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EXISTENCE AND UNIQUENESS OF PERIODIC SOLUTIONS OF LINEAR DIFFERENTIAL EQUATIONS IN BANACH SPACES

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This work concerns the problem

$$P\left(t, \frac{\mathrm{d}}{\mathrm{d}t}, A\right) u(t) = f(t)\,, \tag{0.1}$$

$$u(t + \omega) = u(t)\,, \quad t \in R = (-\infty, \infty)\,,$$

in a complex Banach space B. Here $P(t, \lambda, z)$ is a polynomial in λ, z of degree m in λ with real or complex coefficients, A is a linear closed densely defined operator in B with a domain of definition $D(A)$, A is supposed to be a generator of a strongly continuous group $\{T(s)\}_{s \in R}$ of linear bounded operators defined on B. Consistently, u and f are B-valued functions of $t \in R$ from which f is supopsed to be periodic and u is required to be periodic with the same period.

The problem (0.1) as well as more or less general ones were investigated in the abstract setting by many authors for $m \leqq 2$ (see e.g. [1], [8], [12], [13], [14], [15], [16], [18], [19], [20], [21], [22]).

The case $m > 2$ was treated by Ju. A. Dubinskij ([4], [5], [6]), N. Krylová, O. Vejvoda [11] and M. Sova [17].

Ju. A. Dubinskij proves the existence of a periodic solution to the equation

$$u^{(m)}(t) + \sum_{j=0}^{m-1} A_j\, u^{(j)}(t) = f(t)\,, \quad t \in R\,,$$

where $A_1, A_2, \ldots, A_{m-1}$ are generally unbounded mutually commuting linear operators in a Hilbert space H whose joint spectrum satisfies a certain condition. He uses an explicit formula which is based on the spectral resolution of the operator $\mathrm{i}\,(\mathrm{d}/\mathrm{d}t)$ considered on smooth periodic functions in $L_2(R; H)$. N. Krylová and O. Vejvoda [11] describe the general Poincaré scheme for the equation

$$Du = g + \varepsilon F$$

with a set of general boundary conditions, where D is an m-th order differential operator with respect to $\partial/\partial t$.

In [17], which in revised form is included in [22], Chapter VII, §§ 1–3, M. Sova proves the normal solvability in $L_2([0,2\pi]; H)$ of an operator

$$\frac{d^m}{dt^m} + a_1 \frac{d^{m-1}}{dt^{m-1}} + \ldots + a_{m-2} \frac{d}{dt} + A$$

with a domain consisting of 2π-periodic distributions, supposing that A is a normal linear operator in H and that the spectrum of A satisfies a certain condition. He uses Fourier expansions in $L_2([0,2\pi]; H)$ and the theory of linear operators in H. Section 1 of the present work is auxiliary. In Section 2 we are concerned with the problem (0.1) in B with a given period $\omega > 0$ of the right hand side f. In Secs. 3 and 4 we are also concerned with some generalizations. We follow the idea of R. Hersh's paper [9], where the existence of a solution of the initial-value problem for the equation $(0.1)_1$ with $f = 0$ was proved. Hersh utilized the differential property of the group $\{T(s)\}_{s\in R}$ to transform the problem to the investigation of some properties of complex-valued fundamental solutions of the equation

$$P\left(t, \frac{\partial}{\partial t}, \frac{\partial}{\partial s}\right) v(t, s) = 0 .$$

Later on, his results were generalized and improved in [3], [1], from which [3] by J. A. Donaldson provides the idea for the proof of uniqueness of a solution to (0.1) in our Theorem 2.2.

1. NOTATION, DEFINITIONS, AUXILIARY RESULTS

Let C denote the set of all complex numbers and let B be a Banach space over C with a norm $\|\cdot\|$. (We denote by $\|\cdot\|_E$ the norm in a normed space E if necessary.)

If A is defined on $D(A) \subseteq B$ and if it is B-valued and linear then we denote by $N(A)$, $R(A)$, $\sigma(A)$ and $\varrho(A)$ its kernel, range, spectrum and resolvent set, respectively. If A_1 and A_2 are linear operators from B into itself with $\varrho(A_1) \neq \emptyset$, $\varrho(A_2) \neq \emptyset$ then we say that A_1, A_2 are mutually commuting if so are $R(\lambda_1, A_1)$ $(\equiv (\lambda_1 - A_1)^{-1})$, $R(\lambda_2, A_2)$ for all $\lambda_j \in \varrho(A_j)$, $j = 1, 2$. Operators $A_1, \ldots, A_n$ are said to be mutually commuting if so is any pair A_j, A_k $(1 \leqq j, k \leqq n)$.

A linear operator in B is called the generator of a strongly continuous group of linear bounded operators $\{T(s)\}_{s\in R}$ (in the sequel only a group of operators) if

(i) $\{T(s)\}_{s\in R} \subset L(B)$ ($\equiv$ set of all bounded linear B-valued operators defined on B);

(ii) $D(A) = \{x \in B; \lim_{s\to 0} (1/s)(T(s)x - x) \text{ exists}\}$;

(iii) $Ax = s - \lim_{s\to 0} (1/s)(T(s)x - x)$, $x \in D(A)$.

Theorem 1.1. (Hille-Yosida, see [23], p. 253). *For a linear densely defined operator*

$A : D(A) \subseteq B \to B$ let there exist non-negative constants $c^{\pm}$, $a^{\pm}$ such that

$$\|R(\lambda, -A)^n\|_{L(B)} \leqq \frac{c^+}{(\mathrm{Re}\,\lambda - a^+)^n}, \quad \mathrm{Re}\,\lambda > a^+ ,$$

$$\|R(\lambda, -A)^n\|_{L(B)} \leqq \frac{c^-}{(-\mathrm{Re}\,\lambda - a^-)^n}, \quad \mathrm{Re}\,\lambda < -a^- ,$$

$n = 1, 2, \ldots$. Then A is the infinitesimal generator of a group of operators $\{T(s)\}_{s\in R}$ and there exists a positive constant $K = K(a^+, a^-)$ such that

$$\|T(s)\|_{L(B)} \leqq \begin{cases} K(a^+, a^-)\,\mathrm{e}^{a^+ s}, & s \geqq 0, \\ K(a^+, a^-)\,\mathrm{e}^{-a^- s}, & s \leqq 0, \end{cases}$$

Theorem 1.2. ([23], pp. 237–241). *If A is the generator of a group of operators $\{T(s)\}_{s\in R}$ then*

(i) *$D(A)$ is dense in B;*

(ii) *given $n \in N$, A^n is closed and $D(A^n)$ is a Banach space with respect to the norm $\|x\|_{D(A^n)} = \|x\| + \|A^n x\|$, $x \in D(A^n)$;*

(iii) *$(\mathrm{d}/\mathrm{d}s)\, T(s)\, x = -A\, T(s)\, x$, $x \in D(A)$, $s \in R$,*

(iv) *$\lim_{\lambda \to \pm\infty} \lambda^n R(\lambda, A)^n x = x$, $x \in B$, $n \in N$;*

(v) *$\int_0^\infty \mathrm{e}^{-\lambda s}\, T(s)\, x\, \mathrm{d}s = (\lambda + A)^{-1} x$, $x \in B$, $\mathrm{Re}\,\lambda > a^+$,*
$\int_{-\infty}^0 \mathrm{e}^{\lambda s}\, T(s)\, x\, \mathrm{d}s = (\lambda - A)^{-1} x$, $x \in B$, $\mathrm{Re}\,\lambda > a^-$.

The integrals in (v) as well as below are taken in the sense of Bochner.

For a given interval $I \subseteq R$, a given Banach space E and a non-negative integer k we denote by $C^k(I; E)$ the Banach space of continuous functions $u : I \to E$ which are k times continuously differentiable in I, with

$$\|u\|_{C^k(I,E)} \overset{\mathrm{def.}}{=} \sup\{\|u^{(l)}(t)\|_E;\quad t \in I, \quad l = 0, 1, \ldots, k\} < \infty .$$

If $\omega > 0$ then $C^k_\omega(R; E)$ denotes the subspace of $C^k(R; E)$ of ω-periodic functions. Especially, we shall make use of the spaces $C^k_\omega(R; D(A^l))$, $k = 0, 1, \ldots, l = 1, 2, \ldots,$. In what follows we use a formula of the type

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_a^{b(t)} f(t, \tau)\, \mathrm{d}\tau = \int_a^{b(t)} \frac{\partial f}{\partial t}(t, \tau)\, \mathrm{d}\tau + \frac{\mathrm{d}b}{\mathrm{d}t}(t)\, f(t, b(t))$$

for a vector-valued Bochner integrable function f depending on a parameter t (b being a real-valued-function) whenever its correctness is clear from the context. Here and elsewhere $\mathrm{d}u/\mathrm{d}t$ (or $\partial u/\partial t$) for a Banach space-valued function u stand for the strong (partial) derivative of u with respect to t. Let $M \subseteq R$ be measurable and $p \geqq 1$. Denote

$$L_p(M; E) = \left\{ f : M \to E;\ f \text{ strongly measurable}, \int_M \|f(t)\|_E^p\, \mathrm{d}t < \infty \right\}$$

and

$$\|f\|_{L_p(M,E)} = \left(\int_M \|f(t)\|_E^p \,\mathrm{d}t\right)^{1/p}.$$

Further $C_0^\infty(R^n)$, $n \in N$, denotes the space of functions with compact supports which are infinitely times differentiable. The topology on $C_0^\infty(R^n)$ is that of the inductive limit of the semi-normed spaces $C_0^\infty(K)$, where $K \subset R^n$ is compact.

If $f \in L_1(R^n; E)$ then we define by

$$(Ff)(\xi) \equiv \hat{f}(\xi) = \int_{R^n} \mathrm{e}^{-\mathrm{i}\xi x} f(x)\,\mathrm{d}x$$

the Fourier transform of f, while

$$(F^{-1}f)(x) = \frac{1}{(2\pi)^n}\int_{R^n} \mathrm{e}^{\mathrm{i}x\xi} f(\xi)\,\mathrm{d}\xi$$

is the inverse Fourier transform of f. We have

$$F\,L_1(R^n; E) \subset C(R^n; E)\,, \quad F^{-1}\,L_1(R^n; E) \subset C(R^n; E)\,.$$

It is clear that

$$F\,C_0^\infty(R^n) = Z\,, \quad FZ = C_0^\infty(R^n)\,, \quad F^{-1}\,C_0^\infty(R^n) = Z\,, \quad F^{-1}Z = C_0^\infty(R^n)\,,$$

where Z is the space of functions which are holomorphic in C^n and satisfy

$$|z^\alpha f(z)| \leqq C_\alpha \exp\left(a \sum_{j=1}^n |\mathrm{Im}\, z_j|\right), \quad \alpha \in N^n$$

(see [7] Vol. 1, pp. 175–180). If Z is equipped with a suitable topology then F is a homeomorphism between $C_0^\infty(R^n)$ and Z.

We shall need the following

Theorem 1.3. *Let f be a C-valued function holomorphic in an open set containing*

$$S = \{z \in C^r;\ -b_j^- \leqq \mathrm{Im}\, z_j \leqq b_j^+,\ j = 1, \ldots, r\}\,, \quad (b_j^\pm > 0,\ j = 1, \ldots, r)$$

and let

$$|f(z)| \leqq \text{const.}\,[1 + (\sum_{j=1}^r |z_j|^2)^{1/2}]^{-r-1}\,, \quad z = (z_1, \ldots, z_r) \in S\,.$$

Then Ff is continuous in R and there is a constant C such that

$$|(Ff)(s)| \leqq C \exp\left(\sum_{j=1}^r a_j s_j\right), \quad s = (s_1, \ldots, s_r) \in R^r\,,$$

where $a_j = -b_j^+$ if $s_j \geqq 0$ and $a_j = b_j^-$ if $s_j \leqq 0$, $(j = 1, \ldots, r)$.

The proof is standard and can be performed by successively integrating the function $g_s(z) = \mathrm{e}^{-\mathrm{i}sz} f(z)$ along rectangles with vertices $-M, M, M + \mathrm{i}b_j^+, -M + \mathrm{i}b_j^+$; $-M, -M - \mathrm{i}b_j^-, M - \mathrm{i}b_j^-, M$, $(j = r, r-1, \ldots, 1)$, using the Cauchy integration theorem and letting $M \to \infty$.

Finally, the following theorem will be useful in what follows:

Theorem 1.4. ([7], Vol 3, p. 78). *Let $P = (p_{jk})_{j,k=1}^{m}$ be a matrix with complex elements and let $\Lambda = \max_{1 \leqq j \leqq m} \operatorname{Re} \lambda_j$, $\lambda_1, \ldots, \lambda_m$ being all eigenvalues of P.*

Then

$$\|e^P\| \leqq e^{\Lambda}(1 + 2\|P\| + \ldots + 2^{m-1}\|P\|^{m-1}).$$

(The symbol $\|\cdot\|$ denotes the norm of a matrix induced by the Euclidean norm in R^n.)

In the next two sections we shall construct a solution of the problem (0.1) via a solution of a certain ordinary differential equation in B, the coefficients of which are scalar functions depending on a parameter $\sigma \in C^r$. That is why we formulate the forthcoming auxiliary lemmas.

Lemma 1.1. *Let $r \geqq 1$ be an integer and let $\Omega \subset C^r$. Given functions $q_0(t, \sigma)$, $q_1(t, \sigma), \ldots, q_m(t, \sigma)$, $m \geqq 1$ on $R \times \Omega$, continuous in t and σ, holomorphic with respect to σ at any $\sigma_0 \in \Omega^0$, with $q_m(t, \sigma) \neq 0$ for all $(t, \sigma) \in R \times \Omega$, there exists a unique function $w(t, \tau, \sigma)$, $\tau \leqq t$, $\sigma \in \Omega$ which is m-times continuously differentiable with respect to t and τ, continuous in σ, holomorphic with respect to σ in Ω^0 and satisfies*

$$\sum_{j=0}^{m} q_j(t, \sigma) \frac{d^j w}{dt^j}(t, \tau, \sigma) = 0, \tag{1.1}$$

$$\frac{d^k w}{dt^k}(\tau, \tau, \sigma) = \frac{\delta_{k,m-1}}{q_m(\tau, \sigma)}, \quad k = 0, 1, \ldots, m-1, \quad \tau \leqq t, \quad \sigma \in \Omega. \tag{1.2}$$

In the case that the functions q_j, $j = 1, 2, \ldots, m$, are independent of t, i.e. $q_j(t, \sigma) \equiv q_j(\sigma)$, there exists a unique function $w(t, \sigma)$, $t \in R$, $\sigma \in \Omega$ which is infinitely times differentiable with respect to t, holomorphic with respect to σ in Ω^0 and satisfies

$$\sum_{j=0}^{m} q_j(\sigma) \frac{d^j w}{dt^j}(t, \sigma) = 0, \tag{1.1'}$$

$$\frac{d^k w}{dt^k}(0, \sigma) = \frac{\delta_{k,m-1}}{q_m(\sigma)}, \quad k = 0, 1, \ldots, m-1, \quad t \in R. \tag{1.2'}$$

Moreover, the following relations hold:

$$\left(w(t, \sigma), \frac{dw}{dt}(t, \sigma), \ldots, \frac{d^{m-1}w}{dt^{m-1}}(t, \sigma)\right)^T = \left(\frac{1}{q_m(\sigma)} \exp \frac{t}{q_m(\sigma)} Q(\sigma)\right) (0, 0, \ldots, 0, 1)^T, \tag{1.3}$$

where the superscript T *denotes the transposition,*

$$\left|\frac{d^j w}{dt^j}(t, \sigma)\right| \leqq \frac{e^{t\Lambda(\sigma)}}{|q_m(\sigma)|}\left(1 + \frac{2t}{|q_m(\sigma)|}\|Q(\sigma)\| + \ldots + \frac{(2t)^{m-1}}{|q_m(\sigma)|^{m-1}}\|Q(\sigma)\|^{m-1}\right), \quad j = 0, 1, \ldots, m-1, \tag{1.4}$$

$$(1.5)\qquad \left|\frac{\mathrm{d}^m w}{\mathrm{d}t^m}(t,\sigma)\right| \leqq m|q_m(\sigma)|^{-2} \max_{0\leqq j\leqq m-1} |q_j(\sigma)| \cdot$$

$$\cdot\left(1 + \frac{2t}{|q_m(\sigma)|}\,\|Q(\sigma)\| + \ldots + \frac{(2t)^{m-1}}{|q_m(\sigma)|^{m-1}}\,\|Q(\sigma)\|^{m-1}\right)\mathrm{e}^{tA(\sigma)}, \quad t\in R, \quad \sigma\in\Omega,$$

$$(1.6)\qquad Q(\sigma) = \begin{bmatrix} 0, & q_m(\sigma), & 0, & \ldots, & 0, & 0 \\ 0, & 0, & q_m(\sigma), & \ldots, & 0, & 0 \\ \vdots & & & & & \vdots \\ 0, & 0, & 0, & \ldots, & 0, & q_m(\sigma) \\ -q_0(\sigma), & -q_1(\sigma), & -q_2(\sigma), & \ldots, & -q_{m-2}(\sigma), & -q_{m-1}(\sigma) \end{bmatrix}$$

and

$$(1.7)\quad A(\sigma) = \max_{1\leqq j\leqq m} \operatorname{Re}\lambda_j(\sigma),\ \sigma\in\Omega,$$

$\lambda_j(\sigma)$ being all roots of the polynomial $\sum_{j=0}^{m}\lambda^j q_j(\sigma)$.

Proof. This result is an easy consequence of standard theorems on linear ordinary differential equations.

Let us note that (1.4) follows from (1.3) and Theorem 1.4. The estimate (1.5) is derived from (1.1′) and (1.4).

Lemma 1.2. *Let Ω, $q_j(t,\sigma)$, $j = 0, 1, \ldots, m$ and $w(t,\tau,\sigma)$ be as in Lemma 1.1. Suppose that q_j are periodic in t with a period $\omega > 0$ and that there are constants $C > 0$ and $p_0\in R$ such that*

$$(1.8)\quad \int_0^\infty \left|\frac{\mathrm{d}^j w}{\mathrm{d}t^j}(t, t-\tau, \sigma)\right| \mathrm{d}\tau \leqq C(1+|\sigma|)^{p_0}, \quad t\in R, \quad \sigma\in\Omega, \quad j = 0, 1, \ldots, m-1.$$

Then for any $f\in C_\omega(R;E)$ and any $\sigma\in\Omega$ there exists a unique $v(\cdot,\sigma)\in C_\omega^m(R;E)$ such that

$$(1.9)\qquad \sum_{j=0}^{m} q_j(t,\sigma)\,\frac{\mathrm{d}^j v}{\mathrm{d}t^j}(t,\sigma) = f(t), \quad t\in R,$$

and it is given by

$$(1.10)\quad v(t,\sigma) = \int_0^\infty w(t, t-\tau, \sigma)\, f(t-\tau)\,\mathrm{d}\tau = \int_{-\infty}^{t} w(t,\tau,\sigma)\, f(\tau)\,\mathrm{d}\tau, \quad t\in R.$$

Moreover, $(\mathrm{d}^j v/\mathrm{d}t^j)(t,\sigma)$, $j = 0, 1, \ldots, m-1$, *are continuous in t and σ, holomorphic with respect to σ in Ω^0 and*

$$(1.11)\qquad \left\|\frac{\mathrm{d}^j v}{\mathrm{d}t^j}(t,\sigma)\right\|_E \leqq C(1+|\sigma|)^{p_0}\,\|f\|_{C_\omega(R;E)}, \quad j = 0, 1, \ldots, m-1,$$

$$\left\|\frac{\mathrm{d}^m v}{\mathrm{d}t^m}(t,\sigma)\right\|_E \leqq \max_{\substack{\tau\in[0,\omega]\\ 0\leqq j\leqq m-1}} \frac{|q_j(\tau,\sigma)|}{|q_m(\tau,\sigma)|}\,(1+|\sigma|)^{p_0}$$

holds for all $t\in R$ and $\sigma\in\Omega$.

In the case when the q_j's do not depend on t let us suppose that $\Lambda(\sigma)$ given by (1.7) *satisfies*

$$\Lambda(\sigma) \leqq -d(1 + |\sigma|)^\alpha, \quad \sigma \in \Omega, \tag{1.12}$$

with some constants $d > 0$ and $\alpha \in R$.

Then for any $f \in C_\omega(R; E)$ and any $\sigma \in \Omega$ there is a function $v(\cdot, \sigma) \in C_\omega^m(R; E)$ satisfying (1.9). *This function is given by*

$$v(t, \sigma) = \int_0^\infty w(\tau, \sigma) f(t - \tau)\, \mathrm{d}\tau = \int_{-\infty}^t w(t - \tau, \sigma) f(\tau)\, \mathrm{d}\tau, \quad t \in R, \tag{1.13}$$

with $w(t, \sigma)$ given by (1.3), *the functions $(\mathrm{d}^j v/\mathrm{d}t^j)(t, \sigma)$, $j = 0, 1, \ldots, m$ are continuous in t and σ and holomorphic with respect to σ in Ω^0 and*

$$\left\| \frac{\mathrm{d}^j v}{\mathrm{d}t^j}(t, \sigma) \right\|_E \leqq \sum_{k=1}^m \frac{2^{k-1} \|Q(\sigma)\|^{k-1}}{|q_m(\sigma)|^k d^k (1 + |\sigma|)^{k\alpha}} \|f\|_{C_\omega(R;E)}, \tag{1.14}$$
$$j = 0, 1, \ldots, m - 1,$$

$$\left\| \frac{\mathrm{d}^m v}{\mathrm{d}t^m}(t, \sigma) \right\|_E \leqq |q_m(\sigma)|^{-1} \max_{0 \leqq j \leqq m-1} |q_j(\sigma)| \sum_{k=1}^m \frac{2^{k-1} \|Q(\sigma)\|^{k-1}}{|q_m(\sigma)|^k d^k (1 + |\sigma|)^{k\alpha}} \|f\|_{C_\omega(R;E)}$$

holds for $t \in R$ and $\sigma \in \Omega$, where $Q(\sigma)$ is defined by (1.4).

Proof. By (1.8) the formula (1.10) defines a function $v(t, \sigma)$ which is continuous in t and σ and holomorphic with respect to σ in Ω^0. Moreover,

$$\frac{\mathrm{d}^j v}{\mathrm{d}t^j}(t, \sigma) = \int_{-\infty}^t \frac{\partial^j w}{\partial t^j}(t, \tau, \sigma) f(\tau)\, \mathrm{d}\tau = \int_0^\infty \frac{\partial^j w}{\partial t^j}(t, t - \tau, \sigma) f(t - \tau)\, \mathrm{d}\tau,$$
$$j = 0, 1, \ldots, m - 1,$$

$$\frac{\mathrm{d}^m v}{\mathrm{d}t^m}(t, \sigma) = \frac{1}{q_m(t, \sigma)} f(t) + \int_{-\infty}^t \frac{\partial^m v}{\partial t^m}(t, \tau, \sigma) f(\tau)\, \mathrm{d}\tau,$$

which immediately implies (1.9). Clearly, $(\mathrm{d}^j v/\mathrm{d}t^j)(t, \sigma)$, $j = 1, \ldots, m$, are continuous in t, σ and holomorphic with respect to σ in Ω^0. The assumption (1.8) implies (1.11) by

$$\left\| \frac{\mathrm{d}^j v}{\mathrm{d}t^j}(t, \sigma) \right\|_E \leqq \int_0^\infty \left| \frac{\partial^j w}{\mathrm{d}t^j}(t, t - \tau, \sigma) \right| \mathrm{d}\tau \max_{\tau \in [0, \omega]} \|f(\tau)\|_E, \quad j = 0, 1, \ldots, m,$$

from where (1.11) is obtained immediately by (1.8). The function $v(t, \sigma)$ is ω-periodic in t by virtue of (1.10) and the ω-periodicity of the function $f(t)$.

The case of constant coefficients is similar except for (1.14) which results from (1.13), (1.4) and (1.5). It remains to verify the uniqueness of the ω-periodic solution just found. Let $v(t, \sigma)$ be an ω-periodic solution of

$$P\left(t, \frac{\mathrm{d}}{\mathrm{d}t}, \sigma\right) x(t, \sigma) = 0 . \tag{1.15}$$

If $X(t, \tau, \sigma)$ is a fundamental matrix for (1.15) considered as a system then by the Floquet-Ljapunov theory there exists a continuous non-singular ω-periodic matrix $F(t, \sigma)$ $(F(0, \sigma) = \mathrm{Id})$, such that the substitution $\mathbf{x}(t, \sigma) = F(t, \sigma)\, \mathbf{y}(t, \sigma)$, where

$$\mathbf{x}(t, \sigma) = \left(x(t, \sigma), \frac{\partial x}{\partial t}(t, \sigma), \ldots, \frac{\partial^{m-1} x}{\partial t^{m-1}}(t, \sigma)\right),$$

transforms (1.15) into a system

$$\frac{\partial \mathbf{y}}{\partial t}(t, \sigma) = K(\sigma)\, \mathbf{y}(t, \sigma) \tag{1.16}$$

with $K(\sigma) = \omega^{-1} \ln X(\tau + \omega, \tau, \sigma)$. The function $\mathbf{w}_1(t, \tau, \sigma) = F(t, \sigma)^{-1}\, \mathbf{w}(t, \tau, \sigma)$ satisfies (1.16) and by (1.8) we have

$$\int_0^\infty \left\| \frac{\partial^j \mathbf{w}_1}{\partial t^j}(t + \tau, \tau, \sigma) \right\|_{R^m} \mathrm{d}t < \infty , \quad j = 1, 2, \ldots ,$$

where the estimates for $j \geqq m$ can be reiterated from the equation. But as $K(\sigma)$ is independent of t and $\mathbf{w}_1(\tau, \tau, \sigma) = (0, 0, \ldots, 0, q_m(\tau, \sigma)^{-1})$, the elements of the fundamental matrix $Y(t, \tau, \sigma)$ of (1.16) can be expressed as linear combinations of the components of $\mathbf{w}_1(t, \tau, \sigma)$, $(\partial \mathbf{w}_1/\partial t)(t, \tau, \sigma)$ etc. Thus

$$n \int_0^\omega \|\mathbf{v}(t, \sigma)\|_{R^m} \mathrm{d}t = \int_0^{n\omega} \|\mathbf{v}(t, \sigma)\|_{R^m} \mathrm{d}t = \int_0^{n\omega} \|X(t, 0, \sigma)\, \mathbf{v}(0, \sigma)\|_{R^m} \mathrm{d}t =$$

$$= \int_0^{n\omega} \|F(t, \sigma)\, Y(t, 0, \sigma)\, \mathbf{v}(0, \sigma)\|_{R^m} \mathrm{d}t \leqq \text{const.} \int_0^\infty \|Y(t, 0, \sigma)\, \mathbf{v}(0, \sigma)\|_{R^m} \mathrm{d}t < \infty$$

holds for all $n \in N$. Hence we conclude that $v(t, \sigma) \equiv 0$.

2. THE EQUATION WITH A SINGLE OPERATOR

Let $P(t, \lambda, z) = \sum_{j=0}^{m} Q_j(t, z)\, \lambda^j$, $m \in N$,

where

$$Q_j(z) = \sum_{k=0}^{n_j} q_{jk}(t)\, z^k , \quad n_j \geqq 0 , \quad j = 0, 1, \ldots, m , \quad q_{jk} \in C_\omega(R; C) ,$$

$$k = 0, 1, \ldots, n_j , \quad j = 0, 1, \ldots, m ,$$

$\omega > 0$ being a fixed period. Let us suppose that we have a linear operator A in a complex Banach space B which is the generator of a strongly continuous group

$\{T(s)\}_{s\in R}$ of linear bounded operators in B. Let a function $f \in C_\omega(R; B)$ be given. The purpose of this section is to construct a function $u \in \bigcap_{j=0}^{m} C_\omega^{(j)}(R; D(A^{n_j}))$ satisfying the equation

$$P\left(t, \frac{\mathrm{d}}{\mathrm{d}t}, A\right) u(t) = f(t), \quad t \in R, \tag{2.1}$$

in B. Our idea follows that of R. Hersh, [9] and consists in the following formalism: Having a B-valued function $v(t, s)$ satisfying

$$P\left(t, \frac{\partial}{\partial t}, \frac{\partial}{\partial s}\right) v(t, s) = \delta(s) f(t), \tag{2.2}$$

$$v(t + \omega, s) = v(t, s), \quad t \in R, \quad s \in R \tag{2.3}$$

(here δ means the usual Dirac's δ), set formally

$$u(t) = \int_{-\infty}^{\infty} T(s)\, v(t, s)\, \mathrm{d}s. \tag{2.4}$$

Assuming $v(t, s) \to 0$ as $|s| \to \infty$ in an appropriate way and integrating by parts we have

$$\begin{aligned}
P\left(t, \frac{\mathrm{d}}{\mathrm{d}t}, A\right) u(t) &= P\left(t, \frac{\mathrm{d}}{\mathrm{d}t}, A\right) \int_{-\infty}^{\infty} T(s)\, v(t, s)\, \mathrm{d}s = \\
= \int_{-\infty}^{\infty} P\left(t, \frac{\mathrm{d}}{\mathrm{d}t}, A\right) T(s)\, v(t, s)\, \mathrm{d}s &= \int_{-\infty}^{\infty} P\left(t, \frac{\partial}{\partial t}, \frac{-\partial}{\partial s}\right) T(s) \\
v(t, s)\, \mathrm{d}s = \int_{-\infty}^{\infty} T(s)\, &P\left(t, \frac{\partial}{\partial t}, \frac{\partial}{\partial s}\right) v(t, s)\, \mathrm{d}s = \\
= \int_{-\infty}^{\infty} T(s)\, \delta(s) f(t)\, \mathrm{d}s &= T(0) f(t) = f(t).
\end{aligned} \tag{2.5}$$

In what follows we try to correct the equation (2.2) and the formula (2.4) in such a way to make the arrangements (2.5) justifiable.

Choose numbers $p \geqq -1$ and $b^+ > a^+$, $b^- > a^-$, $b > \max\{b^+, b^-\}$, where a^+, a^- are the numbers from Theorem 1.1. Suppose that $f \in C_\omega(R; D(A^{p+2}))$ and put

$$\varphi^{\pm}(s) = \frac{s^{p+1}}{(p+1)!} \mathrm{e}^{(-b \pm b^{\pm})s}$$

for $s \geqq 0$ and $\varphi^{\pm}(s) = 0$ for $s < 0$, i.e. $\varphi^{\pm}(s)$ is the $(p + 2)$-fold primitive of $\delta(s)$ times $\mathrm{e}^{(-b \pm b^{\pm})s}$.

Writing $v^{\pm}(t, s) = \mathrm{e}^{\pm b^{\pm} s}\, v(t, s)$ we have

$$P\left(t, \frac{\partial}{\partial t}, \frac{\partial}{\partial s}\right) v(t, s) = P\left(t, \frac{\partial}{\partial t}, \frac{\partial}{\partial s} \mp b^{\pm}\right) v^{\pm}(t, s).$$

Further, it can be easily verified that

$$\int_0^\infty \frac{s^{p+1}}{(p+1)!} e^{-bs}(b+A)^{p+2} T(s) f(t)\, ds = f(t)$$

(see the proof of Theorem 2.1). Finally, it appears to be convenient to search for the ω-periodic solution of (2.1) in the form

$$u(t) = \int_{-\infty}^0 e^{b^- s} T(s)\, v^-(t, s)\, ds + \int_0^\infty e^{-b^+ s} T(s)\, v^+(t, s)\, ds, \tag{2.6}$$

where the functions $v^\pm(t, s)$ are the solutions of

$$P\left(t, \frac{\partial}{\partial t}, \frac{\partial}{\partial s} \mp b^\pm\right) v^\pm(t, s) = \varphi^\pm(s)(b + A)^{p+2} f(t), \tag{2.7$^\pm$}$$

$$v^\pm(t + \omega, s) = v^\pm(t, s), \quad t \in R, \quad s \in R$$

with the corresponding signs.

First of all we shall solve the problems (2.7$^\pm$) in the class of functions $v(t, s)$ with continuous bounded derivatives $(\partial^{k+l} v / \partial t^k\, \partial s^l)(t, s)$, $t \in R$, $s \in R$, $k = 0, 1, \ldots, m$, $l = 0, 1, \ldots, n = \max_{0 \leqq j \leqq m} n_j$. Applying the Fourier transform to (2.7$^\pm$), the functions

$$\hat{v}^\pm(t, \sigma) = \int_{-\infty}^\infty e^{-i\sigma s} v^\pm(t, s)\, ds$$

are to satisfy

$$P\left(t, \frac{d}{dt}, i\sigma \mp b^\pm\right) \hat{v}^\pm(t, \sigma) = (b \mp b^\pm + i\sigma)^{-p-2} (b + A)^{p+2} f(t), \tag{2.8$^\pm$}$$

$$\hat{v}^\pm(t + \omega, \sigma) = \hat{v}^\pm(t, \sigma), \quad t \in R, \quad \sigma \in R,$$

with the corresponding signs. Suppose that

$$Q_m(t, i\sigma \mp b^\pm) \neq 0, \quad t \in R, \quad \sigma \in R. \tag{2.9}$$

Then by Lemma 1.1 there are functions $w^\pm(t, \tau, \sigma)$, $\tau \leqq t$, $\sigma \in R$, satisfying

$$P\left(t, \frac{d}{dt}, i\sigma \mp b^\pm\right) w^\pm(t, \tau, \sigma) = 0, \tag{2.10}$$

$$\frac{d^k w^\pm}{dt^k}(\tau, \tau, \sigma) = \frac{\delta_{k,m-1}}{Q_m(\tau, i\sigma \mp b^\pm)}, \quad \tau \leqq t,$$

$$\sigma \in R, \quad k = 0, 1, \ldots, m - 1.$$

Suppose that there exist constants $C > 0$ and $p_0 \in R$ such that

$$\int_0^\infty \left| \frac{d^j w^\pm}{dt^j}(t, t - \tau, \sigma) \right| d\tau \leqq C(1 + |\sigma|)^{p_0}, \quad \sigma \in R, \quad j = 0, 1, \ldots, m - 1. \tag{2.11}$$

Then by Lemma 1.2, ($2.8^{\pm}$) have the solutions

$$\hat{v}^{\pm}(t,\sigma) = (b \mp b^{\pm} + \mathrm{i}\sigma)^{-p-2} \int_0^{\infty} w^{\pm}(t, t-\tau, \sigma) \,. \tag{2.12}$$

$$. (b + A)^{p+2} f(t-\tau)\,\mathrm{d}\tau\,, \quad t \in R\,, \quad \sigma \in R\,,$$

with the corresponding signs.

The functions $(\mathrm{d}^j \hat{v}^{\pm}/\mathrm{d}t^j)(t,\sigma)$, $j = 0, 1, \ldots, m$ are continuous in t and σ and satisfy the inequalities

(2.13)

$$\left\| \frac{\mathrm{d}^j \hat{v}^{\pm}}{\mathrm{d}t^j}(t,\sigma) \right\| \leqq C\left|b \mp b^{\pm} + \mathrm{i}\sigma\right|^{-p-2} (1 + |\sigma|)^{p_0} \left\|(b+A)^{p+2} f\right\|_{C_\omega(R;B)}\,,$$

$$j = 0, 1, \ldots, m-1\,,$$

$$\left\| \frac{\mathrm{d}^m \hat{v}^{\pm}}{\mathrm{d}t^m}(t,\sigma) \right\| \leqq C\left|b \mp b^{\pm} + \mathrm{i}\sigma\right|^{-p-2} (1 + |\sigma|)^{p_0} \,.$$

$$. \max\left\{ \frac{|Q_j(t, \mathrm{i}\sigma \mp b^{\pm})|}{|Q_m(t, \mathrm{i}\sigma \mp b^{\pm})|}\,;\quad t \in [0, \omega]\,,\quad j = 0, 1, \ldots, m-1 \right\}.$$

$$. \left\|(b+A)^{p+2} f\right\|_{C_\omega(R;B)}\,, \quad t \in R\,, \quad \sigma \in R\,.$$

Theorem 2.1. *Let a polynomial $P(t, \lambda, z)$ and an operator A satisfy the assumptions stated at the beginning of this section together with* (2.9) *and* (2.11), *where the functions $w^{\pm}$ are determined by* (2.10).

Let $n = \max_{1 \leqq j \leqq m} n_j$ and suppose that $p \geqq n + p_0$. Moreover, assume that the following implication holds:

(2.14) *If $z : R \times R \to B$ is a function such that*

$$\frac{\partial^{j+k} z}{\partial t^j\, \partial s^k} \in C(R \times R; B)\,, \quad \left\| \frac{\partial^{j+k} z}{\partial t^j\, \partial s^k}(t,s) \right\| \leqq$$

$$\leqq \text{const.} \begin{cases} \mathrm{e}^{b^- s}\,, & s \geqq 0\,, \\ \mathrm{e}^{-b^+ s}\,, & s \leqq 0 \end{cases} \quad j = 0, 1, \ldots, m\,, \quad k = 0, 1, \ldots, n = \max_{1 \leqq j \leqq m} n_j\,,$$

$$P\left(t, \frac{\partial}{\partial t}, \frac{\partial}{\partial s}\right) z(t,s) = 0\,,$$

$$z(t+\omega, s) = z(t,s)\,, \quad t \in R\,, \quad s \in R\,,$$

then $z(t,s) \equiv 0$.

Under these conditions for any $f \in C_\omega(R; D(A^{p+2}))$ ($f \in C_\omega(R; B)$ if $p < -2$) the function u given by (2.6), *where*

$$v^{\pm}(t,s) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}s\sigma}\, \hat{v}^{\pm}(t,\sigma)\,\mathrm{d}\sigma \tag{2.15}$$

with $\hat{v}^{\pm}(t, \sigma)$ *given by* (2.12), *belongs to* $\bigcap_{j=0}^{m} C^j(R; D(A^n))$, *satisfies* (2.1) *and*

$$\max_{0 \leq j \leq m} \left\{ \left\| \frac{d^j u}{dt^j}(t) \right\|_{C_\omega(R; D(A^n))} \right\} \leq K \|f\|_{C_\omega(R; D(A^{p+2}))}$$

holds with a constant K independent of f.

Proof. By the assumption $p \geq n + p_0$ and (2.13) it is clear that

$$\left\| (i\sigma)^k \frac{d^j \hat{v}^{\pm}}{dt^j}(t, \sigma) \right\| \leq \text{const.}\, (1 + |\sigma|)^{-2} \|f\|_{C_\omega(R; D(A^{p+2}))}, \; k = 0, 1, \ldots, n .$$

Note that $\|(b + A)^l x\| \leq \text{const.}\, (\|x\| + \|A^l x\|)$, $x \in D(A^l)$, $l \in N$ by the closed graph theorem.

Thus (2.15) implies that the functions

$$\frac{\partial^{j+k} v^{\pm}}{\partial t^j \, \partial s^k}(t, s), \quad j = 0, 1, \ldots, m, \quad k = 0, 1, \ldots, n$$

are continuous and their norms are bounded by const. $\|f\|_{C_\omega(R; D(A^{p+2}))}$. Hence the integrals

$$\int_{-\infty}^{0} e^{-b^- s} T(s) \frac{\partial^{j+k} v^-}{\partial t^j \, \partial s^k}(t, s) \, ds, \; \int_{0}^{\infty} e^{-b^+ s} T(s) \frac{\partial^{j+k} v^+}{\partial t^j \, \partial s^k}(t, s) \, ds,$$

$$j = 0, 1, \ldots, m, \quad k = 0, 1, \ldots, n,$$

converge and are continuous functions of t.

In particular, the function $u(t)$ given by (2.6) is continuous. Let j and k be integers, $0 \leq j \leq m$, $0 \leq k \leq n$. Let us prove that $u(t) \in D(A^k)$ for all $t \in R$ and that the derivative $(d^j A^k u / dt^j)(t)$ exists and is continuous. Putting $R_\lambda = \lambda R(\lambda, A)$ for $\lambda \in R$, $|\lambda| > \max\{a^+, a^-\}$, we have

$$(2.16) \quad A^k R_\lambda^k u(t) = \int_{-\infty}^{0} e^{b^- s} A^k T(s) R_\lambda^k v^-(t, s) \, ds + \int_{0}^{\infty} e^{-b^+ s} A^k T(s) R_\lambda^k v^+(t, s) \, ds =$$

$$= \int_{-\infty}^{0} \left(\frac{-d}{ds} \right)^k T(s) [e^{b^- s} R_\lambda^k v^-(t, s)] \, ds + \int_{0}^{\infty} \left(\frac{-d}{ds} \right)^k T(s) [e^{-b^+ s} R_\lambda^k v^+(t, s)] \, ds =$$

$$= \sum_{l=0}^{k-1} \left[\left(\frac{-d}{ds} \right)^{k-l-1} T(s) \frac{\partial^l}{\partial s^l} (e^{b^- s} R_\lambda^k v^-(t, s) \right]_{s=-\infty}^{0} +$$

$$+ \sum_{l=0}^{k-1} \left[\left(\frac{-d}{ds} \right)^{k-l-1} T(s) \frac{\partial^l}{\partial s^l} (e^{-b^+ s} R_\lambda^k v^+(t, s)) \right]_{s=0}^{\infty} +$$

$$+ \int_{-\infty}^{0} T(s) \frac{\partial^k}{\partial s^k} (e^{b^- s} R_\lambda^k v^-(t, s)) \, ds + \int_{0}^{\infty} T(s) \frac{\partial^k}{\partial s^k} (e^{-b^+ s} R_\lambda^k v^+(t, s)) \, ds =$$

$$= \sum_{l=0}^{k-1} A^{k-l-1} R_\lambda^k \left[\left(\frac{\partial}{\partial s} + b^- \right)^l v^-(t, s)|_{s=0} - \left(\frac{\partial}{\partial s} - b^+ \right)^l v^+(t, s)|_{s=0} \right] +$$

$$+ R_\lambda^k \int_{-\infty}^0 e^{b^- s} T(s) \left(\frac{\partial}{\partial s} + b^- \right)^k v^-(t, s) \, ds + R_\lambda^k \int_0^\infty e^{-b^+ s} T(s) \left(\frac{\partial}{\partial s} - b^+ \right)^k v^+(t, s) \, ds \, .$$

Let $z^\pm(t, s) = e^{\mp b^\pm s} v^\pm(t, s)$. Since the functions $v^\pm(t, s)$ obviously satisfy $(2.7^\pm)$ with the corresponding signs and the relations

$$\left(\frac{\partial}{\partial s} \mp b^\pm \right) v^\pm(t, s) = e^{\pm b^\pm s} \frac{\partial z^\pm}{\partial s} (t, s)$$

hold, it is clear that

$$P\left(t, \frac{\partial}{\partial t}, \frac{\partial}{\partial s} \right) z^\pm(t, s) = \theta(s) \frac{s^{p+1}}{(p+1)!} e^{-bs} (b + A)^{p+2} f(t) \, , \quad t \in R \, , \quad s \in R \, ,$$

where $\theta(s)$ is the usual Heaviside function. Thus we have

$$P\left(t, \frac{\partial}{\partial t}, \frac{\partial}{\partial s} \right) (z^+(t, s) - z^-(t, s)) = 0 \, , \quad t \in R \, , \quad s \in R \, .$$

The assumption (2.14) implies $z^+(t, s) - z^-(t, s) \equiv 0$, i.e. $e^{-b^+ s} v^+(t, s) = e^{b^- s} v^-(t, s)$, $t \in R$, $s \in R$.

Differentiating successively the last relation by s and putting $s = 0$ we get

$$\left(\frac{\partial}{\partial s} - b^+ \right)^l v^+(t, s)|_{s=0} = \left(\frac{\partial}{\partial s} + b^- \right)^l v^-(t, s)|_{s=0} \, , \quad t \in R \, , \quad l = 0, 1, \ldots, n \, .$$

Hence (2.16) is reduced to

$$A^k R_\lambda^k u(t) = R_\lambda^k \int_{-\infty}^0 e^{b^- s} T(s) \left(\frac{\partial}{\partial s} + b^- \right)^k v^-(t, s) +$$

$$+ R_\lambda^k \int_0^\infty e^{-b^+ s} T(s) \left(\frac{\partial}{\partial s} - b^+ \right)^k v^+(t. s) \, ds \, .$$

Now letting $\lambda \to \infty$ we obtain by Theorem 1.2, (ii) and (iv) that $u(t) \in D(A^k)$ and

$$(2.17) \qquad A^k u(t) = \int_{-\infty}^0 e^{b^- s} T(s) \left(\frac{\partial}{\partial s} + b^- \right)^k v^-(t, s) \, ds +$$

$$+ \int_0^\infty e^{b^+ s} T(s) \left(\frac{\partial}{\partial s} - b^+ \right)^k v^+(t, s) \, ds \, , \quad t \in R .$$

As we may interchange integration and differentiation when differentiating (2.17) we have finally

$$\text{(2.18)} \qquad \frac{\mathrm{d}^j}{\mathrm{d}t^j} A^k u(t) = \int_{-\infty}^{0} \mathrm{e}^{b^- s}\, T(s) \frac{\partial^j}{\partial t^j}\left(\frac{\partial}{\partial s} + b^-\right)^k v^-(t, s)\, \mathrm{d}s + $$

$$+ \int_0^{\infty} \mathrm{e}^{-b^+ s}\, T(s) \frac{\partial^j}{\partial t^j}\left(\frac{\partial}{\partial s} - b^+\right)^k v^+(t, s)\, \mathrm{d}s\,.$$

We have proved $u \in \bigcap_{j=0}^{m} C^j_\omega(R;\, D(A^{n_j}))$. It remains to verify (2.1). By (2.18) and $(2.7^\pm)$, the following arrangements are correct:

$$P\left(t, \frac{\mathrm{d}}{\mathrm{d}t}, A\right) u(t) = \int_{-\infty}^{0} \mathrm{e}^{b^- s}\, T(s)\, P\left(t, \frac{\partial}{\partial t}, \frac{\partial}{\partial s} + b^-\right) v^-(t, s)\, \mathrm{d}s + $$

$$+ \int_0^{\infty} \mathrm{e}^{-b^+ s}\, T(s)\, P\left(t, \frac{\partial}{\partial t}, \frac{\partial}{\partial s} - b^+\right) v^+(t, s)\, \mathrm{d}s = $$

$$= \int_0^{\infty} \frac{s^{p+1}}{(p+1)!}\, \mathrm{e}^{-bs}(b + A)^{p+2}\, T(s) f(t)\, \mathrm{d}s = $$

$$= \int_0^{\infty} \frac{s^{p+1}}{(p+1)!}\, \mathrm{e}^{-bs}\left(b - \frac{\mathrm{d}}{\mathrm{d}s}\right)^{p+2} T(s) f(t)\, \mathrm{d}s = $$

$$= \int_0^{\infty} \frac{s^{p+1}}{(p+1)!}\left(-\frac{\mathrm{d}}{\mathrm{d}s}\right)^{p+2} (\mathrm{e}^{-bs}\, T(s)) f(t)\, \mathrm{d}s = $$

$$= \int_0^{\infty} \frac{\mathrm{d}^{p+1}}{\mathrm{d}s^{p+1}}\left[\frac{s^{p+1}}{(p+1)!}\right]\left(-\frac{\mathrm{d}}{\mathrm{d}s}\right)(\mathrm{e}^{-bs}\, T(s) f(t))\, \mathrm{d}s = $$

$$= -\int_0^{\infty} \frac{\mathrm{d}}{\mathrm{d}s} \exp\left[-(b + A)\, s\right] f(t)\, \mathrm{d}s = f(t)\,, \quad t \in R\,,$$

where we integrated by parts and used Theorem 1.2, (v).

The following theorem provides a more explicit condition for the existence of a periodic solution of (2.1) than (2.14), along with a condition for the uniqueness of the given problem.

Theorem 2.2. *Let a polynomial $P(t, \lambda, z)$ and an operator A satisfy the requirements stated at the beginning of this Section. Let the numbers b^+ and b^- be chosen as in Theorem 2.1. Suppose that there are constants $C > 0$, $p_0 \in R$, $\beta^+ > b^+$ and $\beta^- > b^-$ such that*

$$\text{(2.19)} \qquad Q_m(t, \mathrm{i}\sigma) \neq 0, \quad t \in R, \quad \sigma \in S = \{\sigma \in C;\ -\beta^- \leqq \operatorname{Im} \sigma \leqq \beta^+\}$$

and that the function $w(t, \tau, \sigma)$ determined by the equations

$$P\left(t, \frac{\mathrm{d}}{\mathrm{d}t}, \mathrm{i}\sigma\right) w(t, \tau, \sigma) = 0\,, \quad \frac{\partial^j w}{\partial t^j}(\tau, \tau, \sigma) = \frac{\delta_{j,m-1}}{Q_m(\tau, \sigma)}\,, \quad j = 0, 1, \ldots, m-1$$

satisfies the inequalities

(2.20)

$$\int_0^\infty \left|\frac{\partial^j w}{\partial t^j}(t, t-\tau, \sigma)\right| \mathrm{d}\tau \leqq C(1+|\sigma|)^{p_0} \quad \text{for} \quad j = 0, 1, \ldots, m-1, \quad \sigma \in S.$$

Then the assumption (2.14) *of Theorem* 2.1 *is satisfied and the problem*

$$P\left(t, \frac{\mathrm{d}}{\mathrm{d}t}, A\right) u(t) = 0, \tag{2.21}$$

$$u(t+\omega) = u(t), \quad t \in R$$

has only the trivial solution.

Proof Let the functions $\varphi \in C_0^\infty(R)$ and $f \in C_\omega(R; C)$ be arbitrary.
We shall find a function $v(t, s)$ satisfying

$$P\left(t, \frac{-\partial}{\partial t}, \frac{-\partial}{\partial s}\right) v(t, s) = f(t)\,\varphi(s), \tag{2.22}$$

$$v(t+\omega, s) = v(t, s), \quad t \in R, \quad s \in R,$$

in the classical sense. If we write $v(t, s) = (2\pi)^{-1} \int_{-\infty}^{\infty} \hat{v}(t, \sigma)\, \mathrm{e}^{\mathrm{i}s\sigma}\, \mathrm{d}\sigma$ then the function $\hat{v}$ is to satisfy

$$P\left(t, \frac{-\mathrm{d}}{\mathrm{d}t}, -\mathrm{i}\sigma\right) \hat{v}(t, \sigma) = f(t)\,\hat{\varphi}(\sigma), \tag{2.23}$$

$$\hat{v}(t+\omega, \sigma) = \hat{v}(t, \sigma), \quad t \in R, \quad \sigma \in R.$$

It follows from Lemma 1.2 that

$$\hat{v}(t, \sigma) = \int_0^\infty w(-t, -t-\tau, -\sigma) f(-\tau - t)\, \hat{\varphi}(\sigma)\, \mathrm{d}\tau, \quad t \in R, \quad \sigma \in S,$$

is a solution of (2.23) even for $\sigma \in S$. By [7], Vol. 1, p. 175, the function $\hat{\varphi}(\sigma)$ is holomorphic in all C and

$$(1 + |\sigma|)^k\, |\hat{\varphi}(\sigma)| \leqq C_k \mathrm{e}^{a|\mathrm{Im}\,\sigma|} \tag{2.24}$$

holds for all $\sigma \in C$, $k = 0, 1, \ldots$, with some constants C_k and with $a > 0$ such that $\operatorname{supp} \varphi \subset [-a, a]$. It is clear from (2.20) and Lemma 1.1 that the functions $(\mathrm{d}^j\hat{v}/\mathrm{d}t^j)(t, \sigma)$, $j = 0, 1, \ldots, m$, are holomorphic in S and that

$$\left\|\int_0^\infty \frac{\mathrm{d}^j w}{\mathrm{d}t^j}(t, t-\tau, \sigma) f(t-\tau)\, \mathrm{d}\tau\right\| \leqq C(1+|\sigma|)^p\, \|f\|_{C_\omega(R;C)}$$

holds for all $j = 0, 1, \ldots, m$, $t \in R$ and $\sigma \in S$. This together with (2.24) implies that

$$(1+|\sigma|)^k \left|\frac{\mathrm{d}^j \hat{v}}{\mathrm{d}t^j}(t, \sigma)\right| \leqq M_k \mathrm{e}^{a|\mathrm{Im}\,\sigma|}, \quad \sigma \in S, \quad k = 0, 1, \ldots, \quad j = 0, 1, \ldots\ m$$

with appropriate constants M_k. Therefore, the functions $(\partial^{j+k}v/\partial t^j\, \partial s^k)(t, s)$ are continuous and satisfy (2.22). By Theorem 1.3 for any $\varepsilon > 0$ there exist constants $C^{\pm}(\varepsilon, k)$ such that

$$\left|\frac{\partial^{j+k}v}{\partial t^j\, \partial s^k}(t, s)\right| \leqq C^+(\varepsilon, k)\, \mathrm{e}^{-(\beta^+ - \varepsilon)s}\|f\|_{C_\omega(R;C)}\,, \quad t \in R\,, \quad s \geqq 0\,,$$

$$\left|\frac{\partial^{j+k}v}{\partial t^j\, \partial s^k}(t, s)\right| \leqq C^-(\varepsilon, k)\, \mathrm{e}^{(\beta^- - \varepsilon)s}\|f\|_{C_\omega(R;C)}\,,$$

$$t \in R\,, \quad s < 0\,, \quad j = 0, 1, \ldots, m\,, \quad k = 0, 1, \ldots\,.$$

Now, let $z : R \times R \to B$ be a function such that there exist continuous derivatives $\partial^{j+k}z/\partial t^j\, \partial s^k$ satisfying

$$\left\|\frac{\partial^{j+k}z}{\partial t^j\, \partial s^k}(t, s)\right\| \leqq C^+ \mathrm{e}^{b^+ s}\,, \quad t \in R\,, \quad s \geqq 0\,,$$

$$\left\|\frac{\partial^{j+k}z}{\partial t^j\, \partial s^k}(t, s)\right\| \leqq C^- \mathrm{e}^{-b^- s}, \quad t \in R\,, \quad s < 0$$

for $j = 0, 1, \ldots, m$, $k = 0, 1, \ldots, n$, and

$$P\left(t, \frac{\partial}{\partial t}, \frac{\partial}{\partial s}\right) z(t, s) = 0\,,$$

$$z(t + \omega, s) = z(t, s)\,, \quad t \in R\,, \quad s \in R\,.$$

Choosing $\varepsilon > 0$ so small that $\beta^+ - \varepsilon > b^+$ and $\beta^- - \varepsilon > b^-$ we can write

$$\int_0^\omega \int_{-\infty}^\infty f(t)\, \varphi(s)\, z(t, s)\, \mathrm{d}s\, \mathrm{d}t = \int_0^\omega \int_{-\infty}^\infty P\left(t, \frac{-\partial}{\partial t}, \frac{-\partial}{\partial s}\right)$$

$$v(t, s)\, z(t, s)\, \mathrm{d}s\, \mathrm{d}t = \int_0^\omega \int_{-\infty}^\infty v(t, s)$$

$$P\left(t, \frac{\partial}{\partial t}, \frac{\partial}{\partial s}\right) z(t, s)\, \mathrm{d}s\, \mathrm{d}t = 0\,.$$

Taking into account that $f \in C_\omega(R; C)$ and $\varphi \in C_0^\infty(R)$ are arbitrary we conclude that

$$\langle x^*, z(t, s)\rangle = 0 \quad \text{for all} \quad x^* \in B^* \quad \text{and all} \quad (t, s) \in R^2\,,$$

where B^* is the dual space to B and $\langle \cdot, \cdot \rangle$ denotes the duality between B and B^*. This finally leads to the conclusion $z(t, s) \equiv 0$.

If $u(t)$ is an ω-periodic solution of

$$P\left(t, \frac{\mathrm{d}}{\mathrm{d}t}, A\right) u(t) = 0\,,$$

then according to the above consideration the function $z(t, s) = T(-s)\,u(-t)$ must be identically zero. Thus $u(t)$ must be identically zero as well.

Remark 2.1. In the case when the coefficients of $P(t, \lambda, z)$ are independent of t we can replace the assumptions (2.11) and (2.20) by (1.12), where

$$\Omega = \{\sigma \in C,\ \mathrm{Im}\,\sigma = \pm b^{\pm}\} \quad \text{and} \quad \Omega = S\,, \quad \text{respectively}\,.$$

If the inequality in (1.12) is replaced by $\Lambda(\sigma) \geqq d(1 + |\sigma|)^{\alpha}$ then taking the new variable $\tilde{t} = -t$, the new function $\tilde{\Lambda}(\sigma)$ corresponding to the polynomial $P(t, -\lambda, \mathrm{i}\sigma)$ satisfies $\tilde{\Lambda}(\sigma) \leqq -d(1 + |\sigma|)^{\alpha}$.

Remark 2.2. The assumption (2.19) in fact implies the existence of an everywhere defined, continuous inverse to the operator $Q_m(A)$. Indeed, if $\sigma_1, \sigma_2, \ldots, \sigma_{n_m}$ are all the roots of the polynomial $Q_m(\mathrm{i}\sigma)$ then either $\mathrm{Re}\,\sigma_j > \beta^+ > a^+$ or $\mathrm{Re}\,\sigma_j < -\beta^- < -a^-$. Hence $\sigma_j \in \varrho(A)$, $j = 1, 2, \ldots, n_m$ and $Q_m(A)^{-1} = (\sigma_1 - A)^{-1} \cdot (\sigma_2 - A)^{-1} \ldots (\sigma_{n_m} - A)^{-1} \in L(B)$. Unfortunately, in the case that (2.19) fails to be satisfied at a point $\sigma_0 \in S^0$ our method is not applicable exluding some particular situations. This is illustrated by the following

Example 2.1. Let $P(t, \lambda, z) = Q_1(z)\,\lambda + Q_0(z)$. Let $S_\varepsilon = \{\sigma \in C;\ |\mathrm{Im}\,\sigma| < \varepsilon\}$, where $\varepsilon > 0$. Suppose that there is an $\sigma_0 \in S$ such that $Q_1(\mathrm{i}\sigma) = (\mathrm{i}\sigma - \mathrm{i}\sigma_0)^k\,\tilde{Q}_1(\mathrm{i}\sigma)$, $Q_0(\mathrm{i}\sigma) = (\mathrm{i}\sigma - \mathrm{i}\sigma_0)^l\,\tilde{Q}_0(\mathrm{i}\sigma)$, where k and l are integers, $k \geqq 1$, $l \geqq 0$ and $\tilde{Q}_1(\mathrm{i}\sigma_0) \neq 0$, $\tilde{Q}_0(\mathrm{i}\sigma_0) \neq 0$. If $k \leqq l$, then the equation (1.1) can be transformed into the system

$$\tilde{Q}_1(A)\,\frac{\mathrm{d}u}{\mathrm{d}t}\,(t) + (\mathrm{i}\sigma_0 - A)^{l-k}\,\tilde{Q}_0(A)\,u(t) = v(t)\,,$$

$$(\mathrm{i}\sigma_0 - A)^k\,v(t) = f(t)\,,$$

which can be solved provided the second equation is regularly solvable (the condition on A and f) and that

$$\mathrm{Re}\,\frac{(\mathrm{i}\sigma_0 - \mathrm{i}\sigma)^{l-k}\,\tilde{Q}_0(\mathrm{i}\sigma)}{\tilde{Q}_1(\mathrm{i}\sigma)} \leqq -d(1 + |\sigma|)^{\alpha}\,, \quad (d > 0,\ \alpha \in R)\,, \quad \sigma \in S_\varepsilon\,,$$

what implies $k = l$.

If $k > l$ then obviously $\mathrm{Re}\,((\mathrm{i}\sigma_0 - \mathrm{i}\sigma)^{l-k}\,\tilde{Q}_0(\mathrm{i}\sigma)/\tilde{Q}_1(\mathrm{i}\sigma))$ ranges over all reals when $|\sigma - \sigma_0|$ ranges over any interval $(0, \delta)$, $\delta > 0$. This means that the assumption (2.11) or (1.12) on which our method is based cannot be satisfied.

3. THE EQUATION WITH SEVERAL OPERATORS

Let $A_1, A_2, \ldots, A_r$ be linear, possibly unbounded, and mutually commuting operators in B which generate, respectively, strongly continuous groups $\{T_1(s_1)\}_{s_1 \in R}$, $\{T_2(s_2)\}_{s_2 \in R}, \ldots, \{T_r(s_r)\}_{s_r \in R}$ of linear bounded operators in B. Then also $T_j(s_j) \cdot T_k(s_k)\,x = T_k(s_k)\,T_j(s_j)\,x$ holds for $s_j, s_k \in R$; $j, k = 1, 2, \ldots, r$; $x \in B$ (see [14], Lemma 11).

Let $P(t, \lambda, z_1, \ldots, z_r) = \sum_{j=0}^{m} Q_j(t, z_1, \ldots, z_r) \lambda^j$ be a polynomial of $r + 1$ variables $\lambda, z_1, \ldots, z_r$, the coefficients of which are continuous, ω-periodic complex-valued functions of t. Denote by n_{jk} the degree of the polynomial Q_j with respect to the variable z_k.

Let us consider the problem

$$(3.1) \qquad P\left(t, \frac{\mathrm{d}}{\mathrm{d}t}, A_1, \ldots, A_r\right) u(t) = f(t),$$
$$u(t + \omega) = u(t), \quad t \in R,$$

where $f \in C_\omega(R; B)$. A function $u \in U = \bigcap_{j=0}^{m} C_\omega^{(j)}(R; D(\prod_{k=1}^{r} A^{n_{jk}}))$ satisfying (3.1) is called a solution of (3.1).

Let $a_k^\pm \geqq 0$ be such constants that

$$\|T_k(s_k)\| \leqq \begin{cases} C\mathrm{e}^{a_k^+ s_k}, & s_k \geqq 0, \\ C\mathrm{e}^{-a_k^- s_k}, & s_k \leqq 0, \end{cases}$$

holds for $k = 1, 2, \ldots, r$ with an appropriate positive constant C. Take constants $\beta_k^\pm > b_k^\pm > a_k^\pm$, which are sufficiently close to $a_k^\pm$ $(k = 1, 2, \ldots, r)$, and let M be the set of all pairs (k, l), where $k = (k_1, \ldots, k_{q^+})$, $l = (l_1, \ldots, l_{q^-})$, $k \cup l = \{1, 2, \ldots \ldots, r\}$, $k \cap l = \emptyset$, $(q^+ + q^- = r)$. For $(k, l) \in M$ denote by b_{kl} the $2r$-dimensional vector the k_i-th component of which is $b_{k_i}^+$ and the l_j-th component is $b_{l_j}^-$ $(i = 1, \ldots, q^+,$ $j = 1, \ldots, q^-)$, and by $D_{kl}(z)$ the r-dimensional vector the j-th component of which is equal to $z_j - b_j^+$ if $j \in \{k_1, \ldots, k_{q^+}\}$ and to $z_j + b_j^-$ if $j \in \{l_1, \ldots, l_{q^-}\}$. Further, we define the functions

$$\varphi(s, b_{kl}) = \begin{cases} \prod_{j=1}^{r} \dfrac{s^{p_j + 1}}{(p_j + 1)!} \mathrm{e}^{-bs_j} \prod_{j=1}^{q^+} \mathrm{e}^{b_{k_j}^+ s_{k_j}} \prod_{j=1}^{q^-} \mathrm{e}^{-b_{l_j}^- s_{l_j}} & \text{if} \quad s_i \geqq 0, \quad i = 1, \ldots, r, \\ 0 \quad \text{elsewhere in } R^r, \end{cases}$$

where $p = (p_1, \ldots, p_r)$ is a vector with non-negative integral components, $b > \max_{1 \leqq j \leqq r} \{b_j^+, b_j^-\}$ and $(k, l) \in M$.

Suppose that

$$(3.2) \qquad Q_m(t, \mathrm{i}\sigma) \neq 0, \quad t \in [0, \omega],$$
$$\sigma \in S = \{z \in C^r;\ -\beta_j^- < \operatorname{Im} z_j < \beta_j^+,\ j = 1, \ldots, r\}$$

and let functions $w(t, \tau, \sigma, b_{kl})$ be defined as solutions of the problem

$$(3.3) \quad P\left(t, \frac{\mathrm{d}}{\mathrm{d}t}, D_{kl}(\mathrm{i}\sigma)\right) w(t, \tau, \sigma, b_{kl}) = 0, \quad \frac{\mathrm{d}^j w}{\mathrm{d}t^j}(\tau, \tau, \sigma, b_{kl}) = \frac{\delta_{j,m-1}}{Q_m(\tau, \mathrm{i}\sigma)},$$
$$\tau \leqq t, \quad \sigma \in C^r, \quad j = 0, 1, \ldots, m - 1, \quad (k, l) \in M.$$

Assume that there exist constants C and p_0 such that

$$(3.4) \qquad \int_0^\infty \left|\frac{\mathrm{d}^j w}{\mathrm{d}t^j}(t, t-\tau, \sigma, b_{kl})\right| \mathrm{d}\tau \leqq C(1 + |\sigma|)^{p_0},$$

$$t \in R, \quad j = 1, 2, \ldots, m-1, \quad (k, l) \in M, \quad \sigma \in S.$$

The following theorem, the proof of which follows the lines of the proofs of Theorems 2.1 and 2.2, is valid:

Theorem 3.1. *Let a polynomial $P(t, \lambda, z_1, \ldots, z_r)$ and operators $A_1, A_2, \ldots, A_r$ satisfy the requirements stated at the beginning of this section. Moreover, let (3.2) be fulfilled and let the functions $w(t, \tau, \sigma, b_{kl})$, $(k, l) \in M$ defined as the solutions of (3.3) satisfy (3.4).*

Then there exist $p_j \in N$, $j = 1, \ldots, r$, such that for any $f \in F = C_\omega(R; D(\prod_{j=1}^r A_j^{p_j}))$ there exists one and only one solution of (3.1). This solution is given by

$$u(t) = \frac{1}{(2\pi)^r} \sum_{(\{k_1,\ldots,k_{q^+}\},\{l_1,\ldots,l_{q^-}\})\in M} \underbrace{\int_0^\infty \int_0^\infty}_{q^+\text{-times}}$$

$$\underbrace{\int_{-\infty}^0 \int_{-\infty}^0}_{q^-\text{-times}} \prod_{j=1}^{q^+} \mathrm{e}^{-b_{k_j}{}^+ s_{k_j}} T_{k_j}(s_{k_j}) \prod_{j=1}^{q^-} \mathrm{e}^{b_{l_j}{}^- s_{l_j}} T_{l_j}(s_{l_j})$$

$$\int_{R^r} \mathrm{e}^{is\sigma} \prod_{j=1}^{q^+} (b - b_{k_j}^+ + i\sigma_{k_j})^{-p_{k_j}-2} \prod_{j=1}^{q^-} (b + b_{l_j}^- + i\sigma_{l_j})^{-p_{l_j}-2}\,.$$

$$\int_0^\infty w(t, t-\tau, \sigma, b_{kl}) \prod_{j=1}^r (b + A_j)^{p_j+2} f(t-\tau)\,\mathrm{d}\tau$$

$$\mathrm{d}\sigma\, \mathrm{d}s_{l_1} \ldots \mathrm{d}s_{l_{q^-}}\, \mathrm{d}s_{k_1} \ldots \mathrm{d}s_{k_{q^+}}, \quad t \in R,$$

and satisfies

$$\|u\|_U \leqq K \|f\|_F,$$

where U is the space of solutions, F is the space of the right-hand sides and K is a constant independent of $f \in F$.

4. EXTENSIONS TO MORE GENERAL SPACES

In this section we sketch how to employ the techniques of Donaldson [3] and Carrol [1] to obtain a periodic solution to (2.1) in a linear topological space X. In accordance with [1] we suppose that X is a complete, separated, locally convex topological vector space, $L_s(F)$ denotes continuous linear maps $X \to X$ with the strong operator topology. Further, we assume that a closed densely defined linear operator A in X generates a group $\{T(s)\}_{s\in R} \subset L_s(X)$ and there is a constant $a \geqq 0$ such that for any continuous semi-norm p on X, there exists a continuous seminorm q on X such that $p(T(s)\,x) \leqq e^{a|s|}\,q(x)$ for all $s \in R$ and all $x \in X$. Following Gelfand-Šilov [7], Vol. 2, 3 we choose linear, complete, separated, barreled, locally convex topological spaces Φ and $\hat{\Phi}$ of complex-valued functions, which are topologically isomorphic under the Fourier transform: $F : \Phi \to \hat{\Phi}$, $F^{-1} : \hat{\Phi} \to \Phi$. Further, we assume that polynomials are multipliers in $\hat{\Phi}$. Let $L_s(\Phi, X)$ and $L_s(\hat{\Phi}, X)$ denote continuous linear maps $\Phi \to X$ and $\hat{\Phi} \to X$, respectively, endowed with the strong operator topology, that is, $R_\alpha \to R$ means $\langle R_\alpha, \varphi\rangle \to \langle R, \varphi\rangle$ in X for each $\varphi \in \Phi$. It is clear that the Parseval type formula $\langle \hat{R}, \hat{\varphi}\rangle = 2\pi\langle R, \varphi(-s)\rangle$ for $\varphi \in \Phi$ extends F onto $L_s(\Phi, X)$. It is easily seen that F is a topological isomorphism of $L_s(\Phi, X)$ onto $L_s(\hat{\Phi}, X)$ with $((\partial/\partial s)\,R)\hat{} = i\sigma\hat{R}$.

Now we suppose that we are given a continuous map $J : X \to L_s(\Phi, X)$ defined by $\langle Jx, \varphi\rangle = \int_{-\infty}^{\infty} \varphi(s)\,T(s)\,x\,ds$, $\varphi \in \Phi$, $x \in X$, where the integral sign on the right-hand side stands for the vector-valued integration (see [23], p. 237).

Theorem 4.1. *Let Φ be a linear topological space of complex functions defined on R, with the properties listed above. Let the function $w(t, \tau, \sigma)$ given by*

$$(4.1)\qquad P\left(t, \frac{d}{dt}, i\sigma\right) w(t, \tau, \sigma) = 0\,,$$

$$\frac{d^j w}{dt^j}(\tau, \tau, \sigma) = \frac{\delta_{j,m-1}}{Q_m(\tau, i\sigma)}\,, \quad \tau \leqq t\,, \quad \sigma \in S = \{z \in C;\ |\mathrm{Im}\, z| < b\}\ (b > a)\,,$$

as well as its derivatives $(d^j w/dt^j)(t, \tau, \sigma)$, $j = 0, 1, \ldots, m-1$, generate multiplier functions in $\hat{\Phi}$ which are continuous operator functions in the strong operator topology of $\hat{\Phi}$. Further, suppose that

$$(4.2)\qquad \left|\frac{d^j w}{dt^j}(t, t-\tau, \sigma)\right| \leqq C_j(\sigma)\,e^{-\alpha\tau} \quad \text{for} \quad \tau \leqq t\,, \quad \sigma \in S\,, \quad j = 0, 1, \ldots, m-1\,,$$

where $\alpha > 0$ and C_j's increase at most as polynomials in $|\sigma|$ if $|\sigma| \to \infty$. Finally, let $p \in N$ be sufficiently large and let $(b + A)^p f(t)$ be a continuous function in X. Then there exists a generalized solution $u(t)$ of (2.1) in X in the sense that

$$(4.3)\qquad \sum_{j=1}^{m} \left\langle \frac{d^j}{dt^j}\,\widehat{T(\cdot)u(t)},\ Q_j(t, i\cdot)\,\hat{\varphi}\right\rangle = \langle \widehat{T(\cdot)f(t)}, \hat{\varphi}\rangle$$

holds for all $\hat{\varphi} \in \hat{\Phi}$, the relation taking place in $L_s(\hat{\Phi}, X)$.

Proof. Put

$$u(t) = \int_0^\infty \int_{-\infty}^\infty F^{-1}\left[\frac{1}{(b + i\sigma)^p}\, w(t, t-\tau, \sigma)\right](s)\; T(s)\,(b + A)^p f(\tau)\,ds\,d\tau\,.$$

This is a well defined function in X since by (4.2) and Theorem 1.3

$$\left|F^{-1}\left[\frac{1}{(b+\mathrm{i}\sigma)^p}\, w(t, t-\tau, \sigma)\right](s)\right| \leqq C\, \mathrm{e}^{-(a+\varepsilon)|s|}\, \mathrm{e}^{-\alpha\tau}$$

for $\tau \leqq t$, $s \in R$, whenever $p \in N$ is sufficiently large and $\varepsilon > 0$ sufficiently small. It remains to verify (4.3). This can be accomplished by means of the standard distributional calculus.

Remark 4.1. The preceding theorem and its proof provide only a rough information on the subject. We mention it trying to bring the reader's attention to the interesting technique of [1] and [3], avoiding the details which make it difficult to understand.

5. AN ABSTRACT SECOND-ORDER DIFFERENTIAL EQUATION ARISING FROM THE TELEGRAPH EQUATION WITH AN INNER DAMPING TERM

In this last section we wish to give conditions on continuous ω-periodic positive functions $a(t)$, $b(t)$, $c(t)$, $d(t)$ and an operator $A : B \supset D(A) \to B$ which is the generator of a group $\{T(s)\}_{s\in R}$, in order that the equation

$$\frac{\mathrm{d}^2u}{\mathrm{d}t^2}(t) + (a(t) - b(t)\,A^2)\,\frac{\mathrm{d}u}{\mathrm{d}t}(t) + (c(t) - d(t)\,A^2)\,u(t) = f(t) \tag{5.1}$$

have a solution u in $C^2_\omega(R; B) \cap C^1_\omega(R; D(A^2))$.

As is shown in Sec. 2, this task can be solved by imposing conditions on the functions a, b, c, d such that the solution $w(t, t-\tau, \sigma)$ of the equation

$$\frac{\mathrm{d}^2w}{\mathrm{d}t^2} + (a(t) + b(t)\,\sigma^2)\,\frac{\mathrm{d}w}{\mathrm{d}t} + (c(t) + d(t)\,\sigma^2)\,w = 0\,, \tag{5.2}$$

$$w(\tau, \tau, \sigma) = 0\,, \quad \frac{\mathrm{d}w}{\mathrm{d}t}(\tau, \tau, \sigma) = 1$$

decreases in an appropriate way (e.g. exponentially) when $t - \tau \to \infty$ for $|\sigma| < \beta$ with a $\beta > \gamma = \max\limits_{\pm} \limsup\limits_{s\to\pm\infty} (\ln\|T(s)\|/|s|)$. To this purpose we derive the following

Lemma 5.1. *Let the function $\delta(t) \equiv d(t)\,b(t)^{-1}$ be continously differentiable on R. Suppose there exists a $\beta > 0$ such that*

$$a(t) - \delta(t) - \beta\, b(t) > 0 \tag{5.3}$$

for all $t \in [0, \omega]$. Let γ_0 and γ_1 be positive constants satisfying

$$\gamma_0 \leqq 4\{\delta(t)\,[a(t) - \delta(t) - \beta\, b(t)]\}^{1/2}\,,$$

$$\gamma_1 \geqq 2\left|\frac{\mathrm{d}\delta}{\mathrm{d}t}(t) - \delta(t)^2 + a(t)\,\delta(t)\right|, \quad t \in [0, \omega]\,.$$

If

$$2c_{\min} > (\Delta_c + \gamma_1)^2 \gamma_0^{-2} - \Delta_c\,, \tag{5.4}$$

where $c_{\min} = \min_{t \in [0,\omega]} c(t)$, $c_{\max} = \max_{t \in [0,\omega]} c(t)$ *and* $\Delta_c = c_{\max} - c_{\min}$, *then there exist constants* $C > 0$ *and* $\alpha > 0$ *such that the solution* $w(t, \tau, \sigma)$ *of the equations*

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + (a(t) + b(t)\,\sigma^2)\frac{\mathrm{d}x}{\mathrm{d}t} + (c(t) + d(t)\,\sigma^2)\,x = 0\,, \tag{5.5}$$

$$x(\tau) = 0\,, \quad \frac{\mathrm{d}x}{\mathrm{d}t}(\tau) = 1\,, \quad \tau \leqq t$$

satisfies the inequality

$$\max\left(|w(t, \tau, \sigma)|,\ \left|\frac{\mathrm{d}w}{\mathrm{d}t}(t, \tau, \sigma)\right|\right) \leqq C\mathrm{e}^{-\alpha(t-\tau)} \tag{5.6}$$

for all $t \geqq \tau$ *and all* $\sigma = \sigma_1 + \mathrm{i}\sigma_2$, *where* $\sigma_1 \in R$ *and* $|\sigma_2| \leqq \beta$.

Proof. The equations (5.5) can be written equivalently in the form

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = -\delta x_1 + x_2\,, \tag{5.7}$$

$$\frac{\mathrm{d}x_2}{\mathrm{d}t} = \left[\frac{\mathrm{d}\delta}{\mathrm{d}t} - \delta^2 + (a + b\sigma^2)\,\delta - c - d\sigma^2\right] x_1 +$$
$$+ (\delta - a - b\sigma^2)\,x_2\,, \quad x_1(0) = 0\,, \quad x_2(0) = 1\,,$$

where $x_1 = x$. Let $m > 0$ be an arbitrary constant. Putting $V(x_1, x_2) = \frac{1}{2}m|x_1|^2 + \frac{1}{2}|x_2|^2$ (we omit the arguments) we have

$$\frac{\mathrm{d}V}{\mathrm{d}t} = m\,\mathrm{Re}\left(\bar{x}_1\frac{\mathrm{d}x_1}{\mathrm{d}t}\right) + \mathrm{Re}\left(\bar{x}_2\frac{\mathrm{d}x_2}{\mathrm{d}t}\right). \tag{5.8}$$

Assuming that (x_1, x_2) in (5.8) is a vector-function satisfying (5.7) we get

$$\frac{\mathrm{d}V}{\mathrm{d}t} = -m\delta|x_1|^2 - (r_1 - \delta)\,|x_2|^2 + 2D\,\mathrm{Re}\,(x_1\bar{x}_2)\,,$$

where $r_1 = a + b(\sigma_1^2 - \sigma_2^2)$, $r_0 = c + d(\sigma_1^2 - \sigma_2^2)$ and $D = 2^{-1}(\mathrm{d}\delta/\mathrm{d}t - \delta^2 + r_1\delta - r_0 + m)$. Using the inequality $2|\mathrm{Re}\,(x_1\bar{x}_2)| \leqq \varepsilon|x_1|^2 + \varepsilon^{-1}|x_2|^2$ (which holds for all $\varepsilon > 0$) we get

$$\frac{\mathrm{d}V}{\mathrm{d}t} \leqq -m\,\delta|x_1|^2 - (r_1 - \delta)\,|x_2|^2 + |D|\,\varepsilon|x_1|^2 + \frac{|D|}{\varepsilon}|x_2|^2\,, \tag{5.9}$$

$\varepsilon = \varepsilon(t)$ arbitrary. By (5.3) we have $r_1 - \delta > 0$. Requiring $m\delta - \varepsilon|D| > 0$ and $r_1 - \delta - \varepsilon^{-1}|D| > 0$ the inequalities $|D|\,(r_1 - \delta)^{-1} < \varepsilon < m\,\delta|\mathrm{D}|^{-1}$ have to be satisfied. It is clear that

$$m\,\delta(r_1-\delta)-D^2 \geqq [m\,\delta(r_1-\delta)]^{1/2}\{[m\,\delta(r_1-\delta)]^{1/2}-|D|\} \geqq$$
$$\geqq 4^{-1}\gamma_0 m^{1/2}\left[4^{-1}\gamma_0 m^{1/2}-2^{-1}|m-c|-2^{-1}\left|\frac{\mathrm{d}\delta}{\mathrm{d}t}-\delta^2+a\delta\right|\right] \geqq$$
$$\geqq 4^{-1}\gamma_0 m^{1/2}[4^{-1}\gamma_0 m^{1/2}-4^{-1}\gamma_1-2^{-1}|m-c|]\,.$$

If we put $m = 2^{-1}(c_{\min}+c_{\max})$ then $|m-c| \leqq 2^{-1}\Delta_c$ and by (5.4) $\gamma_0 m^{1/2}-\gamma_1-\Delta_c$ can be bounded below by a positive constant, say α_0. In (5.9) set $\varepsilon = 2^{-1}[m\,\delta|D|^{-1}+|D|\,(r_1-\delta)^{-1}]$.

Then

$$\frac{\mathrm{d}V}{\mathrm{d}t} \leqq -[m\delta(r_1-\delta)-D^2]\{2^{-1}(r_1-\delta)^{-1}|x_1|^2+$$
$$+(r_1-\delta)[m\,\delta(r_1-\delta)+D^2]^{-1}|x_2|^2\}\,.$$

We have proved $m\,\delta(r_1-\delta)-D^2 \geqq 2^{-9/2}\alpha_0\gamma_0(c_{\min}+c_{\max})^{1/2}$ and, particularly, $D^2 \leqq m\,\delta(r_1-\delta)$. Hence $(r_1-\delta)[m\,\delta(r_1-\delta)+D^2]^{-1} \geqq (2m\delta)^{-1}$. Thus we obtain the final estimate

$$\frac{\mathrm{d}V}{\mathrm{d}t}(t) \leqq -\alpha\,V(t)\,, \quad t\in[\tau,\infty)\,,$$

where α is a positive constant which can be computed in terms of the coefficients a, b, c, d and which does not depend on σ if this is contained in the domain $\{|\mathrm{Im}\,\sigma| \leqq \beta\}$. This inequality easily yields our estimate (5.6).

Lemma 5.1 and Theorems 2.1 and 2.2 give a result which is summarized in the following

Theorem 5.1. *Let A be the generator of a group $\{T(s)\}_{s\in R} \subset L(B)$ and let $\beta > \max_{\pm}\limsup_{s\to\pm\infty}(\ln\|T(s)\|/|s|)$. Suppose that a, b, c and d are continuous ω-periodic positive functions on R, which satisfy (5.3) and (5.4), $\delta(t)$ being continuously differentiable on R. Then for any $f\in F = C_\omega(R;\,D(A^4))$ there exists a unique solution $u\in U = C^2_\omega(R;\,B)\cap C^1(R;\,D(A^2))$ of (5.1). Moreover, $\|u\|_U \leqq k\|f\|_F$ holds with a positive constant k.*

Proof. In Theorems 2.1, 2.2 put $n = 2$. Lemma 5.1 ensures that (2.20) holds with $p_0 = 0$. Our assertion follows immediately.

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