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SEMI-VALUATIONS AND d-GROUPS

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1. INTRODUCTION

If A is an integral domain with the quotient field K, the group of divisibility $G(A)$ of A is a partially ordered group $K^*/U(A)$, where K^* is the multiplicative group of K and $U(A)$ denotes the group of units of A, with $a\,U(A) \leqq b\,U(A)$ if and only if $ba^{-1} \in A$. The canonical map $w_A : K \to G(A) \cup \{\infty\}$ with $w_A(x) = \infty$ if and only if $x = 0$ is called a *semi-valuation*. By using this map it is posible to determine several properties of A; for example, a domain A is quasi-local if and only if the inequality $w_A(x) < w_A(y)$ implies $w_A(x + y) = w_A(x)$ for every $x, y, x + y \in K^*$ (see [9]).

On the other hand, there are properties of integral domains which cannot be expressed in terms of properties of groups of divisibility; for example, the property "*A is a Bezout domain*" cannot be determined in terms of $G(A)$ only. Moreover, for several properties of A we do not know if they are equivalent to some properties of $G(A)$, particularly the property "*A is integrally closed*". (It should be noted that there is an error in the note in [3] after Corollary 9.)

In [3] we introduced a method that enables us to determine some properties of a domain A by using the properties of $G(A)$ and a special multivalued addition $\oplus_A$ on $G(A)$ which was substantially dependent on the addition on A. In this paper in Part 3 we continue an investigation of the relation between the properties of $\oplus_A$ and A and, moreover, we do not assume, in general, that a multivalued addition on $G(A)$ is constructed using the addition on A.

In Part 4 we show the proof of the result of R. L. Pendleton that was mentioned without any proof in [9]; namely, that the only directed orders on the group $\boldsymbol{Z}$ of integers which produce groups of divisibility are the two obtained by taking as positive elements either $\boldsymbol{Z}_+$ or $\boldsymbol{Z}_-$.

2. DEFINITIONS AND BASIC FACTS

In this paper all rings and groups are commutative integral domains and abelian groups.

The main tool in this paper is a special partially ordered group, called a d-group, which was introduced by T. Nakano [8]. In order to make this paper self-contained we repeat some basic facts about d-groups (see [8], [3], [5]).

A *d-group* is a partially ordered group $(G, \cdot, \leqq)$ with an element $\infty \notin G$ which admits a multivalued addition $\oplus$ such that

(1) $\alpha \oplus \beta = \beta \oplus \alpha$,

(2) $\alpha \oplus (\beta \oplus \delta) = (\alpha \oplus \beta) \oplus \delta$,

(3) $\alpha \in \beta \oplus \delta$ implies $\beta \in \alpha \oplus \delta$,

(4) $\alpha . (\beta \oplus \delta) = \alpha . \beta \oplus \alpha . \delta$,

(5) $\infty \in \alpha \oplus \beta$ if and only if $\alpha = \beta$,

(6) $\alpha, \beta \geqq \delta$ and $\omega \in \alpha \oplus \beta$ imply $\omega \geqq \delta$,

(7) $\alpha \oplus \beta \neq \emptyset$,

for any $\alpha, \beta, \delta \in G$.

We set $G_+ = G^+ = \{\alpha \in G : \alpha \geqq 1_G = 1\}$. If G is a d-group, a subset $J \subseteq G_+$ is called an *m-ideal* of G_+ provided that $\alpha \oplus \beta \subseteq J$, $\alpha . \omega \in J$, for any $\alpha, \beta \in J$, $\omega \in G_+$. For a d-group G we say that G_+ is *integrally closed* in G if for elements $\alpha_0, \ldots, \alpha_n \in G_+$, $\xi \in G$, such that

$$\xi^{n+1} \in \alpha_n \xi^n \oplus \ldots \oplus \alpha_1 \xi \oplus \alpha_0 \,,$$

it follows that $\xi \in G_+$.

A subgroup H of a d-group G is called *d-convex* if it is convex and $H . G_+ \oplus \oplus H . G_+ = H . G_+$. Any directed convex subgroup (i.e. an *o-ideal* in G) is d-convex ([8], Lemma 5) and for any d-convex subgroup H of G it is easy to see that the factor group G/H plus the infinity element $\infty = \infty H$ becomes a d-group with respect to the addition

$$\alpha H \mathbin{\oplus^{\sim}} \beta H = (\alpha H \oplus \beta H)/H$$

and the factor ordering.

Further, a d-convex subgroup H of G is *prime* if $\alpha H > \beta H$ implies $\alpha H \mathbin{\oplus^{\sim}} \beta H = \{\beta H\}$. For a d-group G we denote by $\mathfrak{C}(G)$ $(\mathfrak{M}(G), \mathfrak{D}(G))$ the set of all d-convex (directed prime d-convex, directed d-convex, respectively) subgroups of G.

If G, G_1 are d-groups with additions $\oplus$, $\oplus_1$, respectively, a map f of $G \cup \{\infty\}$ in $G_1 \cup \{\infty\}$ is called a *d-homomorphism* if f is an order homomorphism (*o-homomorphism*) and

$$f(\alpha \oplus \beta) \subseteq f(\alpha) \oplus_1 f(\beta) \,,$$

$$f(\delta) = \infty \quad \text{if and only if} \quad \delta = \infty \,;$$

f is called a *d-epimorphism* if it is an order epimorphism (*o-epimorphism*) and for $\delta \in f(\alpha) \oplus_1 f(\beta)$ there exist $\alpha_1, \beta_1 \in G$, $\omega \in \alpha_1 \oplus \beta_1$ such that $f(\omega) = \delta$, $f(\alpha_1) = \alpha$, $f(\beta_1) = \beta$; and f is called a *d-isomorphism* if it is an order isomorphism (*o-isomorphism*) and

$$f(\alpha \oplus \beta) = f(\alpha) \oplus_1 f(\beta) \,,$$

$$f(\infty) = \infty \,;$$

in this case we write $G \cong G_1$.

A d-group G is called a *Prüfer d-group* if for any $H \in \mathfrak{M}(G)$, G/H is totally ordered. For any lattice ordered group G (i.e. an *l-group*), $G \cup \{\infty\}$ is a Prüfer d-group with respect to the addition

$$\alpha \oplus_m \beta = \{\omega \in G : \alpha \wedge \beta = \alpha \wedge \omega = \beta \wedge \omega\} ,$$

where $\alpha \wedge \beta = \inf(\alpha, \beta)$ (see [8], [5])..

For a field K, a map w from K onto a partially ordered group G with an element ∞ is called a *semi-valuation* provided that for every $x, y \in K$, the following relations hold.

(i) $w(x \,.\, y) = w(x) + w(y)$,

(ii) $w(x + y) \geqq \delta$ for every $\delta \in G$ such that $\delta \leqq w(x)$, $\delta \leqq w(y)$,

(iii) $w(x) = \infty$ if and only if $x = 0$.

In this case, $A(w) = \{x \in K : w(x) \geqq 0\}$ is a subring in K and there exists an o-isomorphism f of G onto $G(A(w))$ such that $fw = w_{A(w)}$.

Following I. Kaplansky [2] we say that a ring A is a *GCD-domain* if each pair of nonzero elements of A has a greatest common divisor in A, i.e. $G(A)$ is an l-group.

3. SEMI-VALUATIONS AND d-GROUPS

In [3] it has been shown that using a special multivalued addition $\oplus_A$ defined on the divisibility group $G(A)$ of A using the addition $+$ on A it is possible to describe some algebraic properties of A. Namely, for $w_A(a)$, $w_A(b)$, $w_A(c) \in G(A)$ we set $w_A(c) \in w_A(a) \oplus_A w_A(b)$ if there exist $u_1, u_2 \in U(A)$ such that $c = au_1 + bu_2$.

In this section we say that a map w of a field K onto a d-group $(G, \oplus)$ is a *d-valuation* if for every $a, b \in K$ the following relations hold.

(a) $w(a \,.\, b) = w(a) \,.\, w(b)$,

(b) $w(a + b) \in w(a) \oplus w(b)$,

(c) $w(a) = \infty$ if and only if $a = 0$.

It is easy to see that every semi-valuation w is a d-valuation with a d-group $(G(A(w)), \oplus_{A(w)})$ and, conversely, every d-valuation w with a d-group G is a semi-valuation with a group G, since $w(a + b) \in w(a) \oplus w(b)$ implies the property (ii) in virtue of the property (6).

If w_1, w_2 are d-valuations on a field K with d-groups G_1, G_2, respectively, we set $w_1 \geqq w_2$ if there exists a d-homomorphism f of G_1 onto G_2 such that $fw_1 = w_2$. Further, if A is a ring with the quotient field K, we denote by w_A the canonical d-valuation with a d-group $(G(A), \oplus_A)$.

Now, if w is a d-valuation on K with a d-group G, for $H \in \mathfrak{C}(G)$ we set

$$R_w(H) = \{x \in K : w(x) \in H \,.\, G_+\} .$$

Since for $x, y \in R_w(H)$, $w(x \,.\, y) = w(x) \,.\, w(y) \in H \,.\, G_+$, $w(x + y) \in w(x) \oplus w(y) \subseteq H \,.\, G_+ \oplus H \,.\, G_+ = H \,.\, G_+$ hold, we obtain that $R_w(H)$ is a subring in K and since $w(1) = 1$, $1 \in R_w(H)$ holds. We set $R_w(\{1\}) = R_w$.

Moreover, the following proposition holds.

Proposition 3.1. *Let G_1, G_2 be d-groups, $f : G_1 \to G_2$ a d-homomorphism, and let w be a d-valuation on a field K with a d-group G_1. Then fw is a d-valuation. If we suppose that $H \in \mathfrak{C}(G_1)$,* $\operatorname{Ker} f \subseteq H$ *and f is a d-epimorphism, it follows that $f(H) \in \mathfrak{C}(G_2)$ and $R_w(H) = R_{fw}(f(H))$.*

Proof. It is clear that fw is a d-valuation. Suppose that f is a d-epimorphism and $\operatorname{Ker} f \subseteq H$. Then $f(H)$ is a convex subgroup in G_2. Let α, $\beta \in G_2^+$, $\delta_1, \delta_2 \in H$, then for $\delta \in f(\delta_1)\, \alpha \oplus_2 f(\delta_2)\, \beta$ there exist $\alpha_1, \beta_1 \in G_1^+$, $\gamma_1, \gamma_2 \in \operatorname{Ker} f \subseteq H$, $\gamma \in G_1$ such that

$$\gamma \in \gamma_1\alpha_1\delta_1 \oplus_1 \gamma_2\beta_2\delta_2 \subseteq H \,.\, G_1^+ \oplus_1 H \,.\, G_1^+ = H \,.\, G_1^+$$

and $f(\gamma) = \delta$. Thus, $\delta \in f(H \,.\, G_1^+) \subseteq f(H) \,.\, G_2^+$ and $f(H) \in \mathfrak{C}(G_2)$. Let $x \in R_w(H)$. Then $w(x) \in H \,.\, G_1^+$ and $f\, w(x) \in f(H \,.\, G_1^+) \subseteq f(H) \,.\, G_2^+$ and $x \in R_{fw}(f(H))$. Conversely, for $x \in R_{fw}(f(H))$ we have $f\, w(x) \in f(H) \,.\, G_2^+$ and since $\operatorname{Ker} f \subseteq H$ we obtain $w(x) \in H \,.\, G_1^+$. Thus, $R_w(H) = R_{fw}(f(H))$.

Corollary 3.2. *If w is a d-valuation on a field K with a d-group G and if for $H \in \mathfrak{C}(G)$ we denote by w_H the composition of w and the canonical map $G \to G/H$, we have $R_w(H) = R_{w_H}$.*

Proposition 3.3. *Let w be a d-valuation on K with a d-group G and let $H \in \mathfrak{C}(G)$, $A = R_w(H)$. Then $w_H \leqq w_A$.*

Proof. Let $\alpha = w_A(x) \in G(A)$ and let $f(\alpha) = w_H(x)$. Since $R_w(H) = R_{w_H}$, for $u \in U(A)$ we have $w_H(u) = 1$ and the definition of f is correct. For $\gamma \in \alpha \oplus_A \beta$ there exist $x, y, z \in K$, $u_1, u_2 \in U(A)$, such that $w_A(x) = \alpha$, $w_A(y) = \beta$, $w_A(z) = \gamma$, $z = xu_1 + yu_2$. Hence,

$$f(\gamma) = w_H(z) \in w_H(xu_1) \oplus^{\sim} w_H(yu_2) = w_H(x) \oplus^{\sim} w_H(y)\,,$$

and $f(\alpha \oplus_A \beta) \subseteq f(\alpha) \oplus^{\sim} f(\beta)$, where $\oplus^{\sim}$ is the factor addition on G/H. Now, for $\alpha \in G(A)_+$ we have $\alpha = w_A(x)$, $x \in A = R_{w_H}$, and $w_H(x) \in (G/H)_+$. Thus, $f(\alpha) \geqq 1$. It follows that f is a d-homomorphism and $fw_A = w_H$.

If $\oplus$, $\oplus_1$ are additions on a partially ordered group G, we set $\oplus \leqq \oplus_1$ if $\alpha \oplus \beta \subseteq \alpha \oplus_1 \beta$ for every $\alpha, \beta \in G$. With this notation it is easy to see that if w is a d-valuation on a field K with a d-group $(G, \oplus)$ and $A = R_w(H)$ for some $H \in \mathfrak{C}(G)$, the addition $\oplus_A$ is the smallest addition on G/H such that w_H is a d-valuation.

Proposition 3.4. *Let w be a d-valuation on K with a d-group G and let $H \in \mathfrak{D}(G)$. Then $R_w(H)$ is a quotient ring of R_w.*

Proof. By Cor. 3.2, $R_w(H) = R_{w_H}$. Since H is an o-ideal in G, by [6]; Theorem 2.1, R_{w_H} is a quotient ring of R_w.

For rings A, B in a fixed field K we say that B is *well centred* on A, if $A \subseteq B$ and

$B = A \, . \, U(B)$, i.e. the canonical map $f : G(A) \to G(B)$ defined by $f(w_A(x)) = w_B(x)$ is an o-epimorphism.

Lemma 3.5. *Let A be a ring with the quotient field K and let H be a convex subgroup in $G = G(A)$. Then G/H is a group of divisibility of a ring B in K, such that B is well centred on A if and only if $H \in \mathfrak{C}(G, \oplus_A)$.*

Proof. Suppose that there exists a ring B such that $A \subseteq B \subset K$, $G(B) = G/H$ and B is well centred on A. Then $H = \{w_A(x) : x \in U(B)\}$. Let $\alpha, \beta \in G_+$, $\gamma \in H$, $\omega \in \alpha \oplus_A \beta\gamma$. Then there exist $a, b \in A$, $u_1, u_2 \in U(A)$, $c \in U(B)$ such that $w_A(au_1 + bcu_2) = \omega$, $w_A(a) = \alpha$, $w_A(b) = \beta$, $w_A(c) = \gamma$. Thus, $au_1 + bcu_2 \in B$ and there exist $y \in U(B)$, $z \in A$ such that $au_1 + bcu_2 = yz$. Thus, $\omega \geqq w_A(y) \in H$ and $H \in \mathfrak{C}(G, \oplus_A)$ by [8].

Conversely, let $H \in \mathfrak{C}(G, \oplus_A)$. Then $w = fw_A$ is a d-valuation and we set $B = R_w$. By Cor. 3.2, $B = R_{w_A}(H)$ and $A \subseteq B$. For $x \in B$ there exist $a \in A$, $b \in K$ such that $x = a \, . \, b$, $w_A(b) \in H$ and $w(b) = 1$. Thus, $b \in U(B)$ and $B = A\,U(B)$.

We note that there exists a divisibility group G with a nondirected convex subgroup H such that G/H is a group of divisibility (see [4]).

The following proposition is a generalization of [3], Proposition 5.

Proposition 3.6. *Let A be a ring with the quotient field K and let $H \in \mathfrak{C}(G(A), \oplus_A)$. If $\oplus^\sim$ is the factor addition on $G(A)/H$, there exists a ring B in K that is well centred on A and $(G(B), \oplus_B) \cong (G(A)/H, \oplus^\sim)$.*

Proof. By Lemma 3.5, $B = R_w(H)$ is well centred on A, where $w = w_A$, and the map f defined by $f(w_B(x)) = w_H(x)$ is an o-isomorphism. Let $x, y, z \in K$ be such that $w_B(z) \in w_B(x) \oplus_B w_B(y)$. Then there exist $u_1, u_2 \in U(B)$ such that $z = xu_1 + yu_2$ and since $w(U(B)) = H$, we have $w(z) \in w(x)\,\omega_1 \oplus_A w(y)\,\omega_2$ for some $\omega_1, \omega_2 \in H$. Thus, $w(z)\,H \in w(x)\,H \oplus^\sim w(y)\,H$ and since $w(x)\,H = f\,w_B(x)$, we obtain

$$f(w_B(x) \oplus_B w_B(y)) \subseteq f\,w_B(x) \oplus^\sim f\,w_B(y) \, .$$

Conversely, for $f\,w_B(z) \in f\,w_B(x) \oplus^\sim f\,w_B(y)$ there exist $u_1, u_2 \in U(B)$ such that $w(z) \in w(xu_1) \oplus_A w(yu_2)$ and $w_B(z) \in w_B(x) \oplus_B w_B(y)$.

Now, let w be a d-valuation on a field K with a d-group G and let $H \in \mathfrak{D}(G)$. For an m-ideal J in G_+ we set

$$M_w(H, J) = \{x \in K : w(x) \in J \, . \, H\} \, .$$

Then $M_w(H, J) = M$ is an ideal in $R_w(H)$. In fact, let $y \in M$, $x \in R_w(H)$. Then $w(xy) = w(x) \, . \, w(y) \in J \, . \, H \, . \, G_+ \, . \, H \subseteq J \, . \, H$ and $x \, . \, y \in M$. For $x, y \in M$ we have $w(x - y) \in w(x) \oplus w(y) \subseteq J \, . \, H \oplus J \, . \, H$. Then there exist $\delta_1, \delta_2 \in H$, $\gamma_1, \gamma_2 \in J$ such that $w(x - y) \in \gamma_1\delta_1 \oplus \gamma_2\delta_2$. Since H is directed, there exist $\varrho_1, \varrho_2, \varepsilon \in H_+$ such that $\delta_1 = \varrho_1\varepsilon^{-1}$, $\delta_2 = \varrho_2\varepsilon^{-1}$. Then

$$w(x - y) \in \varepsilon^{-1}(\gamma_1\varrho_1 \oplus \gamma_2\varrho_2) \subseteq (J \oplus J)\,\varepsilon^{-1} \subseteq J\varepsilon^{-1} \subseteq J \, . \, H \, ,$$

and $x - y \in M$.

It is easy to see that

$$M_w(H, J) = M_w(\{1\}, J) \, . \, R_w(H) \, .$$

Proposition 3.7. *Let w be a d-valuation on a field K with a d-group $(G, \oplus)$ and let $H \in \mathfrak{C}(G)$. If H is prime, $R_w(H)$ is a quasi-local ring. Conversely, if $\oplus = \oplus_A$ for $A = R_w$ and if $R_w(H)$ is quasi-local, H is prime.*

Proof. First we suppose that $H = \{1\}$ and G is a local d-group, i.e. $\alpha > \beta$ implies $\alpha \oplus \beta = \{\beta\}$. Then $J = \{\alpha \in G : \alpha > 1\}$ is the greatest m-ideal in G_+ and it is easy to see that $M_w(\{1\}, J)$ is the greatest ideal in A. To consider the general case we observe first that if H is prime, G/H is a local d-group and $R_w(H) = R_{w_H}$. By the first part of this proof, R_{w_H} is quasi-local. The converse part follows directly from [3]; Prop. 7.

We note that there exist a local d-group G and $H \in \mathfrak{C}(G)$ that is not prime. In fact, let A be a quasi-local ring which contains at least three noncomparable prime ideals $P_1, P_2, P_3 \subset M$, where M is the maximal ideal of A. We set $S = A - (P_1 \cup P_2) \neq U(A)$ and let H be an o-ideal in $G = G(A)$ such that $G(A_S) = G/H$. Then $(G, \oplus_A)$ is a local d-group and since A_S is well centred on A, by Lemma 3.5 we obtain that $H \in \mathfrak{C}(G, \oplus_A)$. Since $A_S = R_{w_A}(H)$ is not quasi-local, it follows that H is not prime.

Proposition 3.8. *Let w be a d-valuation on a field K with a d-group $(G, \oplus)$ and let $H \in \mathfrak{C}(G)$. If $R_w(H)$ is a Prüfer domain, G/H is a Prüfer d-group. The converse holds if $\oplus = \oplus_A$ for $A = R_w$.*

Proof. First, let $H = \{1\}$ and let $\tilde{H} \in \mathfrak{M}(G, \oplus) = \mathfrak{M}(G/H)$. Then for $\alpha, \beta \in G$ we have $\alpha \oplus_A \beta \subseteq \alpha \oplus \beta$ and by [8]; Lemma 6, $\tilde{H} \in \mathfrak{M}(G, \oplus_A)$. By [3]; Proposition 7, there exists a prime ideal P in A such that $G/\tilde{H} = G(A_P)$. Since A is Prüfer, $G/\tilde{H}$ is totally ordered and $G = G/H$ is a Prüfer d-group.

Now, we consider the general case. By Corollary 3.2, $R_w(H) = R_{w_H}$ and w_H is a d-valuation on K with a d-group G/H. By the first part, G/H is a Prüfer d-group.

Conversely, let $\oplus = \oplus_A$ and let $H \in \mathfrak{C}(G)$ be such that $(G/H, \oplus^\sim)$ is a Prüfer d-group, where $\oplus^\sim$ is the factor addition on G/H. By Lemma 3.5, $B = R_w(H)$ is well centred on A. Let M be a maximal ideal in B and let $\tilde{H}$ be an o-ideal in G/H such that $G(B_M) = (G/H)/\tilde{H}$. If we identify $G(B)$ and G/H, by Prop. 3.6 we obtain that $\oplus^\sim = \oplus_B$. Since B_M is quasi-local, by [3]; Proposition 7, we have $\tilde{H} \in \mathfrak{M}(G/H, \oplus^\sim)$. Since G/H is a Prüfer d-group, $(G/H)/\tilde{H}$ is totally ordered and B_M is a valuation ring. Hence, B is a Prüfer domain.

It should be observed that in the case $\oplus \neq \oplus_A$ the converse implication need not hold in general. In fact, let A be a GCD-domain that is not Bezout, i.e. A is not a Prüfer ring. Let $G = (G(A), \oplus_m)$ and let $w = w_A$. Since

$$w(x + y) \in w(x) \oplus_A w(y) \subset w(x) \oplus_m w(y) \, ,$$

w is a d-valuation with a d-group G. Then G is a Prüfer d-group and A is not a Prüfer ring.

Moreover, the condition $\oplus = \oplus_A$ is not necessary in the converse implication. Indeed, let w be a d-valuation in the field $\boldsymbol{Q}$ of rational numbers with a d-group $G = (\boldsymbol{Z}, \oplus_m)$ such that $R_w = \boldsymbol{Z}_{(2)}$. Since $\mathfrak{C}(G) = \{\{0\}\}$, $R_w(H)$ is a Prüfer domain, but $\oplus_m \neq \oplus_{R_w}$ $(0 \in 0 \oplus_m 0 - 0 \oplus_{R_w} 0)$.

Proposition 3.9. *Let w be a d-valuation on a field K with a d-group $(G, \oplus)$ and let $H \in \mathfrak{C}(G)$. If $(G/H)_+$ is integrally closed in G/H, $R_w(H)$ is integrally closed in K. The converse holds if $\oplus = \oplus_A$ for $A = R_w$.*

Proof. Let $x \in K$ be such that there exist $a_0, \ldots, a_n \in R_w(H)$ with $x^{n+1} = a_0 + a_1 x + \ldots + a_n x^n$. Then $w(a_i) \in H \cdot G_+$ and $w(a_i) H \geqq H$ in G/H. Since

$$w(x)^{n+1} \in w(a_0) \oplus \ldots \oplus w(a_n)\, w(x)^n ,$$

we obtain

$$(w(x)\, H)^{n+1} \in w(a_0)\, H \oplus^{\sim} \ldots \oplus^{\sim} (w(a_n)\, H)\, (w(x)\, H)^n$$

and $w(x)\, H \geqq H$ ($\oplus^{\sim}$ is the factor addition on G/H). Thus, $w(x) \in H \cdot G_+$ and $x \in R_w(H)$.

Conversely, let $\oplus = \oplus_A$ and let $B = R_w(H)$ be integrally closed in K. We suppose that for $\xi \in G$ there exist $\alpha_0, \ldots, \alpha_n \in G$ such that $\alpha_i H \geqq H$ and

$$(\xi H)^{n+1} \in \alpha_0 H \oplus^{\sim} \ldots \oplus^{\sim} \alpha_n \xi^n H ,$$

where $\oplus^{\sim}$ is the factor addition on G/H. If we identity $G(B)$ and G/H, by Prop. 3.6 we obtain $\oplus_B = \oplus^{\sim}$ and there exist $a_0, \ldots, a_n \in K$, $a \in K$, $u_0, \ldots, u_n \in U(B)$ such that

$$w_H(a_i) = \alpha_i \xi^i H , \quad w_H(a) = (\xi H)^{n+1} ,$$

$$a = a_0 u_0 + \ldots + a_n u_n .$$

Let $\xi = w(x)$, $\alpha_i = w(d_i)$, $w(a_i)\, \omega_i = \alpha_i \xi^i$ for some $\omega_i \in H = w(U(B))$. Then there exist $v_i \in U(A)$, y, $y_i \in U(B)$ such that

$$a_i = v_i d_i x^i y_i , \quad a = x^{n+1} y .$$

Thus,

$$x^{n+1} = v_0 d_0 y_0 y^{-1} u_0 + \ldots + v_n d_n y_n y^{-1} u_n x^n ,$$

and since $v_i d_i y_i y^{-1} u_i \in B$, we have $x \in B$. Therefore, $w_H(x) = \xi H \geqq H$ and $(G/H)_+$ is integrally closed.

For every d-valuation w on a field K with a d-group $(G, \oplus)$, the inclusion

$$\mathfrak{C}(G, \oplus) \subseteq \mathfrak{C}(G, \oplus_A)$$

holds for $A = R_w$. It should be noted that, in general, the inclusion is proper. In fact, let $A = \boldsymbol{Z}[X, Y]$ and let H be a nondirected convex subgroup in $G = G(A)$ that is considered in [8]; Appendix 4. Since G/H is a group of divisibility of a domain B in the quotient field K of A such that B is well centred on A, it follows that $H \in \mathfrak{C}(G, \oplus_A)$. Since G is an l-group and H is not directed, H is not an l-ideal in G and $H \notin \mathfrak{C}(G, \oplus_m)$ (see [8]). Thus, $\mathfrak{C}(G, \oplus_m) \neq \mathfrak{C}(G, \oplus_A)$.

J. Ohm announced in [9] without any proof a result of R. L. Pendleton that the only directed orders on the group $\boldsymbol{Z}$ of integers which produce groups of divisibility are the two obtained by taking as positive elements either $\boldsymbol{Z}_+$ or $\boldsymbol{Z}_-$. In this part we shall prove this result.

First, we show a method based on the notion of a d-group that enables us to construct examples of partially ordered groups which are not groups of divisibility. For a partially ordered group G we denote

$$[\alpha, \beta] = \{\gamma \in G : \gamma \leqq \alpha, \beta\}$$

for $\alpha, \beta \in G$, and the symbol $\alpha \parallel \beta$ denotes that α is incomparable with β.

Proposition 4.1. *Let G be a partially ordered group such that there exist $\alpha, \beta \in G_+$ with the properties*

(1) *$\alpha \wedge \beta$ does not exist in G,*

(2) *for every $\xi \in G_+$, $\xi \parallel \alpha, \beta$, there exists $\omega \in [\alpha, \beta]$ such that $\omega \parallel \xi$.*

Then it is not possible to define a structure of a d-group on G and G is not a group of divisibility.

Proof. Suppose that G is a d-group with respect to a multivalued addition $\oplus$. Then for $\alpha, \beta \in G_+$ we have

$$\alpha \oplus \beta \subseteq \{\gamma \in G_+ : [\alpha, \beta] = [\alpha, \gamma] = [\beta, \gamma]\} .$$

Let $\alpha, \beta \in G_+$ be such that (1), (2) hold and let $\xi \in \alpha \oplus \beta$. If $\xi \leqq \alpha$, we have $\xi \in [\alpha, \xi] = [\alpha, \beta]$. For $\omega \in [\alpha, \beta]$ we obtain in virtue of (6) from the definition of a d-group that $\xi \geqq \omega$ and $\xi = \alpha \wedge \beta$, a contradiction. If $\xi > \alpha$, we have $\alpha \in [\xi, \alpha] = [\alpha, \beta]$ and $\alpha = \alpha \wedge \beta$, a contradiction. Thus $\xi \parallel \alpha$ and analogously, $\xi \parallel \beta$. In virtue of (2), there exists $\omega \in [\alpha, \beta]$ such that $\omega \parallel \xi$, and on the other hand, (6) yields $\xi \geqq \omega$, a contradiction. Thus $\alpha \oplus \beta = \emptyset$ and it is not possible to define a multivalued addition on G.

The following examples show that there exist partially ordered (directed) groups which satisfy the conditions of 4.1. Example 1 is the first example of a directed partially ordered group that is not a group of divisibility, which was published by P. Jaffard [1], Example 2 was published in [7].

Example 1. Let $G = \{(a, b) : a, b \in \boldsymbol{Z}, a + b \text{ is even}\}$. The operation and ordering in G are defined component-wise. We set $\alpha = (4, 2)$, $\beta = (3, 3)$. It is clear that $\alpha \wedge \beta$ does not exist in G. Let $\xi \parallel \alpha, \beta$. Then

$$\xi \in \{(x, 1), (1, x), (0, y), (y, 0), (a, b)\} ,$$

where x is odd, y even and $2 \leqq a < 4$, $b > 1$. Then one of the following possibilities occurs:

$$\xi \parallel (0, 2), (2, 0), (1, 1), (3, 1) \in [\alpha, \beta]$$

and α, β satisfy (1), (2).

Example 2. Let $G = \boldsymbol{Z}$ and set

$$P = \{x \in \boldsymbol{Z} : x = 0 \text{ or } x \geqq 2\}.$$

Then P is a set of nonnegative elements for some directed ordering on $\boldsymbol{Z}$. We set $\alpha = 4$, $\beta = 5$. It is clear that α, β satisfy (1). Moreover, there is no element $\xi \in P$ such that $\xi \parallel \alpha, \beta$ and α, β satisfy (2).

On the other hand, it should be observed that there are directed partially ordered groups which are not groups of divisibility and which do not satisfy the conditions of 4.1.

Example 3. Let $G = \boldsymbol{Z} \times \boldsymbol{Z} \times \boldsymbol{Z}$ and define a directed order relation on G in the following way.

$$(x, y, z) \leqq (x_1, y_1, z_1) \quad \text{iff} \quad (x < x_1 \text{ and } y \leqq y_1) \quad \text{or}$$
$$(x \leqq x_1 \text{ and } y < y_1) \quad \text{or} \quad (x = x_1 \text{ and } y = y_1 \text{ and } z = z_1).$$

By [9]; Theorem 5.3, G is not a group of divisibility. Suppose that $\alpha = (\boldsymbol{g}_1, a_1)$, $\beta = (\boldsymbol{g}_2, a_2) \in G_+$ $(\boldsymbol{g}_i \in \boldsymbol{Z} \times \boldsymbol{Z})$ satisfy (1), (2) from 4.1. Let $c \in \boldsymbol{Z}$, $c \neq a_1, a_2$. Then $\xi = (\boldsymbol{g}_1, c) \parallel \alpha, \beta$. The condition (2) implies the existence of an element $\omega = (\boldsymbol{t}, v)$, $\boldsymbol{t} \in \boldsymbol{Z} \times \boldsymbol{Z}$, such that $\omega \parallel \xi$, $\omega \in [\alpha, \beta]$. Thus $\boldsymbol{t} \leqq \boldsymbol{g}_1 \wedge \boldsymbol{g}_2 < \boldsymbol{g}_1$ and $\omega < \xi$, a contradiction.

Unfortunately, in the case of Example 3 we do not know if it is possible to define a multivalued addition on G.

In the rest of this paper we deal with directed orders on $\boldsymbol{Z}$.

Proposition 4.2. *Let $\succeq$ be a directed order relation on $\boldsymbol{Z}$ with the family P of positive elements. Then either $P \subseteq \boldsymbol{Z}_+$ or $P \subseteq \boldsymbol{Z}_-$ and P has only a finite number of atoms. A set $\{m_1, \ldots, m_n\} \subseteq \boldsymbol{Z}_+(\boldsymbol{Z}_-)$ is the set of all atoms for some directed ordering on $\boldsymbol{Z}$ if and only if*

(1) *g.c.d. $\{m_1, \ldots, m_n\} = 1$,*

(2) *$m_i \neq a_1 m_1 + \ldots + a_{i-1} m_{i-1} + a_{i+1} m_{i+1} + \ldots + a_n m_n$ for every $a_k \in \boldsymbol{Z}_+(\boldsymbol{Z}_-)$, $i = 1, \ldots, n$.*

Proof. We admit that $P \nsubseteq \boldsymbol{Z}_+$, $P \nsubseteq \boldsymbol{Z}_-$. Then there exist $m, n \in P$ such that $m < 0$, $n > 0$, where $\leqq$ is the classical ordering on $\boldsymbol{Z}$. Let n_1 be the least element in $\boldsymbol{Z}_+$ such that $n_1 \in P$, then $m + n_1 < n_1$, $m + n_1 \in P$, a contradiction. We may assume that $P \subseteq \boldsymbol{Z}_+$. Let k_P be the least element in $\boldsymbol{Z}_+$ such that $k_P \neq 0$, $k_P \in P$. Since $\succeq$ is directed, there exists $a \in P$ such that $b = a + 1 \in P$. We consider the following family of $k_P + 1$ elements of P:

$$k_P a, \ (k_P - 1) a + b, \ldots, a + (k_P - 1) b, \ k_P b.$$

For $q \geqq k_P b$ there exist $n, z \in \boldsymbol{Z}_+$ such that

$$q = k_P \,.\, n + z\,, \quad 0 \leqq z < k_P\,.$$

Since $a \,.\, k_P \leqq a \,.\, k_P + z < b \,.\, k_P$, we have $a \,.\, k_P + z \in P$ and $q = (a \,.\, k_P + z) + (n - a) \,.\, k_P \in P$. Hence, there exists a least element n_P in $\boldsymbol{Z}_+$ such that $q \in P$ for every $q \geqq n_P$.

Now, let

$$m_1 < m_2 < \ldots < m_n < \ldots$$

be an infinite family of atoms in P. Then there exists n such that $m_n - m_1 \geqq n_P$ and in this case we have $m_n \succ m_1$, a contradiction. Thus, in P there is only a finite number of atoms $m_1 < \ldots < m_n$. If m is the unique atom in P, we have $q \succeq m$ for every $q \in P$. If $m \neq 1$, there exists an element q that is the least element in $\boldsymbol{Z}_+$ with $q \in P$ and for which there exists $n \geqq 0$ with $nm < q < (n + 1)\, m$. In this case we have $q - m \in P$ and $q - m < q$, $(n - 1)\, m < q - m < nm$, a contradiction. Thus $m = 1$ and $P = \boldsymbol{Z}_+$.

Now, let $\{m_1, \ldots, m_n\} \subset \boldsymbol{Z}_+$ be the set of atoms for some directed ordering on $\boldsymbol{Z}$ with the set P of positive elements. We show that

$$(+) \qquad P = \{a_1 m_1 + \ldots + a_n m_n : a_i \in \boldsymbol{Z}_+\}\,.$$

Indeed, let $q \in P$ and set

$$T = \{(a_1, \ldots, a_n) \in \boldsymbol{Z}_+^n : q \succeq a_1 m_1 + \ldots + a_n m_n\}$$

with the component-wise ordering. Let $(a_1, \ldots, a_n) \in T$ be a maximal element of T. If $q \neq a_1 m_1 + \ldots + a_n m_n$, there exists $1 \leqq j \leqq n$ such that

$$q - (a_1 m_1 + \ldots + a_n m_n) \succeq m_j\,,$$

and $(a_1, \ldots, a_n) < (a_1, \ldots, a_j + 1, \ldots, a_n) \in T$, a contradiction. Thus $q = a_1 m_1 + \ldots + a_n m_n$. Since $1 \in P - P$, we have $1 = a_1 m_1 + \ldots + a_n m_n$ for some $a_i \in \boldsymbol{Z}$ and, especially, g.c.d. $\{m_1, \ldots, m_n\} = 1$. If $m_i = a_1 m_1 + \ldots + a_{i-1} m_{i-1} + a_{i+1} m_{i+1} + \ldots + a_n m_n$ for some $a_j \in \boldsymbol{Z}_+$, then we have $m_i \prec m_k$ for $k \neq i$ such that $a_k \neq 0$, a contradiction.

Conversely, let $\{m_1, \ldots, m_n\} \subset \boldsymbol{Z}_+$ be a family satisfying (1), (2) and let P be defined by $(+)$. Since $P + P \subseteq P$, $P \cap (-P) = 0$, $\boldsymbol{Z} = P - P$, the set P is the set of positive elements for a directed ordering $\succeq$ on $\boldsymbol{Z}$. We suppose that $0 \prec q \preceq m_i$, $q = a_1 m_1 + \ldots + a_n m_n$. Then

$$(-a_1)\, m_1 + \ldots + (1 - a_i)\, m_i + \ldots + (-a_n)\, m_n \in P$$

and in virtue of (2) it is easy to see that $a_i = 1$, $a_k = 0$, $k \neq i$. Thus, m_i is an atom in P and it is clear that $\{m_1, \ldots, m_n\}$ is the set of all atoms in P.

In what follows we denote by $\succeq$ a directed ordering on $\boldsymbol{Z}$ with the family $\{m_1, \ldots, m_n\} \subset \boldsymbol{Z}_+$ of all atoms in the set P of nonnegative elements.

(4.3) *There is no proper o-ideal in* $(\boldsymbol{Z}, \geqq)$.

Indeed, let H be an o-ideal in $(\boldsymbol{Z}, \geqq)$, $H \neq \{0\}$, and let $m_1, \ldots, m_t$ be all atoms of P which are contained in H. Then

$$H_+ = \{a_1 m_1 + \ldots + a_t m_t : a_i \in Z_+\} .$$

For, $q \in H_+ \subseteq P$ fulfils $q = a_1 m_1 + \ldots + a_n m_n$. If $a_j > 0$ for $t < j \leqq n$, then $0 \prec m_j \prec q$ and since H is convex, we have $m_j \in H$, a contradiction.

Now, let $q \in \boldsymbol{Z}$. Then $q = a_1 m_1 + \ldots + a_n m_n$ for some $a_i \in \boldsymbol{Z}$ and there exist $b_1, \ldots, b_t \in \boldsymbol{Z}_+$ such that

$$\sum_{i=1}^{t} (a_i + b_i)\, m_i + \sum_{j=t+1}^{n} a_j m_j \geqq n_P ,$$

where n_P is from the proof of 4.2. Then $b_1 m_1 + \ldots + b_t m_t \in H$ and in the factor group $(\boldsymbol{Z}, \geqq)/H$ we have

$$q + H = \sum_{i=1}^{n} a_i m_i + H = \sum_{i=1}^{t} (a_i + b_i)\, m_i + \sum_{j=t+1}^{n} a_j m_j + H \geqq H .$$

Therefore, $((\boldsymbol{Z}, \geqq)/H)_+ = (\boldsymbol{Z}, \geqq)/H$ and $H = \boldsymbol{Z}$.

(4.4) *There is only a finite number of pairwise incomparable elements in P.*

Indeed, if $X = \{x_1, x_2, \ldots\}$ is an infinite family of pairwise incomparable elements of P, there are indexes i, j such that $x_i > x_j + n_P$ and we have $x_i \succ x_j$, a contradiction.

(4.5) *If* $(\boldsymbol{Z}, \geqq)$ *is a group of divisibility of a domain A, A is a local ring and* $\dim A = 1$.

Indeed, if J is an ideal in A, we set

$$X = \{w_A(x) : x \in J,\ w_A(x) \text{ is minimal in } w_A(J)\} .$$

By 4.4, X is finite, $X = \{w_A(x_1), \ldots, w_A(x_s)\}$. Then $(x_1, \ldots, x_s) \subseteq J$. Let $x \in J$. Since $(P, \geqq)$ satisfies d.c.c., there exists $1 \leqq i \leqq s$ such that $w_A(x) \geqq w_A(x_i)$ and we have $x = x_i a \in J$ for some $a \in A$. Hence, J is finitely generated and A is Noetherian. By 4.3, A is a local ring and $\dim A = 1$.

Now, we suppose that $G = (\boldsymbol{Z}, \geqq)$ is a group of divisibility of a domain A with the quotient field K. By 4.5, there is a unique prime ideal M in A. Let v be a valuation on the field K with a valuation ring R_v and maximal ideal M_v such that

$$A \subseteq R_v , \quad M_v \cap A = M .$$

By [10]; Appendix 2, Prop. 2,

$$\dim v + \dim_A v \leqq \dim A ,$$

where $\dim v$ is the rank of v and $\dim_A v$ is the transcendental degree of the field

R_v/M_v over the field A/M. Since $\dim A = 1 \leqq \dim v$, we have

$$\dim_A v = 0 .$$

It follows that v is a prime A-divisor of a field K and by [10]; Appendix, Corollary 2, v is a discrete rank 1 valuation. We may assume that $G(R_v) = (Z, \leqq)$. For $w_A(x) \in G$ we set $\alpha(w_A(x)) = v(x)$, then α is an o-homomorphism of G onto $G(R_v)$. Since Ker $\alpha \neq G$, by 4.3 we have Ker $\alpha = \{0\}$ and α is injective. Let $\alpha(1) = k_0$ $(\neq 0)$, then

$$\alpha(n) = \alpha(1 + \dots + 1) = k_0 \, . \, n$$

and

$$v(x) = \alpha(w_A(x)) = k_0 \, . \, w_A(x) \, .$$

Since v is surjective, we have $k_0 = 1$ and $v(x) = w_A(x)$. Let $n \in Z_+$ and let $x_0 \in K$ be such that $v(x_0) = 1$. Then

$$w_A(x_0^n + 1) = v(x_0^n + 1) = \min \{v(x_0^n), \, v(1)\} = 0$$

and $x_0^n + 1 = a \in A$, $x_0^n = a - 1 \in A$. Then

$$0 \prec w_A(x_0^n) = v(x_0^n) = n$$

and we have $P = Z_+$.

We note that we do not know whether $(Z, \geqq)$ satisfies the conditions of Proposition 4.1 for every directed ordering $\geqq$. It should be observed that if P has exactly two atoms, $(Z, \geqq)$ satisfies the conditions of 4.1.

Indeed, let $m_1 < m_2$ be the two atoms in P and set $\alpha = m_1 + m_2$, $\beta = n \, . \, m_1$, where n is the minimal natural number such that $n \, . \, m_1 \succ m_2$. If $\beta \succ \alpha$, we have $(n - 1) \, m_1 \succ m_2$, a contradiction with the minimality of n. If $\beta \prec \alpha$, we have $0 \prec (n - 1) \, m_1 \prec m_2$, a contradiction. Thus $\alpha \parallel \beta$. We suppose that there exists $\alpha \wedge \beta$. Since $m_2 \in P \cap [\alpha, \beta]$, we have $\alpha \wedge \beta \succ m_2$, $\alpha \wedge \beta \prec m_1 + m_2$ and $m_1 + m_2 - \alpha \wedge \beta \in P$, $0 < m_1 + m_2 - \alpha \wedge \beta < m_1$. Since m_1 is the least element in Z_+ contained in $P - \{0\}$, we have a contradiction. Now, let $\xi \in P$, $\xi \parallel \alpha, \beta$. We assume that $\xi \succ m_1$, $\xi \succ m_2$. Then $\xi - m_1 \in P$ and it follows that $\xi \succ 2m_1$. By induction we may obtain $\xi \succ km_1$ for every $k \in Z_+$ and hence $\xi > s$ for every $s \in Z_+$, a contradiction. Thus, $\xi \parallel m_1$ or $\xi \parallel m_2$ and $(Z, \geqq)$ satisfies the conditions of Proposition 4.1.

We note that in terms of groups $(Z, \geqq)$ an example of a divisibility group G may be constructed for which there exists a nondirected convex subgroup H such that G/H is not a group of divisibility. Indeed, let $\{p_1, \dots, p_n\}$ be a set of prime numbers in Z_+ and set

$$m_i = p_1 \cdots p_{i-1} \cdot p_{i+1} \cdots p_n \, .$$

Then the family $\{m_1, \dots, m_n\}$ satisfies the conditions of 4.2 and there is a directed order relation $\geqq$ on Z with atoms $m_1, \dots, m_n$. We set

$$H = \{(a_1, \dots, a_n) \in Z^n : a_1 m_1 + \dots + a_n m_n = 0\} \, .$$

Then H is a nondirected convex subgroup in a group (of divisibility) Z^n and the map

$$\alpha : Z^n/H \to (Z, \geqq)$$

$$\alpha((x_1, \ldots, x_n) + H) = x_1 m_1 + \ldots + x_n m_n ,$$

is an o-isomorphism. Hence, Z^n/H is not a group of divisibility.

Problem. Does there exist a directed partially ordered group which admits a structure of a d-group and which is not a group of divisibility?

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