bar{u}$. Since $w'(\varepsilon) \in \ker (B - \varrho I)$ for each $\varepsilon \in (0, \varepsilon_*]$, $w'(\varepsilon)$ must be a multiple by a scalar, possibly dependent on ε, of $\bar{u}$; that is

$$w'(\varepsilon) = \mathscr{s}(\varepsilon)\bar{u} \tag{3.10}$$

in $(0, \varepsilon_*]$, where $\mathscr{s}(\varepsilon)$ is some well defined function on this domain. Consider now the first component of the n-vectors on both sides of (3.3). Since $\bar{\varrho}'(\varepsilon) = 0$ in $(0, \varepsilon_*]$, we deduce, using the notations of (2.4) and (3.5), that

$$a^{\mathrm{T}} w'(\varepsilon) + \varepsilon \bar{v}^{\mathrm{T}} w'(\varepsilon) + v^{\mathrm{T}} z(\varepsilon) = 0 , \tag{3.11}$$

where $\bar{v}$ is the $(n - 1)$-vector formed from the second through the nth component of v. For any $\varepsilon \in (0, \varepsilon_*]$ for which $v^{\mathrm{T}} z(\varepsilon) > 0$, (3.11) implies that $(a^{\mathrm{T}} + \varepsilon \bar{v}^{\mathrm{T}}) w'(\varepsilon) <$ < 0. But then, since $a^{\mathrm{T}} + \varepsilon \bar{v}^{\mathrm{T}} \geqq 0$, $w'(\varepsilon)$ must have at least one component which is negative. Whence, because $\bar{u}$ is a nonzero nonnegative vector, it is readily argued from (3.10) that $w'(\varepsilon) \leqq 0$. Suppose therefore that for some $\tilde{\varepsilon} \in (0, \varepsilon_*]$, $v^{\mathrm{T}} z(\tilde{\varepsilon}) = 0$, and consider the function

$$\phi(\varepsilon): = v^{\mathrm{T}} z(\varepsilon) \tag{3.12}$$

in $(0, \varepsilon_*]$. Clearly, $\phi(\varepsilon)$ as a scalar product of two nonnegative vectors is a nonnegative function. Furthermore, by the assumptions of the theorem, $\phi(\varepsilon)$ is differentiable in the interval $(0, \varepsilon_*]$, with

$$\phi'(\varepsilon) = v^{\mathrm{T}} z'(\varepsilon) = \alpha(\varepsilon)\, \bar{v}^{\mathrm{T}} \bar{u} \tag{4.13}$$

by (3.5) and (3.10), so that since $\phi(\tilde{\varepsilon}) = 0$, we must have that $\phi'(\tilde{\varepsilon}) \leqq 0$. If $\bar{v}^{\mathrm{T}}\bar{u} > 0$, then, evidently, from (3.13) and (3.10), $w'(\tilde{\varepsilon}) \leqq 0$. If, however, $\bar{v}^{\mathrm{T}}\bar{u} = 0$, then by (3.13) $\phi'(\varepsilon) = 0$ throughout the interval $(0, \tilde{\varepsilon}]$, in which case ϕ is a constant in this range and so, because $\phi(\tilde{\varepsilon}) = 0$,

$$\phi(\varepsilon) = v^{\mathrm{T}} z(\varepsilon) = 0$$

for all $\varepsilon \in (0, \tilde{\varepsilon}]$. We have thus shown that for any $\varepsilon \in (0, \tilde{\varepsilon}]$,

$$A\, z(\varepsilon) = (A + \varepsilon e^{(1)} v^{\mathrm{T}})\, z(\varepsilon) = \bar{\varrho}(\varepsilon)\, z(\varepsilon) = \varrho\, z(\varepsilon)\,, \tag{3.14}$$

where the last equality in (3.14) follows from (3.8). But then, since the Perron root of A is simple (and since the same normalization strategy is applied throughout) it follows that $z(\varepsilon) = z(0)$ for all $\varepsilon \in (0, \tilde{\varepsilon}]$. Hence in this interval $w'(\varepsilon) = 0$ and so (3.1) holds (also) in the case when $\bar{\varrho}(\varepsilon) = \varrho(B)$ for some $\tilde{\varepsilon} \in (0, \varepsilon_0]$. □

Theorem 3.1 exhibits that with an appropriate eigenvector normalization scheme, nonnegative matrices whose Perron roots are simple, when positively perturbed, manifest a 'pay-off' relationship between the Perron root and the Perron vector – of the form described in the introductory section.

Some of the expressions developed in the course of proving the theorem can be exploited to obtain a further insight into the nature of the decrease (or lack of) of the different components of the Perron vector in various situations. We shall consider three cases. For the sake of convenience, we retain (in all cases) the notations used in Theorem 2.

Case 1. *Suppose that* $v^{\mathrm{T}} z(0) = 0$. Since $z_k(0) \geqq z_k(\varepsilon)$ for all $\varepsilon \in [0, \varepsilon_0]$ and for all $1 \leqq k \leqq n$, it follows that $v^{\mathrm{T}} z(\varepsilon) = 0$ for all ε in this range. But then from (3.4) $\varrho := \bar{\varrho}(0) = \bar{\varrho}(\varepsilon) =$ const. throughout the interval. Moreover, as in (3.14), we observe that for each $\varepsilon \in [0, \varepsilon_0]$,

$$A\, z(\varepsilon) = \varrho\, z(\varepsilon)\,,$$

implying, by the simplicity of the Perron root of A (and the previously mentioned normalization of Perron vectors) that

$$z(0) = z(\varepsilon) \quad \text{for all} \quad \varepsilon \in [0, \varepsilon_0]\,.$$

Case 2. *Suppose that* $v^{\mathrm{T}} z(0) > 0$, *that the Perron root of* $A + e^{(1)} v^{\mathrm{T}}$ *(as well as that of* A*) is simple and that* $\varrho := \varrho(A) = \varrho(A + e^{(1)} v^{\mathrm{T}})$. Under the latter assumption it is evident that $\bar{\varrho}(\varepsilon)$ (when considered as an eigenvalue of some small, not necessarily positive, perturbation of A) is constant in some neighbourhood of $\varepsilon = 0$, so that from (3.4), $l^{\mathrm{T}}(0)\, e^{(1)} = 0$. Whence, there exists a nonzero n-vector x such that

$$(3.15) \qquad (A - \varrho I)\, x = e^{(1)}\,,$$

yielding the factorization

$$(A - \varrho I + e^{(1)} v^{\mathrm{T}}) = (A - \varrho I)\,(I + x v^{\mathrm{T}})\,.$$

Thus, if y is a vector which satisfies

$$(3.16) \qquad (I + x v^{\mathrm{T}})\, y = z(0)\,,$$

then

$$(A - \varrho I + e^{(1)} v^{\mathrm{T}})\, y = 0\,,$$

so that, because the Perron root of $A + e^{(1)} v^{\mathrm{T}}$ is simple, $y = \alpha\zeta = \zeta$, where $\zeta = (\zeta_1, \ldots, \zeta_n)^{\mathrm{T}}$ denotes the Perron vector of $A + e^{(1)} v^{\mathrm{T}}$. It follows from (3.16) that

$$(3.17) \qquad \zeta = z(0) - (v^{\mathrm{T}}\zeta)\, x\,.$$

Partition the vector x into $x^{\mathrm{T}} = (0, \bar{x}^{\mathrm{T}})$, where $\bar{x}$ is an $(n-1)$-vector, in which case from (3.15) and (2.4) we see that $\bar{x} \in \ker(B - \varrho I)$. (Whence, additionally $\varrho = \varrho(B)$.) We first note that $v^{\mathrm{T}}\zeta \neq 0$, otherwise if $v^{\mathrm{T}}\zeta = 0$ then (3.17) would reduce to $\zeta = z(0)$, implying that $v^{\mathrm{T}}\zeta = v^{\mathrm{T}} z(0) > 0$, which is at once a contradiction. Next, as $\zeta_1 = z_1(0) = 1$, it follows from (2.3) that $z_k(0) \geqq \zeta_k$ for $k = 2, \ldots, n$, so that by (3.17) $x^{\mathrm{T}} = (0, \bar{x}^{\mathrm{T}})$ is a nonnegative vector. Finally, upon premultiplying both sides of (3.17) by v^{T} and substituting in that same equation the expression obtained for $v^{\mathrm{T}}\zeta$, we obtain that

$$\zeta = z(0) - \frac{v^{\mathrm{T}} z(0)}{1 + v^{\mathrm{T}} x}\, x\,.$$

The (numerical) coefficient of x in this equation is positive, and hence, with the exception of the first component of ζ which is the same as that of $z(0)$, the *question* for which indices k lying between 2 and n,

$$(3.18) \qquad z_k(0) > \zeta_k\,,$$

entirely depends on the nonzero entries of $\bar{x}$, and hence on the nonzero entries of a Perron vector of B which is unique up to a multiple by a positive constant. In any event the example

$$\begin{bmatrix} \frac{1}{2} & (\frac{1}{2}) + \varepsilon & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1/(1 + 2\varepsilon) \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1/(1 + 2\varepsilon) \\ 1 \end{bmatrix}$$

illustrates that for no k, $2 \leqq k \leqq n$, (3.18) need hold.

Case 3. *Suppose that $v^{\mathrm{T}} z(0) > 0$ but $\bar{\varrho}(\varepsilon)$ is not a constant in an interval $[0, \varepsilon_0]$ in which A_ε has a simple Perron root.* Then there exists an $\varepsilon_1 \in (0, \varepsilon_0)$ such that simultaneously $\bar{\varrho}(\varepsilon_1) > \bar{\varrho}(0) = \varrho(A) \geqq \varrho(B)$ and $\bar{\varrho}'(\varepsilon_1) > 0$. It follows from (3.6) that

$$(3.19) \qquad -w'(\varepsilon_1) = \bar{\varrho}'(\varepsilon_1)\left(\bar{\varrho}(\varepsilon_1) I - B\right)^{-1} w(\varepsilon) \geqq \frac{\bar{\varrho}'(\varepsilon_1)}{\bar{\varrho}(\varepsilon_1)} w(\varepsilon),$$

where the inequality on the RHS of (3.19) has been obtained by the Neumann expansion of $(\varrho(\varepsilon_1) I - B)^{-1}$. From (3.5) and the fact that in (3.19) $\bar{\varrho}'(\varepsilon_1)/\bar{\varrho}(\varepsilon_1) > 0$ it follows that for each index k, $2 \leqq k \leqq n$, for which $z_k(0) > 0$, $z_k(0) > z_k(\varepsilon_0)$.

Evidently, Case 3 together with Theorem 3.1 provides us with a further, and perhaps deeper, proof of Theorem 2.1 as well as of (2.2), since if A is irreducible then $A + \varepsilon e^{(1)} v^{\mathrm{T}}$ (is irreducible for all $\varepsilon > 0$ and hence) has a simple Perron root in $[0, \varepsilon)$ and $z_k(0) > 0$ for all $2 \leqq k \leqq n$. We further remark that the results in Case 3 could be sligthly strengthened by the observation that if $\varepsilon > 0$ but $\varepsilon \notin [0, \varepsilon_0]$ and $\varrho(A_\varepsilon) > \varrho(B)$, then necessarily the Perron root of A_ε is simple.

In the next section we shall present some observations relating to the perturbation of a nonnegative matrix whose Perron root is simple in more than one row. The difficulties arising in the analysis of these more general perturbations will be raised.

4. CONCLUDING OBSERVATIONS AND REMARKS

We begin with a generalization of Theorem 2.1.

Observation 4.1. *Let A be an $n \times n$ nonnegative irreducible matrix and let $\mathcal{J}$ be a nonempty subset of $\{1, 2, \ldots, n\}$. For an index $i \in \mathcal{J}$, denote by $v_{(i)}$ any nonzero nonnegative n-vector. Let $x = (x_1, \ldots, x_n)^{\mathrm{T}}$ and $y = (y_1, \ldots, y_n)^{\mathrm{T}}$ denote, respectively, Perron vectors of A and $A + \sum_{i \in \mathcal{J}} e^{(i)} v_{(i)}^{\mathrm{T}}$. Then for any index $j \notin \mathcal{J}$,*

$$(4.1) \qquad \frac{y_j}{x_j} < \max_{i \in \mathcal{J}} \frac{y_i}{x_i}.$$

Proof. Assume that there exists an index $m \notin \mathcal{J}$ such that

$$\frac{y_m}{x_m} = \max_{1 \leqq k \leqq n} \frac{y_k}{x_k}.$$

Then exactly as in the proof of Theorem 2.1 a contradiction is obtained. □

The problem of identifying the index $i_0 \in \mathcal{J}$ for which the ratio on the RHS of (4.1) is maximal does not seem to be simple, and examples can be constructed to show various modes of behaviour. Thus further investigation is needed to obtain a deeper insight into this problem.

An 'appropriate' re-arrangement of the inequalities in (4.1) allows (just as in Theorem 2.1) a weakening of the condition that A in the above observation is irreducible to the conditions that both A and $A + \sum e^{(i)} v_{(i)}^{\mathrm{T}}$ possess a simple Perron root.

Observation 4.2. *Let A be an $n \times n$ nonnegative irreducible matrix and let u and v be nonzero nonnegative n-vectors. Suppose that the matrix*

$$\hat{A} := A + e^{(i)}u^{\mathrm{T}} - e^{(j)}v^{\mathrm{T}}$$

is nonnegative and irreducible, and let ϱ and $x = (x_1, \ldots, x_n)^{\mathrm{T}}$ and $\hat{\varrho}$ and $y = (y_1, \ldots, y_n)^{\mathrm{T}}$ denote the Perron roots and Perron vectors of A and $\hat{A}$, respectively. Then

(i) *if $\varrho \leqq \hat{\varrho}$, then*

$$\frac{y_k}{x_k} < \frac{y_i}{x_i}$$

for all $k \neq i$, $1 \leqq k \leqq n$, while

(ii) *if $\hat{\varrho} \leqq \varrho$, then*

$$\frac{y_k}{x_k} > \frac{y_j}{x_j}$$

for all $k \neq j$, $1 \leqq k \leqq n$.

Whence, if $\varrho = \hat{\varrho}$, then

$$\frac{y_j}{x_j} < \frac{y_k}{x_k} < \frac{y_i}{x_i}$$

for all $k \neq i, j$; $1 \leqq k \leqq n$.

Proof. Without loss of generality we may assume that $i = 1$, $j = 2$ and $x_1 = y_1 = 1$. Partition the vectors x, y, v and $e^{(2)}$ as $x^{\mathrm{T}} = (1, \bar{x}^{\mathrm{T}})$, $y^{\mathrm{T}} = (1, \bar{y}^{\mathrm{T}})$, $v^{\mathrm{T}} = (v_1, \bar{v}^{\mathrm{T}})$ and $(e^{(2)})^{\mathrm{T}} = (0, \bar{e}_2^{\mathrm{T}})$ and consider the partition of A given in (2.4). (We shall only prove here Statement (i), as the proof of Statement (ii) follows along similar lines.) From the relation $Ax = \varrho x$ we obtain that

$$\bar{x} = (\varrho I - B)^{-1},$$

and from $\hat{A}y = \hat{\varrho}y$ we have that

$$(b - v_1\bar{e}_2) \,.\, 1 + (B - \bar{e}_2\bar{v}^{\mathrm{T}})\,\bar{y} = \hat{\varrho}\bar{y}\,.$$

From the second of these equalities we further obtain that

$$\bar{y} = (\hat{\varrho}I - B + \bar{e}_2\bar{v}^{\mathrm{T}})^{-1}(b - v_1\bar{e}_2)\,, \tag{4.2}$$

where here I denotes the identity matrix of order $n - 1$. Since $\varrho I - B$ and $\hat{\varrho}I - B + \bar{e}_2 v^{\mathrm{T}}$ are nonsingular M-matrices (or in the language of Fiedler and Pták [2], matrices in class K) such that

$$\varrho I - B \leqq \hat{\varrho}I - B + \bar{e}_2\bar{v}^{\mathrm{T}}\,,$$

componentwise, by [2, Thm. 4.2]

$$0 \leqq (\hat{\varrho}I - B + \bar{e}_2\bar{v}^{\mathrm{T}})^{-1} \leqq (\varrho I - B)^{-1}\,, \tag{4.3}$$

and hence

$$\bar{y} \leqq (\varrho I - B)^{-1} (b - v_1 \bar{e}_2) \leqq (\varrho I - B)^{-1} b = \bar{x} . \tag{4.4}$$

If $v_1 \neq 0$ then the last inequality is strict because $(\varrho I - B)^{-1} > 0$ due to irreducibility of B. While if $v_1 = 0$, then $\bar{v} \neq 0$, so that $\bar{e}_2 \bar{v}^T$ is not the zero matrix, then the second inequality in (4.3) is componentwise strict, showing that $\bar{y} < \bar{x}$ by (4.2), (4.3) and (4.4). □

The adaptation of the statement of Observation 4.2 to the case where the irreducibility of the matrices A and $A + e^{(i)}u^T - e^{(j)}v^T$ is replaced by the assumption that their respective Perron root is simple is not quite obvious. It is easy to prove that

$$\varrho < \hat{\varrho} \rightrightarrows y_k x_i \leqq y_i x_k , \quad k = 1, \ldots, n$$

or that

$$\varrho > \hat{\varrho} \rightrightarrows y_k x_j \geqq y_j x_k , \quad k = 1, \ldots, n ,$$

but counterexamples show that for $\varrho = \hat{\varrho}$ no such statement is possible.

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