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ABOUT THE SIXTH HILBERT'S PROBLEM

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1. INTRODUCTION

In his lecture at the 2-nd International Congress of Mathematicians in the year 1900 (see [1]), David Hilbert formulated his 6th problem:

„Durch die Untersuchungen über die Grundlagen der Geometrie wird uns die Aufgabe nahe gelegt, *nach diesem Vorbilde diejenigen physikalischen Disciplinen axiomatisch zu behandeln, in denen schon heute die Mathematik eine hervorragende Rolle spielt*: *dies sind in erster Linie die Wahrscheinlichkeitsrechnung und die Mechanik.*

... Über die Grundlagen der Mechanik liegen von physikalischer Seite bedeutende Untersuchungen vor; ...; es ist daher sehr wünschenswert, wenn auch von den Mathematikern die Erörterung der Grundlagen der Mechanik aufgenommen würde. ..."

Hilbert also indicated what the solution of this problem in his opinion should contain. He said among other:

„Auch wird der Mathematiker, wie er es in der Geometrie getan hat, nicht bloß die der Wirklichkeit nahe kommenden, sondern überhaupt alle logisch möglichen Theorien zu berücksichtigen haben und stets darauf bedacht sein, einen vollständigen Überblick über die Gesamtheit der Folgerungen zu gewinnen, die das gerade angenommene Axiomensystem nach sich zieht.

Ferner fällt dem Mathematiker in Ergänzung der physikalischen Betrachtungsweise die Aufgabe zu, jedes Mal genau zu prüfen, ob das neu adjungierte Axiom mit den früheren Axiomen nicht in Widerspruch steht. Der Physiker sieht sich oftmals durch die Ergebnisse seiner Experimente gezwungen, zwischendurch und während der Entwickelung seiner Theorie neue Annahmen mit den früheren Axiomen lediglich auf eben jene Experimente oder auf ein gewisses physikalisches Gefühl beruft — ein Verfahren, welches beim streng logischen Aufbau einer Theorie nicht statthaft ist. Der gewünschte Nachweis der Widerspruchslosigkeit aller gerade gemachten Annahmen erscheint mit auch deshalb von Wichtigkeit, weil das Bestreben, einen solchen

Nachweis zu führen, uns stets am wirksamsten zu einer exakten Formulierung der Axiome selbst zwingt.“

Since the time of this Hilbert's lecture, the axiomatic probability theory has been built. In the almanac [2], in which the statement of the solution of Hilbert's problems is discussed, among the references to the 6th problem only papers of probability theory are quoted. There is only a remark that a great amount of axiomatic explanations of various sections of physics exists. As examples of an axiomatic theory of the classical mechanics Hamel's and Marcolongo's papers are mentioned. But in the book of Marcolongo (see [7]) only the statics is built axiomatically. In the book of Hamel (see [9]), the matter is discussed from the physical point of view; from the mathematical point of view this book does not satisfy today's demands on exactness and logical completeness. Also more recent books, e.g. [10], are written mainly from the physical point of view.

A paper that meets Hilbert's demand most closely is – as far as I was able to find out – the paper [8]. Theorems are here deduced from exactly formulated axioms in a purely logical way. Nonetheless, any logical analysis of the system of axioms (such as the proof that the system is not contadictory, that axioms are independent etc.) is again missing here and could be probably hardly accomplished due to a considerable complexity of the system.

There are, however, some papers, in which some special topics of mechanics are presented axiomatically and dicussed from the mathematical point of view. Thus the notion of the resultant of two forces is axiomatically defined and studied in the papers [3], [4], [5] and [6]. Moreover, in the papers [4], [5], and [6] a logical analysis of axioms is carried out. In the book [7], an axiomatic system of statics is given.

In this paper we give a system of axioms for the mechanics of a system of material points, we prove that this system is not contradictory, deduce some fundamental theorems and prove the independence of the individual axioms. The axioms are formulated in terms of geometrical and physical notions without referring to coordinates (even though coordinates are of course used when theorems are formulated and demonstrated). By the formulation of axioms we avoided the assumptions about the existence of derivatives of the functions taken into account, because such assumptions seem to be artifical and unnatural. The assertions concerning the existence of derivatives are proved as theorems.

2. SYSTEM OF AXIOMS

Let us have a threedimensional euclidean space E_3, that is, a metric space with a metric ϱ, in which it is possible to introduce reference systems (called *cartesian*) which map the space onto the set of triples of real numbers in such a way that if $A = [a_1, a_2, a_3]$, $B = [b_1, b_2, b_3]$, then $\varrho(A, B) = \left(\sum_{i=1}^{3} (b_i - a_i)^2\right)^{1/2}$. We consider the geometry of E_3, as known. Let us recall only that *vectors* are defined in E_3. A vector is given by a pair of points A, B and two pairs (A, B) and (C, D) define the same vector if and only if there exists a point E such that $\varrho(A, E) = \varrho(E, D) = \frac{1}{2}\varrho(A, D)$. $\varrho(B, E) = \varrho(E, C) = \frac{1}{2}\varrho(B, C)$. If a vector $\boldsymbol{U}$ is defined by a pair of points A, B, we shall write $\boldsymbol{U} = AB$. The set of all vectors of the space E_3 will be denoted by V_3. If $\boldsymbol{U} \in V_3$, $\boldsymbol{V} \in V_3$, then $\boldsymbol{U} + \boldsymbol{V} \in V_3$ is defined. If $\boldsymbol{U} \in V_3$ and a is a real number, then $a\boldsymbol{U} \in V_3$ is defined. If $\boldsymbol{U} = AB$, then the number $\varrho(A, B)$ is called the *magnitude* $|\boldsymbol{U}|$ of the vector $\boldsymbol{U}$. If $|\boldsymbol{U}| = 1$ then $\boldsymbol{U}$ is called a *unit vector*. If $\boldsymbol{V} = a\boldsymbol{U}$, where $a \geqq 0$ and $\boldsymbol{U}$ is a unit vector, then $\boldsymbol{U}$ is called the *direction* of the vector $\boldsymbol{V}$. The null vector will be denoted by $\boldsymbol{0}$.

By a *system of vectors* $\{\boldsymbol{U}_j\}_{j\in J}$ we mean an arbitrary set J (possibly empty) together with a map $J \to V_3$ such that $j \in J \mapsto \boldsymbol{U}_j \in V_3$. If J has only one element, we shall sometimes write only $\{\boldsymbol{U}\}$, if no error can occur. If J is empty, the system of vectors will be called empty as well. If $\{\boldsymbol{U}_k\}_{k\in K}$, $\{\boldsymbol{V}_l\}_{l\in L}$ are systems of vectors and $K \cap L = \emptyset$, then by $\{\boldsymbol{U}_k\}_{k\in K} \cup \{\boldsymbol{V}_l\}_{l\in L}$ we mean the system $\{\boldsymbol{W}_j\}_{j\in K\cup L}$, where $\boldsymbol{W}_j = \boldsymbol{U}_j$ if $j \in K$ and $\boldsymbol{W}_j = \boldsymbol{V}_j$ if $j \in L$.

We introduce the following denotations: Let $\mathscr{T}_1$ be an open interval of real numbers and let us have maps

$$B\colon \mathscr{T}_1 \to E_3\,, \qquad C\colon \mathscr{T}_1 \to E_3\,.$$

If $t \in \mathscr{T}_1$, we put

$$\Delta_B(C; t) = C(t) - B(t)\,. \tag{2.1}$$

If $t \in \mathscr{T}_1$, $\tau > 0$, $t + \tau \in \mathscr{T}_1$, we put

$$\mathbf{V}_B(C; t, \tau) = \frac{\Delta_B(C; t + \tau) - \Delta_B(C; t)}{\tau}\,. \tag{2.2}$$

The vector $\mathbf{V}(C; t, \tau)$ will be called the *mean velocity* of the point C with regard to B in the interval $\langle t, t + \tau\rangle$. If $t_1 \in \mathscr{T}_1$, $\tau > 0$, $t_1 < t_2$, $t_2 + \tau \in \mathscr{T}_1$, we put

$$\mathbf{A}_B(C; t_1, t_2, \tau) = \frac{\mathbf{V}_B(C; t_2, \tau) - \mathbf{V}_B(C; t_1, \tau)}{t_2 - t_1}\,. \tag{2.3}$$

The vector $\mathbf{A}_B(C; t_1, t_2, \tau)$ will be called the *mean acceleration* of the point C with regard to B between the intervals $\langle t_1, t_1 + \tau\rangle$ and $\langle t_2, t_2 + \tau\rangle$.

For vectors $\boldsymbol{U}, \boldsymbol{V}, \boldsymbol{W}$ we have the primitive notion *the force $\boldsymbol{W}$ is the resultant of forces $\boldsymbol{U}, \boldsymbol{V}$* (in symbols $\boldsymbol{W} = \boldsymbol{U} \oplus \boldsymbol{V}$). [1])

By R we shall denote the set of all real numbers, by R^+ the set of all positive real numbers.

We suppose that we have an open interval $\mathscr{T} \subset R$ whose elements are called *time instants*. Further, we have a set $\mathscr{M}$ whose elements are called *particles*. With any

[1]) We do not define the notion of force. The expression *the force $\boldsymbol{x}$ is the resultant of forces $\boldsymbol{y}, \boldsymbol{z}$* (or $\boldsymbol{x} = \boldsymbol{y} \oplus \boldsymbol{z}$) must be considered a propositional function with the domain $V_3 \times V_3 \times V_3$.

particle $\alpha \in \mathscr{M}$ we associate a positive real number m_α, called the *mass of the particle* α. For each particle $\alpha \in \mathscr{M}$ we have a map $\mathsf{P}_\alpha : \mathscr{T} \to E_3$. If $\alpha \in \mathscr{M}$, $t \in \mathscr{T}$, then the point $\mathsf{P}_\alpha(t) \in E_3$ is called the *position of the particle* α *at the instant* t. If $\alpha \in \mathscr{M}$, $\beta \in \mathscr{M}$, $\boldsymbol{F} \in V_3$, $t \in \mathscr{T}$, then we have the primitive notion *the particle* β *effects the particle* α *at the instant* t *by the force* $\boldsymbol{F}$ (in symbols $\mathscr{G}(\alpha, \beta, t, \boldsymbol{F})$).

A material point is defined as a pair $\{A, m\}$, where $A \in E_3$, $m \in R^+$.

If we have an open interval $\mathscr{T}_1 \subset \mathscr{T}$, a map $Y\colon \mathscr{T}_1 \to E_3$, a number $m \in R^+$, and if we have for every $t \in \mathscr{T}_1$ a system of vectors $\{\boldsymbol{F}_j(t)\}_{j\in J}$ (the set J being the same for all $t \in \mathscr{T}_1$), then we introduce the primitive notion *the motion of the material point* $\{Y(t, m\}$ *in the interval* $\mathscr{T}_1$ *can be interpreted by the operation of the system of forces* $\{\boldsymbol{F}_j(t)\}_{j\in J}$ (in symbols $\mathscr{F}(\{Y(t), m\}, \{\boldsymbol{F}_j(t)\}_{j\in J}, \mathscr{T}_1)$).

For these primitive notions we suppose the validity of the following axioms:

I. The effect of a force

Axiom I.1. *If the motion of a material point* $\{B(t), m\}$ *in an open interval* $\mathscr{T}_1 \subset \mathscr{T}$ *can be interpreted by the operation of a system of forces* $\{\boldsymbol{F}_j(t)\}_{j\in J}$, *if the motion of a material point* $\{C(t), m\}$ *in* $\mathscr{T}_1$ *can be interpreted by the operation of a system of forces* $\{\boldsymbol{F}_j(t)\}_{j\in J} \cup \{\boldsymbol{G}(t)\}$ *and if the vectors* $\boldsymbol{G}(t)$ *have for all* $t \in \mathscr{T}_1$ *the same direction* $\boldsymbol{u}$, *then for every* $t_1 \in \mathscr{T}_1, \tau > 0, t_1 < t_2, t_2 + \tau \in \mathscr{T}_1$ *there exists a number* $k \geqq 0$ *such that*

$$\mathbf{A}_B(C; t_1, t_2, \tau) = k \,.\, \boldsymbol{u} \,.$$

Axiom I.2. *Suppose that the motion of the material point* $\{B(t), m\}$ *in the open interval* $\mathscr{T}_1 \subset \mathscr{T}$ *can be interpreted by the operation of the system of forces* $\{\boldsymbol{F}_j(t)\}_{j\in J}$, *the motion of the material point* $\{C(t), m\}$ *by the operation of the system of forces* $\{\boldsymbol{F}_j(t)\}_{j\in J} \cup \{\boldsymbol{G}(t)\}$, *where the vectors* $\boldsymbol{G}(t)$ *have for all* $t \in \mathscr{T}_1$ *the same direction* $\boldsymbol{u}$. *Let* $t_1 \in \mathscr{T}_1$, $\tau > 0$, $t_1 < t_2$, $t_2 + \tau \in \mathscr{T}_1$ *and let*

$$\mathbf{A}_B(C; t_1, t_2, \tau) = k \,.\, \boldsymbol{u} \,, \quad k \geqq 0 \,.$$

Then

$$p_1 \leqq |\boldsymbol{G}(t)| \leqq p_2 \quad \textit{for all} \quad t \in \langle t_1, t_2 + \tau \rangle$$

implies

$$p_1 \leqq m|\mathbf{A}_B(C; t_1, t_2, \tau)| \leqq p_2 \,.$$

Axiom I.3. *Suppose that the motion of the material point* $\{B(t), m\}$ *in the open interval* $\mathscr{T}_1 \subset \mathscr{T}$ *can be interpreted by the operation of the system of forces* $\{\boldsymbol{F}_j(t)\}_{j\in J}$, *the motion of the material point* $\{C(t), m\}$ *by the operation of the system of forces* $\{\boldsymbol{F}_j(t)\}_{j\in J} \cup \{\boldsymbol{G}(t)\}$, *where the vectors* $\boldsymbol{G}(t)$ *have for all* $t \in \mathscr{T}_1$ *the same direction* $\boldsymbol{u}$. *Then, if the function* $t \mapsto |\boldsymbol{G}(t)|$ *is bounded on every compact subinterval of* $\mathscr{T}_1$, *the function* $t \mapsto |\Delta_B(C; t)|$ *is also bounded on every compact subinterval of* $\mathscr{T}_1$.

II. The composition of forces

Axiom II.1. *If for every t from an open interval $\mathscr{T}_1 \subset \mathscr{T}$, the force $\boldsymbol{G}(t)$ is the resultant of forces $\boldsymbol{G}_1(t)$, $\boldsymbol{G}_2(t)$, then the motion of a material point $\{B(t), m\}$ in $\mathscr{T}_1$ can be interpreted by the operation of the system of forces $\{\boldsymbol{F}_j(t)\}_{j\in J} \cup \{\boldsymbol{G}_1(t)\} \cup \cup \{\boldsymbol{G}_2(t)\}$ if and only if it can be interpreted in $\mathscr{T}_1$ by the operation of the system of forces $\{\boldsymbol{F}_j(t)\}_{j\in J} \cup \{\boldsymbol{G}(t)\}$.*

Axiom II.2. *If $\boldsymbol{F}$, $\boldsymbol{G}$ are vectors, then there exists a vector $\boldsymbol{H}$ such that the force $\boldsymbol{H}$ is the resultant of forces $\boldsymbol{F}$, $\boldsymbol{G}$.*

Axiom II.3. *Suppose that the motion of the material point $\{B(t), m\}$ in the open interval $\mathscr{T}_1 \subset \mathscr{T}$ can be interpreted by the operation of the system of forces $\{\boldsymbol{F}_j(t)\}_{j\in J}$. Let $\boldsymbol{G}$ be a vector and let points $C(t)$, $t \in \mathscr{T}_1$ satisfy*

$$m\mathbf{A}_B(C; t_1, t_2, \tau) = \boldsymbol{G} \quad \text{for every} \quad t_1 \in \mathscr{T}_1,\ t_2 < t_2,\ \tau > 0,\ t_2 + \tau \in \mathscr{T}_1 .$$

Let the function $t \mapsto \Delta_B(C; t)$ be continuous on $\mathscr{T}_1$. Then the motion of the material point $\{C(t), m\}$ in the interval $\mathscr{T}_1$ can be interpreted by the operation of the system of forces $\{\boldsymbol{F}_j(t)\}_{j\in J} \cup \{\boldsymbol{G}\}$.

III. The decomposition of motions

Axiom III.1. *If $\mathscr{T}_1 \subset \mathscr{T}$ is an open interval and if there exist (for all $t \in \mathscr{T}_1$) points $C(t)$ such that the motion of the material point $\{C(t), m\}$ in $\mathscr{T}_1$ can be interpreted by the operation of the system of forces $\{\boldsymbol{F}_j(t)\}_{j\in J} \cup \{\boldsymbol{G}(t)\}$, where the vectors $\boldsymbol{G}(t)$ have the same direction for all $t \in \mathscr{T}_1$ and the function $t \mapsto |\boldsymbol{G}(t)|$ is continuous on $\mathscr{T}_1$, then there exist (for all $t \in \mathscr{T}_1$) points $B(t)$ such that the motion of the material point $\{B(t), m\}$ in $\mathscr{T}_1$ can be interpreted by the operation of the system of forces $\{\boldsymbol{F}_j(t)\}_{j\in J}$.*

IV. Law of inertia

Axiom IV.1. *If the motion of a material point $\{B(t), m\}$ in an open interval $\mathscr{T}_1 \subset \mathscr{T}$ can be interpreted by the operation of the empty system of forces and if $t_1 \in \mathscr{T}_1$, $t_1 < t_2$, $\tau > 0$, $t_2 + \tau \in \mathscr{T}_1$, then*

$$\varrho(B(t_1 + \tau), B(t_1)) = \varrho(B(t_2 + \tau), B(t_2)) .$$

Axiom IV.2. *If the motion of the material point $\{B(t), m\}$ in the open interval $\mathscr{T}_1 \subset \mathscr{T}$ can be interpreted by the operation of the system of forces $\{\boldsymbol{0}\}$, then it can be interpreted by the operation of the empty system of forces.*

Axiom IV.3. *If the motion of the material point $\{B(t), m\}$ in the open interval $\mathscr{T}_1 \subset \mathscr{T}$ can be interpreted by the operation of the empty system of forces, then the function $t \mapsto B(t)$ is continuous on $\mathscr{T}_1$.*

V. Gravitational law

Axiom V.1. *Let $\alpha \in \mathcal{M}$ be a particle with a mass m_α. Let, for every $\beta \in \mathcal{M} \setminus \{\alpha\}$ and every $t \in \mathcal{T}_1$, $\boldsymbol{F}_\beta(t)$ be a vector such that the particle β effects the particle α at the instant t by the force $\boldsymbol{F}_\beta(t)$. Then the motion of the material point $\{\mathsf{P}_\alpha(t), m_\alpha\}$ can be in every open interval $\mathcal{T}_1 \subset \mathcal{T}$ interpreted by the operation of the system of forces $\{\boldsymbol{F}_\beta(t)\}_{\beta \in \mathcal{M} \setminus \{\alpha\}}$.*

Axiom V.2. *If $\alpha \in \mathcal{M}$, $\beta \in \mathcal{M}$, $t \in \mathcal{T}$ and $\mathsf{P}_\alpha(t) \neq \mathsf{P}_\beta(t)$, then there exists a vector $\boldsymbol{F}$ such that the particle β effects the particle α at the instant t by the force $\boldsymbol{F}$.*

Axiom V.3. *If the particle β effects the particle α at the instant t by the force $\boldsymbol{F}$ and $\mathsf{P}_\alpha(t) \neq \mathsf{P}_\beta(t)$, then the unit vector $\mathsf{P}_\alpha(t)\, \mathsf{P}_\beta(t) / \varrho(\mathsf{P}_\alpha(t), \mathsf{P}_\beta(t))$ is the direction of the vector $\boldsymbol{F}$.*

Axiom V.4. *There exists a positive real number $\varkappa$ such that if the particle β effects the particle α at the instant t by the force $\boldsymbol{F}$ and if $\mathsf{P}_\alpha(t) \neq \mathsf{P}_\beta(t)$, then*

$$|\boldsymbol{F}| = \varkappa \frac{m_\alpha m_\beta}{(\varrho(\mathsf{P}_\alpha(t), \mathsf{P}_\beta(t)))^2}.$$

Axiom V.5. *If $\alpha \in \mathcal{M}$, $\beta \in \mathcal{M}$, then the function $t \mapsto \Delta_{\mathsf{P}_\alpha}(\mathsf{P}_\beta; t)$ is continuous on $\mathcal{T}$.*

VI. Axioms of existence

Axiom VI.1. *The set $\mathcal{M}$ of all particles is non-empty.*

Axiom VI.3. *The set $\mathcal{M}$ of all particles is finite.*

Axiom VI.3. *If $t \in \mathcal{T}$ and α, β are two different particles, then $\mathsf{P}_\alpha(t) \neq \mathsf{P}_\beta(t)$.*

3. AUXILIARY THEOREMS

Before analysing our system of axioms, we prove two auxiliary theorems.

Theorem 3.1. *Let s be a real function defined on an interval K (open, semiopen or closed) of real numbers. Suppose that to every $T \in K$ there exist numbers $\varepsilon_T > 0$, $\delta_T > 0$ such that*

$$(3.1) \qquad |s(t) - s(T)| < \varepsilon_T \quad \textit{if} \quad |t - T| < \delta_T\,, \quad t \in K\,,$$

and that there exist real numbers M_1, M_2 such that

$$(3.2) \qquad M_1(t_2 - t_1) \leqq \frac{s(t_2 + \tau) - s(t_2)}{\tau} - \frac{s(t_1 + \tau) - s(t_1)}{\tau} \leqq M_2(t_2 - t_1)$$

if $t_1 < t_2$, $\tau > 0$, $t_1 \in K$, $t_2 + \tau \in K$. *Then*

(a)
$$M_1\tau \frac{n-1}{2} + \frac{s(t_0+\tau) - s(t_0)}{\tau} \leqq \frac{s(t_0+n\tau) - s(t_0)}{n\tau} \leqq$$
$$\leqq M_2\tau \frac{n-1}{2} + \frac{s(t_0+\tau) - s(t_0)}{\tau}$$

if n *is a natural number and* $t_0 \in K$, $t_0 + n\tau \in K$, $\tau > 0$;

(b)
$$M_1\sigma \frac{n-1}{2n} + \frac{s\left(t_0 + \frac{\sigma}{n}\right) - s(t_0)}{\frac{\sigma}{n}} \leqq \frac{s(t_0+\sigma) - s(t_0)}{\sigma} \leqq$$
$$\leqq M_2\sigma \frac{n-1}{2n} + \frac{s\left(t_0 + \frac{\sigma}{n}\right) - s(t_0)}{\frac{\sigma}{n}}$$

if n *is a natural number*, $t_0 \in K$, $t_0 + \sigma \in K$, $\sigma > 0$;

(c)
$$M_1\tau \leqq \frac{s(t_0+\tau) - s(t_0)}{\tau} - \frac{s(t_0-\tau) - s(t_0)}{-\tau} \leqq M_2\tau$$

if $\tau > 0$, $t_0 - \tau \in K$, $t_0 + \tau \in K$;

(d) *at every* $t_0 \in K$ *there exists a finite*
$$v(t_0) = \dot{s}(t_0) = \lim_{\substack{t \to t_0 \\ t \in K}} \frac{s(t) - s(t_0)}{t - t_0};$$

(e) *the function* v *is continuous on* K;

(f)
$$s(t_0) + v(t_0)\,\sigma + \tfrac{1}{2}M_1\sigma^2 \leqq s(t_0+\sigma) \leqq s(t_0) + v(t_0)\,\sigma + \tfrac{1}{2}M_2\sigma^2$$

if $\sigma > 0$, $t_0 \in K$, $t_0 + \sigma \in K$;

(g)
$$v(t_0) + M_1\sigma \leqq v(t_0+\sigma) \leqq v(t_0) + M_2\sigma$$

if $\sigma > 0$, $t_0 \in K$, $t_0 + \sigma \in K$.

Proof. (a) If k is a natural number, $1 < k \leqq n$, and if we write in (3.2) t_0 instead of t_1 and $t_0 + (k-1)\tau$ instead of t_2, we obtain

$$M_1(k-1)\,\tau + \frac{s(t_0+\tau) - s(t_0)}{\tau} \leqq \frac{s(t_0+k\tau) - s(t_0+(k-1)\,\tau)}{\tau} \leqq$$
$$\leqq M_2(k-1)\,\tau + \frac{s(t_0+\tau) - s(t_0)}{\tau}.$$

We can see that this formula holds also for $k = 1$. By summing for k from 1 to n we obtain

$$M_1\tau\,\frac{n(n-1)}{2} + n\,\frac{s(t_0+\tau)-s(t_0)}{\tau} \leqq \frac{s(t_0+n\tau)-s(t_0)}{\tau} \leqq$$

$$\leqq M_2\tau\,\frac{n(n-1)}{2} + n\,\frac{s(t_0+\tau)-s(t_0)}{\tau}.$$

(b) is obtained by putting $\sigma = n\tau$ in (a).

(c) If we write in (3.2) t_0 instead of t_2 and $t_0 - \tau$ instead of t_1, we obtain

$$M_1\tau \leqq \frac{s(t_0-\tau)-s(t_0)}{\tau} - \frac{s(t_0-\tau+\tau)-s(t_0-\tau)}{\tau} \leqq M_2\tau\,;$$

because

$$\frac{s(t_0-\tau+\tau)-s(t_0-\tau)}{\tau} = \frac{s(t_0-\tau)-s(t_0)}{-\tau},$$

we obtain (c).

(d) We will prove first of all that

$$(3.3) \qquad p_2(t_0) = \limsup_{\tau\to 0+} \frac{s(t_0+\tau)-s(t_0)}{\tau} < +\infty$$

if $t_0 \in K$ and t_0 is not the right hand side boundary point of K. Indeed, let $\varepsilon_T > 0$, $\delta_T > 0$ be numbers satisfying (3.1) for $T = t_0$. We can choose δ_T so small that $t_0 + \delta_T \in K$. Suppose that $\limsup_{\tau\to 0+} (s(t_0+\tau) - s(t_0))/\tau = \infty$. Then there exists a number τ_0 such that $0 < \tau_0 < \delta_T$ and that

$$\frac{s(t_0+\tau_0)-s(t_0)}{\tau_0} > 2\varepsilon_T\delta_T^{-1} + |M_1|\,\delta_T\,.$$

There exists a natural number n such that $\frac{1}{2}\delta_T \leqq n\tau_0 < \delta_T$. By (a), we obtain for such a number n the inequalities

$$\frac{s(t_0+n\tau_0)-s(t_0)}{n\tau_0} \geqq \frac{s(t_0+\tau_0)-s(t_0)}{\tau_0} + M_1\tau_0\,\frac{n-1}{2} > 2\varepsilon_T\delta_T^{-1} +$$

$$+ |M_1|\,\delta_T + \tfrac{1}{2}M_1 n\tau_0 - \tfrac{1}{2}M_1\tau_0 \geqq$$

$$\geqq 2\varepsilon_T\delta_T^{-1} + |M_1|\,\delta_T - \tfrac{1}{2}|M_1|\,\delta_T - \tfrac{1}{2}|M_1|\,\delta_T = 2\varepsilon_T\delta_T^{-1}\,,$$

therefore

$$s(t_0+n\tau_0) - s(t_0) > 2\varepsilon_T\delta_T^{-1}n\tau_0 \geqq 2\varepsilon_T\delta_T^{-1}\,.\,\tfrac{1}{2}\delta_T = \varepsilon_T\,;$$

but this is a contradiction with (3.1), because $n\tau_0 < \delta_T$. Hence (3.3) is proved.

Now we will prove that

$$(3.4) \qquad p_1(t_0) = \liminf_{\tau \to 0+} \frac{s(t_0 + \tau) - s(t_0)}{\tau} > -\infty$$

if $t_0 \in K$ and t_0 is not the right hand side boundary point of K. Indeed, let $\varepsilon_T > 0$, $\delta_T > 0$ be again numbers satisfying (3.1) for $T = t_0$ and $t_0 + \delta_T \in K$. Suppose that $\liminf_{\tau \to 0+} ((s(t_0 + \tau) - s(t_0))/\tau) = -\infty$. Then there exists a number τ_0 such that $0 < \tau_0 < \delta_T$ and that

$$\frac{s(t_0 + \tau_0) - s(t_0)}{\tau_0} < -2\varepsilon_T \delta_T^{-1} - |M_2| \, \delta_T .$$

There exists a natural number n such that $\frac{1}{2}\delta_T \leqq n\tau_0 < \delta_T$. By (a), we obtain for such a number n the inequalities

$$\frac{s(t_0 + n\tau_0) - s(t_0)}{n\tau_0} \leqq \frac{s(t_0 + \tau_0) - s(t_0)}{\tau_0} + M_2 \tau_0 \frac{n-1}{2} <$$

$$< -2\varepsilon_T \delta_T^{-1} - |M_2| \, \delta_T + \tfrac{1}{2} M_2 n\tau_0 - \tfrac{1}{2} M_2 \tau_0 \leqq$$

$$\leqq -2\varepsilon_T \delta_T^{-1} - |M_2| \, \delta_T + \tfrac{1}{2}|M_2| \, \delta_T + \tfrac{1}{2}|M_2| \, \delta_T = -2\varepsilon_T \delta_T^{-1} ,$$

therefore

$$s(t_0 + n\tau_0) - s(t_0) < -2\varepsilon_T \delta_T^{-1} n\tau_0 .$$

The number on the right hand side being negative, we obtain

$$|s(t_0 + n\tau_0) - s(t_0)| > 2\varepsilon_T \delta_T^{-1} n\tau_0 \geqq 2\varepsilon_T \delta_T^{-1} \cdot \tfrac{1}{2}\delta_T = \varepsilon_T ;$$

but this is a contradiction with (3.1), because $n\tau_0 < \delta_T$. Hence (3.4) is proved.

Now we will prove that there exists a finite $\lim_{\tau \to 0+} ((s(t_0 + \tau) - s(t_0))/\tau)$ at every point $t_0 \in K$ which is not the right hand boundary point of K. With regard to (3.3) and (3.4), it is sufficient to prove $p_1(t_0) = p_2(t_0)$. Let us suppose that $p_1(t_0) < p_2(t_0)$ for some $t_0 \in K$. Let us denote $H = p_2(t_0) - p_1(t_0)$. There exists a number $\eta > 0$ such that $t_0 + \eta \in K$ and

$$(3.5) \quad p_1(t_0) - \tfrac{1}{4}H < \frac{s(t_0 + \tau) - s(t_0)}{\tau} < p_2(t_0) + \tfrac{1}{4}H , \quad \text{whenever} \quad 0 < \tau < \eta .$$

Let us choose a number N such that

$$(3.6) \qquad 0 < N < \min\left(\eta, \frac{H}{1 + 2(|M_1| + |M_2|)}\right).$$

Further, let us choose a number δ such that

$$(3.7) \qquad 0 < \delta < \min\left(\tfrac{1}{3}N, \frac{HN}{4(H + 2|M_2| \, N}, \frac{HN}{4(2H + |M_1| \, N + |M_2| \, N}\right).$$

There exist numbers τ_1, τ_2 such that

$$\frac{s(t_0 + \tau_1) - s(t_0)}{\tau_1} < p_1(t_0) + \tfrac{1}{4}H = p_2(t_0) - \tfrac{3}{4}H, \quad 0 < \tau_1 < \delta, \tag{3.8}$$

$$\frac{s(t_0 + \tau_2) - s(t_0)}{\tau_2} > p_2(t_0) - \tfrac{1}{4}H, \quad 0 < \tau_2 < \delta. \tag{3.9}$$

There exists a natural number n_2 such that $N - \delta < n_2\tau_2 \leqq N$. Further, there exists a natural number n such that $n_2\tau_2 - \delta \leqq n_1\tau_1 < n_2\tau_2$. Consequently, with regard to (3.7) we obtain

$$0 < N - 2\delta < n_2\tau_2 - \delta \leqq n_1\tau_1 < n_2\tau_2 \leqq N. \tag{3.10}$$

Let us now denote

$$\delta_0 = n_2\tau_2 - n_1\tau_1; \tag{3.11}$$

thus

$$0 < \delta_0 \leqq \delta. \tag{3.12}$$

By (a) and (3.9) we now have

$$\frac{s(t_0 + n_2\tau_2) - s(t_0)}{n_2\tau_2} \geqq \frac{s(t_0 + \tau_2) - s(t_0)}{\tau_2} + \tfrac{1}{2}M_1 n_2\tau_2 - \tfrac{1}{2}M_1\tau_2 >$$
$$> p_2(t_0) - \tfrac{1}{4}H + \tfrac{1}{2}M_1 n_2\tau_2 - \tfrac{1}{2}M_1\tau_2,$$

therefore (3.11) yields

$$s(t_0 + n_2\tau_2) > s(t_0) + n_2\tau_2(p_2(t_0) - \tfrac{1}{4}H) + \tfrac{1}{2}M_1(n_2\tau_2)^2 - \tfrac{1}{2}M_1(n_2\tau_2)\,\tau_2 =$$
$$= s(t_0) + n_1\tau_1(p_2(t_0) - \tfrac{1}{4}H) + \delta_0(p_2(t_0) - \tfrac{1}{4}H) + \tfrac{1}{2}M_1(n_2\tau_2)^2 - \tfrac{1}{2}M_1(n_2\tau_2)\,\tau_2.$$

In a similar way, (a) and (3.8) imply

$$\frac{s(t_0 + n_1\tau_1) - s(t_0)}{n_1\tau_1} \leqq \frac{s(t_0 + \tau_1) - s(t_0)}{\tau_1} + \tfrac{1}{2}M_2 n_1\tau_1 - \tfrac{1}{2}M_2\tau_1 <$$
$$< p_2(t_0) - \tfrac{3}{4}H + \tfrac{1}{2}M_2 n_1\tau_1 - \tfrac{1}{2}M_2\tau_1,$$

therefore

$$s(t_0 + n_1\tau_1) < s(t_0) + n_1\tau_1(p_2(t_0) - \tfrac{3}{4}H) + \tfrac{1}{2}M_2(n_1\tau_1)^2 - \tfrac{1}{2}M_2(n_1\tau_1)\,\tau_1.$$

Subtracting the two inequalities and using (3.10) we obtain

$$s(t_0 + n_2\tau_2) - s(t_0 + n_1\tau_1) > \tfrac{1}{2}n_1\tau_1 H + \delta_0(p_2(t_0) - \tfrac{1}{4}H) - \tfrac{1}{2}|M_1|\,N^2 -$$
$$- \tfrac{1}{2}|M_2|\,N^2 - \tfrac{1}{2}|M_1|\,N\delta - \tfrac{1}{2}|M_2|\,N\delta > \tfrac{1}{2}H(N - 2\delta) + \delta_0(p_2(t_0) - \tfrac{1}{4}H) -$$
$$- \tfrac{1}{2}|M_1|\,N^2 - \tfrac{1}{2}|M_2|\,N^2 - \tfrac{1}{2}|M_1|\,N\delta - \tfrac{1}{2}|M_2|\,N\delta = \delta_0(p_2(t_0) - \tfrac{1}{4}H) +$$
$$+ \tfrac{1}{2}N(H - |M_1|\,N - |M_2|\,N) - \delta(H + \tfrac{1}{2}|M_1|\,N + \tfrac{1}{2}|M_2|\,N).$$

By (3.6) we have $2N(|M_1| + |M_2|) < N(1 + 2(|M_1| + |M_2|) < H$, and therefore we obtain

$$s(t_0 + n_2\tau_2) - s(t_0 + n_1\tau_1) >$$
$$> \delta_0(p_2(t_0) - \tfrac{1}{4}H) + \tfrac{1}{4}NH - \delta(H + \tfrac{1}{2}|M_1| N + \tfrac{1}{2}|M_2| N) ;$$

by (3.7) we have $\delta(H + \frac{1}{2}|M_1| N + \frac{1}{2}|M_2| N) < \frac{1}{8}HN$ and therefore

$$s(t_0 + n_2\tau_2) - s(t_0 + n_1\tau_1) > \delta_0(p_2(t_0) - \tfrac{1}{4}H) + \tfrac{1}{8}NH .$$

As we have $n_2\tau_2 = n_1\tau_1 + \delta_0$ by (3.11), we obtain with regard to (3.12):

$$\frac{s(t_0 + n_1\tau_1 + \delta_0) - s(t_0 + n_1\tau_1)}{\delta_0} > p_2(t_0) - \tfrac{1}{4}H + \tfrac{1}{8}\delta_0^{-1}NH \geqq$$
$$\geqq p_2(t_0) - \tfrac{1}{4}H + \tfrac{1}{8}\delta^{-1}NH .$$

By (3.2) and (3.10) we hence obtain

$$\frac{s(t_0 + \delta_0) - s(t_0)}{\delta_0} \geqq \frac{s(t_0 + n_1\tau_1 + \delta_0) - s(t_0 + n_1\tau_1)}{\delta_0} - M_2 n_1 \tau_1 >$$
$$> p_2(t_0) - \tfrac{1}{4}H + \tfrac{1}{8}\delta^{-1}NH - |M_2| N ;$$

but by (3.7) we have $\delta^{-1}NH > 4(H + 2|M_2| N)$ and therefore

$$\frac{s(t_0 + \delta_0) - s(t_0)}{\delta_0} > p_2(t_0) - \tfrac{1}{4}H + \tfrac{1}{2}H + |M_2| N - |M_2| N = p_2(t_0) + \tfrac{1}{4}H ,$$

which is a contradiction with (3.5), because $\delta_0 < \eta$ by (3.12), (3.7) and (3.6). Thus we have proved $p_1(t_0) = p_2(t_0)$.

Now we will prove that there exists a finite $\lim_{\tau \to 0_-} ((s(t_0 + \tau) - s(t_0))/\tau)$ at every $t_0 \in K$ which is not the left hand boundary point of K. Indeed, put $\tilde{K} = \{t \in R\colon -t \in K\}$, $\tilde{s}(t) = s(-t)$, $\tilde{t}_0 = -t_0$. The function $\tilde{s}$ satisfies (3.1) and (3.2) on the interval $\tilde{K}$, therefore there exists a finite $\lim_{\tau \to 0_+} ((\tilde{s}(\tilde{t}_0 + \tau) - \tilde{s}(t_0))/\tau)$ and obviously $\lim_{\tau \to 0_-} ((s(t_0 + \tau) - s(t_0))/\tau) = \lim_{\tau \to 0_+} ((\tilde{s}(\tilde{t}_0 + \tau) - \tilde{s}(t_0))/\tau)$.

Now we can already prove (d). Indeed, if t_0 is the left or the right hand side boundary point of K, then $\lim_{t \to t_0, t \in k} ((s(t) - s(t_0))/(t - t_0)) = \lim_{\tau \to 0_+} ((s(t_0 + \tau) - s(t_0))/\tau)$ or $\lim_{t \to t_0, t \in K} ((s(t) - s(t_0))/(t - t_0)) = \lim_{\tau \to 0_-} ((s(t_0 + \tau) - s(t_0))/\tau)$, respectively. If t_0 is an interior point of K, then, by (c), we have $\lim_{\tau \to 0_+} ((s(t_0 + \tau) - s(t_0))/\tau) = \lim_{\tau \to 0_-} ((s(t_0 + \tau) - s(t_0))/\tau)$.

(e) If $t_0 \in K$, $t \in K$, $t_0 + \tau \in K$, $t + \tau \in K$, then by (3.2) we have

$$\left| \frac{s(t + \tau) - s(t)}{\tau} - \frac{s(t_0 + \tau) - s(t_0)}{\tau} \right| \leqq M|t - t_0|,$$

where $M = \max(|M_1|, |M_2|)$. By passing to the limit we obtain

$$|v(t) - v(t_0)| \leqq M|t - t_0|,$$

therefore v is continuous at t_0.

(f) If $t_0 \in K$, $\sigma > 0$, $t_0 + \sigma \in K$, we obtain from (b) with regard to (d) by passing to the limit for $n \to \infty$:

$$\tfrac{1}{2}M_1\sigma + v(t_0) \leqq \frac{s(t_0 + \sigma) - s(t_0)}{\sigma} \leqq \tfrac{1}{2}M_2\sigma + v(t_0);$$

from these inequalities we easily obtain (f).

(g) If $t_0 \in K$, $\sigma > 0$, $t_0 + \sigma \in K$, we can write

$$v(t_0 + \sigma) = \lim_{\tau \to 0_+} \frac{s(t_0 + \sigma - \tau) - s(t_0 + \sigma)}{-\tau}.$$

But

$$\frac{s(t_0 + \sigma - \tau) - s(t_0 + \sigma)}{-\tau} = \frac{s(t_0 + \sigma) - s(t_0 + \sigma - \tau)}{\tau};$$

if we suppose $0 < \tau < \sigma$ and if we write in (3.2) $t_0 + \sigma - \tau$ instead of t_2, t_0 instead of t_1, we obtain

$$\frac{s(t_0 + \tau) - s(t_0)}{\tau} + M_1(\sigma - \tau) \leqq$$

$$\leqq \frac{s(t_0 + \sigma) - s(t_0 + \sigma - \tau)}{\tau} \leqq \frac{s(t_0 + \tau) - s(t_0)}{\tau} + M_2(\sigma - \tau);$$

by passing to the limit $\tau \to 0_+$ we obtain (g).

Theorem 3.2. *Let us have an open interval $K \subset R$ and maps*

$$B: K \to E_3, \quad C: K \to E_3.$$

Suppose that the function $t \mapsto |\Delta_B(C; t)|$ is bounded on every compact interval $K_1 \subset K$ and that there exists a vector $\boldsymbol{u}$ such that

(3.13) $$\mathbf{A}_B(C; t_1, t_2, \tau) = \boldsymbol{u}$$

for every $t_1 \in K$, $\tau > 0$, $t_1 < t_2$, $t_2 + \tau \in K$. Let $t_0 \in K$. Then there exists a vector $\boldsymbol{v}$ such that

(3.14) $$\Delta_B(C; t) = \Delta_B(C; t_0) + (t - t_0)\,\boldsymbol{v} + \tfrac{1}{2}(t - t_0)^2\,\boldsymbol{u}$$

for every $t \in K$.

Proof. Let us choose a cartesian reference system $\mathscr{S}$ and let $\Delta_B(C; t) = (s_1(t), s_2(t), s_3(t))$, $\boldsymbol{u} = (u_1, u_2, u_3)$ in $\mathscr{S}$. Then, by (2.3) and (2.2), (3.13) can be written in the form

$$\frac{s_l(t_2 + \tau) - s_l(t_2)}{\tau} - \frac{s_l(t_1 + \tau) - s_l(t_1)}{\tau} = u_l(t_2 - t_1)\,, \quad l = 1, 2, 3\,.$$

Therefore, (3.2) is fulfilled for the functions s_l with $M_1 = M_2 = u_l$. Further, $|s_l(t)| \leqq \leqq ((s_1(t))^2 + (s_2(t))^2 + (s_3(t))^2)^{1/2} = |\Delta_B(C; t)|$ and thus the functions s_l are bounded on every compact interval $K_1 \subset K$ and hence (3.1) holds on K. Therefore, by Theorem 3.1 (d), (f), we obtain

$$s_l(t_0 + \sigma) = s_l(t_0) + v_l(t_0)\,\sigma + \tfrac{1}{2}u_l\sigma^2 \quad \text{if} \quad t_0 + \sigma \in K\,, \quad \sigma > 0\,.$$

Thus, if we write $t_0 + \sigma = t$, $\boldsymbol{v} = (v_1(t_0), v_2(t_0), v_3(t_0))$, we obtain (3.14) in K for the case $t > t_0$.

If $t < t_0$, choose a number $\tilde{t} \in K$, $\tilde{t} > t_0$. Then we have

$$\Delta_B(C; \tilde{t}) = \Delta_B(C; t_0) + (\tilde{t} - t_0)\,\boldsymbol{v} + \tfrac{1}{2}(\tilde{t} - t_0)^2\,\boldsymbol{u}\,. \tag{3.15}$$

As $t < t_0 < \tilde{t}$, there exists a vector $\tilde{\boldsymbol{v}}$ such that we also have

$$\Delta_0(C; t_0) = \Delta_B(C; t) + (t_0 - t)\,\tilde{\boldsymbol{v}} + \tfrac{1}{2}(t_0 - t)^2\,\boldsymbol{u}\,, \tag{3.16}$$

$$\Delta_B(C; \tilde{t}) = \Delta_B(C; t) + (\tilde{t} - t)\,\tilde{\boldsymbol{v}} + \tfrac{1}{2}(\tilde{t} - t)^2\,\boldsymbol{u}\,. \tag{3.17}$$

From (3.16) and (3.17) we obtain

$$\Delta_B(C; \tilde{t}) = \Delta_B(C; t_0) + (\tilde{t} - t_0)\,\tilde{\boldsymbol{v}} + \tfrac{1}{2}\boldsymbol{u}(\tilde{t}^2 - t_0^2 + 2tt_0 - 2t\tilde{t})\,;$$

if we compare it with (3.15), we obtain

$$(\tilde{t} - t_0)\,\boldsymbol{v} + \tfrac{1}{2}(\tilde{t} - t_0)^2\,\boldsymbol{u} = (\tilde{t} - t_0)\,\tilde{\boldsymbol{v}} + \tfrac{1}{2}\boldsymbol{u}(\tilde{t}^2 - t_0^2 + 2tt_0 - 2t\tilde{t})$$

and therefore

$$\tilde{\boldsymbol{v}} = \boldsymbol{v} + (t - t_0)\,\boldsymbol{u}\,.$$

If we substitute it into (3.16), we obtain (3.14) for $t < t_0$. It is obvious that (3.14) holds also for $t = t_0$.

4. CONSISTENCY OF THE SYSTEM OF AXIOMS

We will prove that the system formed by axioms I.1–I.3, II.1–II.3, III.1, IV.1–IV.3, V.1–V.5, VI.1–VI.3 is consistent, if the theory of real numbers and the geometry of the three-dimensional euclidean space are consistent. We shall carry this proof out if we construct, in terms of objects of the three-dimensional euclidean space and real numbers, a model fulfilling all these axioms.

First of all, given a three-dimensional euclidean space E_3 with the metric ϱ, let us choose a cartesian reference system $\mathscr{S}$ in it. If $\boldsymbol{U}, \boldsymbol{V}, \boldsymbol{W}$ are vectors in E_3, we shall define that *the force* $\boldsymbol{W}$ *is the resultant of forces* $\boldsymbol{U}, \boldsymbol{V}$ if and only if

$$\boldsymbol{W} = \boldsymbol{U} + \boldsymbol{V}. \tag{4.1}$$

Let us choose a natural number n and n real positive numbers $m_1, \ldots, m_n$. For $i = 1, \ldots, n$ we will define *the particle* α_i as the couple of numbers $\{i, m_i\}$. The number m_i will be defined as *the mass* of the particle $\alpha_i = \{i, m_i\}$. Our system will have n particles $\alpha_1, \ldots, \alpha_n$.

Further, let us choose $6n$ real numbers

$$a_{ir}, v_{ir}\,; \quad i = 1, \ldots, n\,; \quad r = 1, 2, 3\,; \tag{4.2}$$

such that

$$(a_{i1}, a_{i2}, a_{i3}) \neq (a_{h1}, a_{h2}, a_{h3}) \quad \text{for} \quad i \neq h\,. \tag{4.3}$$

Moreover, let us choose a real number $\varkappa > 0$ and a real number $\bar{t}$. Let us consider the system of differential equations[2])

$$m_i \ddot{x}_{ir} = \sum_{\substack{h=1\\h\neq i}}^{n} \varkappa \frac{m_i m_h (x_{hr} - x_{ir})}{\left(\sum_{l=1}^{3} (x_{hl} - x_{il})^2\right)^{3/2}}\,; \quad i = 1, \ldots, n\,; \quad r = 1, 2, 3\,. \tag{4.4}$$

This system has a solution in an open interval $\mathscr{T}$, containing the number $\bar{t}$, and this solution is formed by functions $x_{1r}, \ldots, x_{nr}$; $r = 1, 2, 3$; fulfiling the initial conditions

$$x_{ir}(\bar{t}) = a_{ir}\,, \quad \dot{x}_{ir}(\bar{t}) = v_{ir} \tag{4.5}$$

and the inequality

$$(x_{i1}(t), x_{i2}(t), x_{i3}(t)) \neq (x_{h1}(t), x_{h2}(t), x_{h3}(t)) \quad \text{for} \quad i \neq h\,, \quad t \in \mathscr{T}\,. \tag{4.6}$$

We will choose the interval $\mathscr{T}$ as the set of time instants. We define the position of the particle $\alpha_i = \{i, m_i\}$ at instant $t \in \mathscr{T}$ to be the point of E_3 whose coordinates in the reference system $\mathscr{S}$ are $[x_{i1}(t), x_{i2}(t), x_{i3}(t)]$.

If $\boldsymbol{F}$ is a vector whose coordinates in the reference system $\mathscr{S}$ are (f_1, f_2, f_3) and if $t \in \mathscr{T}$, we define that *the particle* α_h *effects the particle* α_i *at the instant* t *by the force* $\boldsymbol{F}$ if and only if $i \neq h$ and

$$f_r = \varkappa m_i m_h \frac{x_{hr}(t) - x_{ir}(t)}{\left(\sum_{l=1}^{3} (x_{hl}(t) - x_{il}(t))^2\right)^{3/2}}\,; \quad r = 1, 2, 3\,. \tag{4.7}$$

Let us have an open interval $\mathscr{T}_1 \subset \mathscr{T}$, a map $Y : \mathscr{T}_1 \to E_3$, a positive real number m and for every $t \in \mathscr{T}_1$ a system of vectors $\{\boldsymbol{F}_j(t)\}_{j \in J}$ (the set J being the same for

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[2]) In the case $n = 1$ the system will be

$$m_1 \ddot{x}_{1r} = 0\,; \quad r = 1, 2, 3\,.$$

all $t \in \mathcal{T}_1$). In the reference system $\mathcal{S}$ let $Y(t) = [y_1(t), y_2(t), y_3(t)]$, $\boldsymbol{F}_j(t) = (f_{j1}(t), f_{j2}(t), f_{j3}(t))$ for $j \in J$, $t \in \mathcal{T}_1$. We will say that *the motion of the material point* $\{Y(t), m\}$ *in the interval* $\mathcal{T}_1$ *can be interpreted by the operation of the system of forces* $\{\boldsymbol{F}_j(t)\}_{j \in J}$ if and only if the following conditions take place:

(4.8a) the set J is finite;[3])

(4.8b) the functions $\sum_{j \in J} f_{jl}$ for $l = 1, 2, 3$ are continuous on $\mathcal{T}_1$;

(4.8c) the functions y_1, y_2, y_3 have continuous derivatives of the second order on $\mathcal{T}_1$;

(4.8d) $$m\ddot{y}_l = \sum_{j \in J} f_{jl} \quad \text{on} \quad \mathcal{T}_1 \quad \text{for} \quad l = 1, 2, 3\,.$$

Theorem 4.1. *The above described model fulfils all axioms* I.1–I.3, II.1–II.3, III.1, IV.1–IV.3, V.1–V.5, VI.1–VI.3.

Proof. Axioms I.1 and I.2. Suppose that (see the begining of Chapter 2)

(4.9) $$\mathcal{F}(\{B(t), m\}, \{\boldsymbol{F}_j(t)\}_{j \in J}, \mathcal{T}_1)\,,$$

(4.10) $$\mathcal{F}(\{C(t), m\}, \{\boldsymbol{F}_j(t)\}_{j \in J} \cup \{\boldsymbol{G}(t)\}, \mathcal{T}_1)\,,$$

(4.11) $$\boldsymbol{G}(t) = g(t)\,\boldsymbol{u}\,, \quad |\boldsymbol{u}| = 1\,, \quad g(t) \geqq 0 \quad \text{for} \quad t \in \mathcal{T}_1\,.$$

By (4.8a), the set J is finite. By (4.8b), the functions $\sum_{j \in J} \boldsymbol{F}_j$, $g\boldsymbol{u} + \sum_{j \in J} \boldsymbol{F}_j$ and therefore also the function g are continuous on $\mathcal{T}_1$. By (4.8c), the functions B, C have continuous derivatives of the second order on $\mathcal{T}_1$ and by (4.8d) we have

$$m\ddot{B} = \sum_{j \in J} \boldsymbol{F}_j\,, \quad m\ddot{C} = g\boldsymbol{u} + \sum_{j \in J} \boldsymbol{F}_j \quad \text{on} \quad \mathcal{T}_1$$

and therefore

$$m(\ddot{C} - \ddot{B}) = g\boldsymbol{u} \quad \text{on} \quad \mathcal{T}_1\,.$$

If we choose a number $t_0 \in \mathcal{T}_1$, we obtain by (2.1)

$$m\,\Delta_B(C; t) = m\,\Delta_B(C; t_0) + m\dot{\Delta}_B(C; t_0)\,(t - t_0) + \int_{t_0}^{t} \left(\int_{t_0}^{\sigma_2} g(\sigma_1)\,\mathrm{d}\sigma_1 \right) \mathrm{d}\sigma_2\,\boldsymbol{u} \quad \text{for} \quad t \in \mathcal{T}_1\,.$$

If now $t \in \mathcal{T}_1$, $\tau > 0$, $t + \tau \in \mathcal{T}_1$, we obtain by (2.2)

$$m\,\mathbf{V}_B(C; t, \tau) = m\,\dot{\Delta}_B(C; t_0) + \frac{1}{\tau} \left(\int_{t_0}^{t+\tau} \left(\int_{t_0}^{\sigma_2} g(\sigma_1)\mathrm{d}\sigma_1 \right) \mathrm{d}\sigma_2\,\boldsymbol{u} - \int_{t_0}^{t} \left(\int_{t_0}^{\sigma_2} g(\sigma_1)\mathrm{d}\sigma_1 \right) \mathrm{d}\sigma_2\,\boldsymbol{u} \right) = m\,\dot{\Delta}_B(C; t_0) + \frac{1}{\tau} \int_{t}^{t+\tau} \left(\int_{t_0}^{\sigma_2} g(\sigma_1)\mathrm{d}\sigma_1 \right) \mathrm{d}\sigma_2\,\boldsymbol{u}\,.$$

[3]) The empty set is also considered finite. If the set J is empty, we put $\sum_{j \in J} f_{jl} = 0$.

If $t_1 \in \mathscr{T}_1$, $\tau > 0$, $t_1 < t_2$, $t_2 + \tau \in \mathscr{T}_1$, we obtain by (2.3) and the above relations

$$m\mathbf{A}_B(C; t_1, t_2, \tau) = \frac{1}{t_2 - t_1} \left\{ \frac{1}{\tau} \int_{t_2}^{t_2+\tau} \left(\int_{t_0}^{\sigma_2} g(\sigma_1) \mathrm{d}\sigma_1 \right) \mathrm{d}\sigma_2 - \right.$$

$$\left. - \frac{1}{\tau} \int_{t_1}^{t_1+\tau} \left(\int_{t_0}^{\sigma_2} g(\sigma_1) \mathrm{d}\sigma_1 \right) \mathrm{d}\sigma_2 \right\} \boldsymbol{u} .$$

If we substitute $\sigma_3 = \sigma_2 + t_1 - t_2$ in the first integral and write σ_3 instead of σ_2 in the second one, we obtain

$$(4.12) \qquad m\, \mathbf{A}_B(C; t_1, t_2, \tau) = \frac{1}{t_2 - t_1} \left\{ \frac{1}{\tau} \int_{t_1}^{t_1+\tau} \left(\int_{t_0}^{\sigma_3 + t_2 - t_1} g(\sigma_1) \mathrm{d}\sigma_1 \right) \mathrm{d}\sigma_3 - \right.$$

$$\left. - \frac{1}{\tau} \int_{t_1}^{t_1+\tau} \left(\int_{t_0}^{\sigma_3} g(\sigma_1) \mathrm{d}\sigma_1 \right) \mathrm{d}\sigma_3 \right\} \boldsymbol{u} = \frac{1}{t_2 - t_1} \frac{1}{\tau} \int_{t_1}^{t_1+\tau} \left(\int_{\sigma_3}^{\sigma_3 + t_2 - t_1} g(\sigma_1) \mathrm{d}\sigma_1 \right) \mathrm{d}\sigma_3\, \boldsymbol{u}.$$

Thus, Axiom I.1 is fulfilled with

$$k = \frac{1}{m} \frac{1}{t_2 - t_1} \frac{1}{\tau} \int_{t_1}^{t_1+\tau} \left(\int_{\sigma_3}^{\sigma_3 + t_2 - t_1} g(\sigma_1) \mathrm{d}\sigma_1 \right) \mathrm{d}\sigma_3 .$$

Now, provided $t_1 \leqq \sigma_3 \leqq t_1 + \tau$ (i.e. σ_3 is within the limits of integration by σ_3), we have $\sigma_3 \leqq \sigma_3 + t_2 - t_1 \leqq t_2 + \tau$, therefore $\langle \sigma_3, \sigma_3 + t_2 - t_1 \rangle \subset \langle t_1, t_2 + \tau \rangle$. If we now have

$$p_1 \leqq |G(t)| = g(t) \leqq p_2 \quad \text{for all} \quad t \in \langle t_1, t_2 + \tau \rangle ,$$

then

$$p_1(t_2 - t_1) \leqq \int_{\sigma_3}^{\sigma_3 + t_2 - t_1} g(\sigma_1)\, \mathrm{d}\sigma_1 \leqq p_2(t_2 - t_1)$$

and

$$p_1(t_2 - t_1) = \frac{1}{\tau} \int_{t_1}^{t_1+\tau} p_1(t_2 - t_1) \leqq \frac{1}{\tau} \int_{t_1}^{t_1+\tau} \left(\int_{\sigma_3}^{\sigma_3 + t_2 - t_1} g(\sigma_2) \mathrm{d}\sigma_2 \right) \mathrm{d}\sigma_3 \leqq$$

$$\leqq \frac{1}{\tau} \int_{t_1}^{t_1+\tau} p_2(t_2 - t_1) = p_2(t_2 - t_1) ,$$

therefore, by (4.11) and (4.12), we obtain

$$p_1 \leqq m|\mathbf{A}_B(C; t_1, t_2, \tau)| = \frac{1}{t_2 - t_1} \frac{1}{\tau} \int_{t_1}^{t_1+\tau} \left(\int_{\sigma_3}^{\sigma_3 + t_2 - t_1} g(\sigma_1) \mathrm{d}\sigma_1 \right) \mathrm{d}\sigma_3 \leqq p_2 .$$

Thus, Axiom I.2 is fulfilled.

Axiom I.3. Suppose that (4.9), (4.10) and (4.11) hold. Then, by (4.8c), the functions B, C are continuous on $\mathscr{T}_1$, therefore by (2.1) the function $t \mapsto \Delta_B(C; t)$ is also continuous on $\mathscr{T}_1$ and hence bounded on every compact subinterval of $\mathscr{T}_1$.

Axioms II.1 and II.2 are implied by (4.1) and (4.8).

Axiom II.3. Suppose that (4.9) holds. Let G be a vector, let $C(t) \in E_3$ if $t \in \mathscr{T}_1$ and let

$$m\, \mathbf{A}_B(C; t_1, t_2, \tau) = G \quad \text{for every} \quad t_1 \in \mathscr{T}_1,\ t_1 < t_2,\ \tau > 0,\ t_2 + \tau \in \mathscr{T}_1 .$$

Let the function $t \mapsto \Delta_B(C; t)$ be continuous on $\mathscr{T}_1$. Then the function $t \mapsto |\Delta_B(C; t)|$ is bounded on every compact interval $K_1 \subset \mathscr{T}_1$ and therefore, by Theorem 3.2, there exists a vector v such that

$$\Delta_B(C; t) = \Delta_B(C; t_0) + (t - t_0)\, v + \frac{1}{2m}(t - t_0)^2\, G \quad \text{for all} \quad t \in \mathscr{T}_1 ,$$

or, according to (2.1),

(4.13)

$$C(t) = B(t) + \Delta_B(C; t_0) + (t - t_0)\, v + \frac{1}{2m}(t - t_0)^2\, G \quad \text{for all} \quad t \in \mathscr{T}_1 .$$

Now, (4.9) together with (4.8) implies that the set J is finite, the function $t \mapsto \sum_{j \in J} F_j(t)$ is continuous on $\mathscr{T}_1$ and

$$m\ddot{B} = \sum_{j \in J} F_j . \tag{4.14}$$

(4.13) and (4.14) imply

$$m\ddot{C} = \sum_{j \in J} F_j + G ,$$

therefore $\mathscr{F}(\{C(t), m\}, \{F_j(t)\}_{j \in J} \cup \{G\}, \mathscr{T}_1)$ holds by (4.8).

Axiom III.1. Suppose that (4.10) and (4.11) hold and that the function $t \mapsto |G(t)|$ is continuous on $\mathscr{T}_1$. Then, by (4.8), the set J is finite and the function $t \mapsto G(t) + \sum_{j \in J} F_j(t)$ is continuous on $\mathscr{T}_1$. Because the function $t \mapsto |G(t)|$ is continuous on $\mathscr{T}_1$ as well, the function $t \mapsto g(t)\, u = G(t)$ and therefore also $t \mapsto \sum_{j \in J} F_j(t)$ is continuous on $\mathscr{T}_1$. Define the function B by the differential equation

$$m\ddot{B} = \sum_{j \in J} F_j .$$

In accordance with (4.8), the material points $\{B(t), m\}$ fulfil (4.9). Thus Axiom III.1 is fulfilled.

Axioms IV.1 – IV.3. If $\mathscr{F}(\{B(t), m\}, \mathfrak{M}, \mathscr{T}_1\}$ holds, where $\mathfrak{M}$ is the empty system of forces, then by (4.8d) we have $\ddot{B} = 0$ on $\mathscr{T}_1$ and therefore there exists a point Q and a vector v such that

$$B(t) = Q + tv \quad \text{for all} \quad t \in \mathscr{T}_1 .$$

From this equality we easily deduce Axioms IV.1 and IV.3.

Axiom IV.2 follows from the fact that $\sum_{j \in J} F_j = 0$ in the case of the empty J and also in the case of the system $\{0\}$.

Axiom V.1. The formulae (4.4) and (4.6) imply the continuity of the functions $t \mapsto \mathsf{P}_\alpha(t)$. Therefore, by (4.6) and (4.7), if $\mathscr{G}(\alpha_i, \alpha_j, t, \boldsymbol{F}_{ij}(t))$ for all $t \in \mathscr{T}_1$, the functions $t \mapsto \boldsymbol{F}_{ij}(t)$ are continuous on $\mathscr{T}_1$. Thus, (4.4), (4.7) and (4.8) imply Axiom V.1.

Axioms V.2 – V.4 follow from (4.7).

Axiom V.5 follows from (2.1) and from the continuity of the functions $t \mapsto \mathsf{P}_\alpha(t)$.

Axioms VI.1 and VI.2 follow from the definition of particles.

Axiom VI.3 follows from (4.6).

We have proved Theorem 4.1 and therefore also the consistency of our system of axioms. In fact, we have proved the following stronger

Theorem 4.2. *Let us have three-dimensional euclidean space E_3 and a cartesian reference system $\mathscr{S}$ in it. Further, let us have a natural number n and $6n$ real numbers a_{ir}, v_{ir} $(i = 1, \ldots, n; r = 1, 2, 3)$, a real number $\varkappa > 0$ and a real number $\bar{t}$. Then there exists a model fulfilling all the axioms* I.1, …, VI.3 *with the following properties*:

(a) *There is exactly n particles $\alpha_1, \ldots, \alpha_n$ in the model.*

(b) *The set $\mathscr{T}$ of time instants is an open interval containing $\bar{t}$.*

(c) *For every $t \in \mathscr{T}$, the position $\mathsf{P}_{\alpha_i}(t)$ of the particle α_i is a point of the space E_3.*

(d) *If $[x_{i1}(t), x_{i2}(t), x_{i3}(t)]$ are the coordinates of the point $\mathsf{P}_{\alpha_i}(t)$ in the reference system $\mathscr{S}$, then the functions x_{ir} have continuous derivatives of the second order, fulfil the system of differential equations* (4.4) *and the initial conditions* (4.5).

In the next chapters we shall see that the model constructed in this chapter is essentially the single model fulfilling all axioms I.1, …, VI.3.

5. THE EFFECT OF A FORCE OF A CONSTANT DIRECTION. CONSEQUENCES OF THE AXIOMS OF THE GROUP I

In this section we shall use only Axioms I.1 – I.3.

Theorem 5.1. *Suppose that the motion of a material point $\{B(t), m\}$ in an open interval $\mathscr{T}_1 \subset \mathscr{T}$ can be interpreted by the operation of a system of forces $\{\boldsymbol{F}_j(t)\}_{j \in J}$, the motion of a material point $\{C(t), m\}$ by the operation of a system of forces $\{\boldsymbol{F}_j(t)\}_{j \in J} \cup \{\boldsymbol{G}(t)\}$ and that all vectors $\boldsymbol{G}(t)$, $t \in \mathscr{T}_1$, have a common direction $\boldsymbol{u}$. Let the function $t \mapsto |\boldsymbol{G}(t)|$ be bounded on every compact subinterval of $\mathscr{T}_1$ (this is certainly fulfilled if $t \mapsto \boldsymbol{G}(t)$ is continuous on $\mathscr{T}_1$). Then there exists a real function s defined on $\mathscr{T}_1$ and vectors $\boldsymbol{v}$, $\boldsymbol{w}$ with the following properties*:

(a) $$C(t) = B(t) + s(t)\,\boldsymbol{u} + t\boldsymbol{v} + \boldsymbol{w} \quad \textit{for all} \quad t \in \mathscr{T}_1\,;$$

(b) *the function s has a continuous derivative $v = \dot{s}$ on $\mathscr{T}_1$*;

(c) *if* $t_0 \in \mathscr{T}_1$, $t \in \mathscr{T}_1$, $t_0 \leqq t$, *then*

$$(5.1) \qquad v(t_0) + \frac{1}{m}\underline{\int_{t_0}^{t}} |G(\sigma)| \, d\sigma \leqq v(t) \leqq v(t_0) + \frac{1}{m}\overline{\int_{t_0}^{t}} |G(\sigma)| \, d\sigma \,,$$

where the symbols $\underline{\int}$, $\overline{\int}$ *denote the lower and the upper Riemann's integral;*

(d) *if the function* $t \mapsto |G(t)|$ *is continuous on* $\mathscr{T}_1$ *and* $t_0 \in \mathscr{T}_1$, *then*

$$(5.2) \qquad v(t) = v(t_0) + \frac{1}{m}\int_{t_0}^{t} |G(\sigma)| \, d\sigma \,.$$

Proof. Choose a cartesian reference system $\mathscr{S}$ such that in this system

$$(5.3) \qquad \boldsymbol{u} = (1, 0, 0) \,.$$

In the reference system $\mathscr{S}$ the vectors $G(t)$ have the coordinates

$$(5.4) \qquad G(t) = (|G(t)|, 0, 0) \,.$$

In the system $\mathscr{S}$ let

$$(5.5) \qquad C(t) = [c_1(t), c_2(t), c_3(t)] \,, \quad B(t) = [b_1(t), b_2(t), b_3(t)]$$

and denote

$$(5.6) \qquad s_l(t) = c_l(t) - b_l(t) \,; \quad l = 1, 2, 3 \,; \quad t \in \mathscr{T}_1 \,.$$

If $t_1 \in \mathscr{T}_1$, $\tau > 0$, $t_1 < t_2$, $t_2 + \tau \in \mathscr{T}_1$, then, by Axiom I.1, there exists a number $k(t_1, t_2, \tau) \geqq 0$ such that

$$(5.7) \qquad \mathbf{A}_B(C; t_1, t_2, \tau) = k(t_1, t_2, \tau)\, \boldsymbol{u} \,,$$

or, with regard to (2.3), (2.2), (2.1), (5.6), (5.5) and (5.3), we have

$$(5.8a) \qquad \frac{s_1(t_2 + \tau) - s_1(t_2)}{\tau} - \frac{s_1(t_1 + \tau) - s_1(t_1)}{\tau} = k(t_1, t_2, \tau)\,(t_2 - t_1) \,,$$

$$(5.8b) \qquad \frac{s_l(t_2 + \tau) - s_l(t_2)}{\tau} - \frac{s_l(t_1 + \tau) - s_l(t_1)}{\tau} = 0 \,; \quad l = 2, 3 \,.$$

We will prove first of all that the functions s_l have continuous derivatives on $\mathscr{T}_1$. It is sufficient to prove that they have continuous derivatives on every open bounded interval $\mathscr{T}_2$ such that $\bar{\mathscr{T}}_2 \subset \mathscr{T}_1$. Let $\mathscr{T}_2$ be such an interval.

The function $t \mapsto |G(t)|$ being bounded on $\mathscr{T}_2$, (5.7), (5.3) and Axiom I.2 imply that there exists a number M such that

$$0 \leqq |\mathbf{A}_B(C; t_1, t_2, \tau)| = k(t_1, t_2, \tau) \leqq M \quad \text{if}$$

$$t_1 \in \mathscr{T}_2 \,, \quad t_1 < t_2 \,, \quad \tau > 0 \,, \quad t_2 + \tau \in \mathscr{T}_2 \,,$$

and (5.8a) then implies

$$(5.9)\quad -M(t_2 - t_1) \leqq \frac{s_1(t_2 + \tau) - s_1(t_2)}{\tau} - \frac{s_1(t_1 + \tau) - s_1(t_1)}{\tau} \leqq M(t_2 - t_1)\,.$$

Axiom I.3 implies that the function $t \mapsto |\Delta_B(C; t)|$ is bounded on $\mathscr{T}_2$. Since $|s_l(t)| \leqq \leqq |\Delta_B(C; t)|$ by (5.6), (5.5) and (2.1), the functions s_l are also bounded on $\mathscr{T}_2$, Now, (5.9), (5.8b) and Theorem 3.1 (d), (e) imply that the functions s_l have continuous derivatives on $\mathscr{T}_2$. Therefore, they have continuous derivatives $v_l = \dot{s}_l$ on the whole $\mathscr{T}_1$.

By passing to the limit $\tau \to 0+$ in (5.8b), we now obtain

$$(5.10)\qquad \dot{s}_l(t) = v_l(t) = a_l\,,$$

$$s_l(t) = a_l t + w_l \quad \text{for any} \quad t \in \mathscr{T}_1\,; \quad l = 1, 2\,;$$

where a_2, a_3, w_2, w_3 are constant. If we now put $s(t) = s_1(t)$, $\boldsymbol{v} = (0, a_2, a_3)$, $\boldsymbol{w} = = (0, q_2, q_3)$, then (2.1), (5.5), (5.6) and (5.10) imply (a). As $s_1 = s$ has a continuous derivative, we have proved also (b).

Let us now have $t_0 \in \mathscr{T}_1$, $t \in \mathscr{T}_1$, $t_0 < t$. Choose a division

$$(5.11)\qquad t_0 = \tau_0 < \tau_1 < \tau_2 < \ldots < \tau_n = t$$

of the interval $\langle t_0, t\rangle$ and denote

$$(5.12)\qquad g_i = \inf_{\sigma \in \langle \tau_{i-1}, \tau_i\rangle} |\boldsymbol{G}(\sigma)|\,, \quad G_i = \sup_{\sigma \in \langle \tau_{i-1}, \tau_i\rangle} |\boldsymbol{G}(\sigma)|\,; \quad i = 1, 2, \ldots, n$$

(g_i and G_i are all finite, as the function $t \mapsto |\boldsymbol{G}(t)|$ is bounded on every compact subinterval of $\mathscr{T}_1$). Axiom I.2 implies

$$\frac{g_i}{m} \leqq |\mathbf{A}_B(C; t_1, t_2, \tau)| \leqq \frac{G_i}{m} \quad \text{if} \quad t_1 \in \langle \tau_{i-1}, \tau_i\rangle\,, \quad t_1 < t_2\,, \quad \tau > 0\,,$$
$$t_2 + \tau \in \langle \tau_{i-1}, \tau_i\rangle$$

and for these t_1, t_2, τ, (5.7) and (5.3) imply

$$\frac{g_i}{m} \leqq k(t_1, t_2, \tau) \leqq \frac{G_i}{m}\,.$$

Consequently, from (5.8a) we obtain

$$\frac{g_i}{m}(t_2 - t_1) \leqq \frac{s_1(t_2 + \tau) - s_1(t_2)}{\tau} - \frac{s_1(t_1 + \tau) - s_1(t_1)}{\tau} \leqq \frac{G_i}{m}(t_2 - t_1)\,.$$

The function s_1 is bounded on $\langle \tau_{i-1}, \tau_i\rangle$ and therefore Theorem 3.1(g) implies

$$\frac{g_i}{m}(\tau_i - \tau_{i-1}) \leqq v_1(\tau_i) - v_1(\tau_{i-1}) \leqq \frac{G_i}{m}(\tau_i - \tau_{i-1})\,.$$

Summing from $i = 1$ to $i = n$, we obtain

$$\sum_{i=1}^{n} \frac{g_i}{m} (\tau_i - \tau_{i-1}) \leqq v_1(t) - v_1(t_0) \leqq \sum_{i=1}^{n} \frac{G_i}{m} (\tau_i - \tau_{i-1}).$$

These inequalities remain true, if we pass on the left hand side to the supremum, on the right hand side to the infimum with regard to all the divisions (5.11). Thus we obtain

$$\underline{\int}_{t_0}^{t} \frac{|G(\sigma)|}{m} \, d\sigma \leqq v_1(t) - v_1(t_0) \leqq \overline{\int}_{t_0}^{t} \frac{|G(\sigma)|}{m} \, d\sigma .$$

Because $s = s_1$ and therefore $v = v_1$, we have proved (c) for $t_0 < t$. It is evident that (c) is true for $t = t_0$ as well.

If the function $t \mapsto |G(t)|$ is continuous on $\mathscr{T}_1$ and $t_0 \leqq t$, (5.2) is an easy consequence of (5.1). If $t < t_0$, then by (5.1) we have

$$v(t_0) = v(t) + \frac{1}{m} \int_{t}^{t_0} |G(\sigma)| \, d\sigma$$

and this again implies (5.2), because $\int_{t_0}^{t} |G(\sigma)| \, d\sigma = - \int_{t}^{t_0} |G(\sigma)| \, d\sigma$.

Theorem 5.2. *Suppose that the motion of the material point $\{B(t), m\}$ in the open interval $\mathscr{T}_1 \subset \mathscr{T}$ can be interpreted by the operation of the system of forces $\{F_j(t)\}_{j \in J}$, the motion of the material point $\{C(t), m\}$ by the operation of the system of forces $\{F_j(t)\}_{j \in J} \cup \{G\}$, the vector G being constant. Then the vector G is uniquely determined. If $t_1 \in \mathscr{T}_1$, $t_1 < t_2$, $\tau > 0$, $t_2 + \tau \in \mathscr{T}_1$, then*

$$G = m \, . \, \mathbf{A}_B(C; t_1, t_2, \tau) . \tag{5.13}$$

Proof. We can write $G = |G| \, u$, where $|u| = 1$. Let $t_1 \in \mathscr{T}_1$, $t_1 < t_2$, $\tau > 0$, $t_2 + \tau \in \mathscr{T}_1$. Then, by Axiom I.1, there exists a real number $k \geqq 0$ such that

$$\mathbf{A}_B(C; t_1, t_2, \tau) = ku .$$

Axiom I.2 implies

$$m|\mathbf{A}_B(C; t_1, t_2, \tau)| = |G|$$

and therefore

$$k = |\mathbf{A}_B(C; t_1, t_2, \tau)| = \frac{1}{m} |G| .$$

6. COMPOSITION OF FORCES. CONSEQUENCES OF THE AXIOMS OF THE GROUPS I AND II

In this section we shall use only the axioms of the groups I, II and the following

Supposition E. *If we have an open interval $\mathscr{T}_1 \subset \mathscr{T}$, then there exists a positive real number m, points $B(t) \in E_3$ for any $t \in \mathscr{T}_1$, a finite set J (possibly empty)* *and a system of vectors* $\{\boldsymbol{F}_j(t)\}_{j\in J}$, $t \in \mathscr{T}_1$, *such that the motion of the material point* $\{B(t), m\}$ *in the interval* $\mathscr{T}_1$ *can be interpreted by the operation of the system of forces* $\{\boldsymbol{F}_j(t)\}_{j\in J}$.

The supposition E is an evident consequence of the axioms of the groups V and VI. Indeed, by Axiom VI.1, there exists a particle α. Put $B(t) = \mathsf{P}_\alpha(t)$, $m = m_\alpha$, $J = \mathscr{M} \setminus \{\alpha\}$; by Axiom VI.3, $\mathsf{P}_\alpha(t) \neq \mathsf{P}_\beta(t)$ for any $\beta \in J$; therefore, by Axiom V.2, for every $\beta \in \mathscr{M}$, $t \in \mathscr{T}_1$, there exists a vector $\boldsymbol{F}_\beta(t)$ such that $\mathscr{G}(\alpha, \beta, t, \boldsymbol{F}_\beta(t))$. By Axiom VI.2, the set J is finite and by Axiom V.1, $\mathscr{F}(\{B(t), m\}, \{\boldsymbol{F}_\beta(t)\}_{\beta\in J}, \mathscr{T}_1)$.

Theorem 6.1. *If* $\boldsymbol{F}, \boldsymbol{G}, \boldsymbol{H}$ *are vectors, then the force* $\boldsymbol{H}$ *is the resultant of forces* $\boldsymbol{F}, \boldsymbol{G}$ *if and only if* $\boldsymbol{H} = \boldsymbol{F} + \boldsymbol{G}$.

Proof. Let us choose an open bounded interval $\mathscr{T}_1 \subset \mathscr{T}$. By the supposition E, there exists a positive real number m, points $B(t) \in E_3$ for all $t \in \mathscr{T}_1$, a finite set J and a system of vectors $\{\boldsymbol{F}_j(t)\}_{j\in J}$, $t \in \mathscr{T}_1$, such that $\mathscr{F}(\{B(t), m\}, \{\boldsymbol{F}_j(t)\}_{j\in J}, \mathscr{T}_1)$. Choose a number $t_0 \in \mathscr{T}_1$ and define

$$C(t) = B(t) + \frac{1}{2m}(t - t_0)^2 \boldsymbol{F}, \quad t \in \mathscr{T}_1, \tag{6.1}$$

$$D(t) = C(t) + \frac{1}{2m}(t - t_0)^2 \boldsymbol{G} = B(t) + \frac{1}{2m}(t - t_0)^2 (\boldsymbol{F} + \boldsymbol{G}), \quad t \in \mathscr{T}_1. \tag{6.2}$$

If $t_1 \in \mathscr{T}_1$, $t_1 < t_2$, $\tau > 0$, $t_2 + \tau \in \mathscr{T}_1$, then by (2.3), (2.2) and (2.1) we obtain

$$m\,\mathbf{A}_B(C; t_1, t_2, \tau) = \boldsymbol{F}, \quad m\,\mathbf{A}_C(D; t_1, t_2, \tau) = \boldsymbol{G}.$$

Axiom II.3 then yields

$$\begin{gathered} \mathscr{F}(\{C(t), m\}, \{\boldsymbol{F}_j(t)\}_{j\in J} \cup \{\boldsymbol{F}\}, \mathscr{T}_1), \\ \mathscr{F}(\{D(t), m\}, \{\boldsymbol{F}_j(t)\}_{j\in J} \cup \{\boldsymbol{F}\} \cup \{\boldsymbol{G}\}, \mathscr{T}_1). \end{gathered} \tag{6.3}$$

But by (6.2), we also have

$$m\,\mathbf{A}_B(C; t_1, t_2, \varepsilon) = \boldsymbol{F} + \boldsymbol{G},$$

and Axiom II.3 implies

$$\mathscr{F}(\{D(t), m\}, \{\boldsymbol{F}_j(t)\}_{j\in J} \cup \{\boldsymbol{F} + \boldsymbol{G}\}, \mathscr{T}_1).$$

Now, by Axiom II.2, there exists a vector $\boldsymbol{H}$ such that $\boldsymbol{H} = \boldsymbol{F} \oplus \boldsymbol{G}$. By (6.3) and by Axiom II.1 we obtain

$$\mathscr{F}(\{B(t), m\}, \{\boldsymbol{F}_j(t)\}_{j\in J} \cup \{\boldsymbol{H}\}, \mathscr{T}_1),$$

and we conclude from Theorem 5.2 that $\boldsymbol{H} = \boldsymbol{F} + \boldsymbol{G}$.

Theorem 6.1 and Axiom II.1 imply

Theorem 6.2. *Let us have an open interval* $\mathscr{T}_1 \subset \mathscr{T}$ *and two systems of vectors* $\{\boldsymbol{F}_j(t)\}_{j\in J}$, $\{\boldsymbol{G}_k(t)\}_{k\in K}$, $t \in \mathscr{T}_1$, *where the sets* J, K *are finite, and let*

$$\sum_{j\in J} \boldsymbol{F}_j(t) = \sum_{k\in K} \boldsymbol{G}_k(t) \quad \textit{for all} \quad t \in \mathscr{T}_1.$$

Then the motion of a material point $\{B(t), m\}$ *in* $\mathscr{T}_1$ *can be interpreted by the operation of the system of forces* $\{F_j(t)\}_{j\in J}$ *if and only if it can be interpreted by the operation of the system* $\{G_k(t)\}_{k\in K}$.

7. CONSEQUENCES OF THE AXIOMS OF THE GROUPS I, II AND III. NEWTON'S SECOND LAW

In this chapter we shall use the axioms of the groups I, II, III and the supposition E.

Theorem 7.1. *Let us have an open interval* $\mathscr{T}_1 \subset \mathscr{T}$, *areal number* $m > 0$ *and points* $C(t) \in E_3$ *for all* $t \in \mathscr{T}_1$. *Suppose that the motion of the material point* $\{C(t), m\}$ *in* $\mathscr{T}_1$ *can be interpreted by the operation of a system of forces* $\{G_j(t)\}_{j\in J}$, *where the set J is finite. Let the map* $t \mapsto \sum_{j\in J} G_j(t)$ *be continuous on* $\mathscr{T}_1$ *and let* $t_0 \in \mathscr{T}_1$. *Then there exist vectors* $u \in V_3$, $v \in V_3$ *and points* $B(t) \in E_3$, $t \in \mathscr{T}_1$, *such that the motion of the material point* $\{B(t), m\}$ *in* $\mathscr{T}_1$ *can be interpreted by the operation of the empty system of forces and that*

$$(7.1)\quad C(t) = B(t) + u + tv + \frac{1}{m}\int_{t_0}^{t}\left(\int_{t_0}^{\sigma_2} \sum_{j\in J} G_j(\sigma_1)\,d\sigma_1\right) d\sigma_2 \quad \textit{fot all} \quad t \in \mathscr{T}_1 .$$

Proof. Denote $G(t) = \sum_{j\in J} G_j(t)$. Choose a cartesian reference system $\mathscr{S}$ and let in $\mathscr{S}$

$$G(t) = (g_1(t), g_2(t), g_3(t)),$$
$$w_1 = (1, 0, 0), \quad w_2 = (0, 1, 0), \quad w_3 = (0, 0, 1).$$
$$w_4 = (-1, 0, 0), \; w_5 = (0, -1, 0), \; w_6 = (0, 0, -1).$$

Let us denote[4])

$$h_i = g_i^+ \text{ for } i = 1, 2, 3; \; h_i = g_{i-3}^- \text{ for } i = 4, 5, 6.$$

Then

$$(7.2)\qquad \sum_{j\in J} G_j(t) = \sum_{i=1}^{6} h_i(t)\, w_i, \quad |w_i| = 1, \quad h_i(t) \geqq 0 \quad \text{for} \quad t \in \mathscr{T}_1 .$$

The functions h_i are continuous on $\mathscr{T}_1$, because the map $t \mapsto \sum_{j\in J} G_j(t)$ is continuous. Theorem 6.2 implies that $\mathscr{F}(\{C(t), m\}, \{G_j(t)\}_{j\in J}, \mathscr{T}_1)$ if and only if $\mathscr{F}(\{C(t), m\}, \{h_i(t)\, w\}_{1\leqq i\leqq 6}, \mathscr{T}_1)$. Denote

$$\mathfrak{M}_\nu = \{h_i(t)\, w_i\}_{1\leqq i\leqq \nu}\,; \quad \nu = 0, 1, \ldots, 6\,;$$

(in particular, $\mathfrak{M}_0$ denotes the empty system of vectors).

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[4]) If f is a real function, then

$$f^+ = \max(f, 0), \quad f^- = -\min(f, 0).$$

Axiom III.1 implies that if there exist points $B_\nu(t)$ such that $\mathscr{F}(\{B_\nu(t), m\}, \mathfrak{M}_\nu, \mathscr{T}_1)$, $\nu = 1, 2, \ldots, 6$; then there exist points $B_{\nu-1}(t)$ such that $\mathscr{F}(\{B_{\nu-1}(t), m\}, \mathfrak{M}_{\nu-1}, \mathscr{T}_1)$. Therefore we deduce by induction that there exist points

$$B_6(t) = C(t),\ B_5(t),\ B_4(t), \ldots, B_1(t),\ B_0(t)\ ; \quad t \in \mathscr{T}_1\ ;$$

such that $\mathscr{F}(\{B_\nu(t), m\}, \mathfrak{M}_\nu, \mathscr{T}_1)$; $\nu = 0, 1, \ldots, 6$. Theorem 5.1 now implies that there exist vectors $\boldsymbol{u}_\nu$, $\boldsymbol{v}_\nu$ and real functions s_ν such that

$$B_\nu(t) = B_{\nu-1}(t) + s_\nu(t)\, \boldsymbol{w}_\nu + t\boldsymbol{v}_\nu + \boldsymbol{u}_\nu\ ; \quad t \in \mathscr{T}_1\ ; \quad \nu = 1, \ldots, 6\ ;$$

where

$$s_\nu(t) = \frac{1}{m} \int_{t_0}^{t} \left(\int_{t_0}^{\sigma_2} t_\nu(\sigma_1)\, \mathrm{d}\sigma_1 \right) \mathrm{d}\sigma_2\ . \tag{7.3}$$

Therefore, we have

$$C(t) = B(t) + \sum_{i=1}^{6} s_i(t)\, \boldsymbol{w}_i + t\boldsymbol{v} + \boldsymbol{u}\ , \tag{7.4}$$

where $\boldsymbol{v} = \sum_{i=1}^{6} \boldsymbol{v}_i$, $\boldsymbol{u} = \sum_{i=1}^{6} \boldsymbol{u}_i$, $B(t) = B_0(t)$ and hence $\mathscr{F}(\{B(t), m\}, \mathfrak{M}_0, \mathscr{T}_1)$. Now, using (7.3) and (7.2), we obtain

$$\sum_{i=1}^{6} s_i(t)\, \boldsymbol{w}_i = \sum_{i=1}^{6} \frac{1}{m} \int_{t_0}^{t} \left(\int_{t_0}^{\sigma_2} h_i(\sigma_1)\, \mathrm{d}\sigma_1 \right) \mathrm{d}\sigma_2\, \boldsymbol{w}_i =$$

$$= \frac{1}{m} \int_{t_0}^{t} \left(\int_{t_0}^{\sigma_2} \sum_{i=1}^{6} h_i(\sigma_1)\, \boldsymbol{w}_i\, \mathrm{d}\sigma_1 \right) \mathrm{d}\sigma_2 = \frac{1}{m} \int_{t_0}^{t} \left(\int_{t_0}^{\sigma_2} \sum_{j \in J} \boldsymbol{G}_j(\sigma_1)\, \mathrm{d}\sigma_1 \right) \mathrm{d}\sigma_2\ .$$

Thus, (7.4) becomes (7.1).

8. NEWTON'S FIRST LAW

Axioms II.3, IV.1, IV.2, and IV.3 imply

Theorem 8.1. *If the motion of a material point $\{B(t), m\}$ in an open interval $\mathscr{T}_1 \subset \mathscr{T}$ can be interpreted by the operation of the empty system of forces, then there exists a point $Q \in E_3$ and a vector $\boldsymbol{v}$ such that*

$$B(t) = Q + t\boldsymbol{v} \quad \text{for} \quad t \in \mathscr{T}_1\ . \tag{8.1}$$

Proof. I. Let $\mathfrak{M}_0$ be the empty system of vectors and let $\mathscr{F}(\{B(t), m\}, \mathfrak{M}_0, \mathscr{T}_1)$. Choose a number $t_1 \in \mathscr{T}_1$ and a number $\tau > 0$ such that $t_1 + \tau \in \mathscr{T}_1$ and denote

$$\boldsymbol{v}_1 = B(t_1 + \tau) - B(t_1)\ . \tag{8.2}$$

If we now have $t_2 \in \mathscr{T}_1$, $t_2 + \tau \in \mathscr{T}_1$, denote

$$\boldsymbol{v}_2 = B(t_2 + \tau) - B(t_2)\ . \tag{8.3}$$

We will prove first of all $\boldsymbol{v}_1 = \boldsymbol{v}_2$.

Indeed, Axiom IV.1 implies $|v_1| = |v_2|$, i.e.

(8.4) $$v_1 \, . \, v_1 = v_2 \, . \, v_2 \, .$$

Let w be an arbitrary vector and define

$$C(t) = B(t) + tw \quad \text{for} \quad t \in \mathscr{T}_1 \, .$$

Since $\mathbf{A}_B(C; t_3, t_4, \tau) = 0$ provided $t_3 \in \mathscr{T}_1$, $\tau > 0$, $t_3 < t_4$, $t_4 + \tau \in \mathscr{T}_1$, Axiom II.3 implies $\mathscr{F}(\{C(t), m\}, \{0\}, \mathscr{T}_1)$. Now Axiom IV.2 implies $\mathscr{F}(\{C(t), m\}, \mathfrak{M}_0, \mathscr{T}_1)$ and Axiom IV.1 implies

$$\varrho(C(t_1 + \tau), C(t_1)) = \varrho(C(t_2 + \tau), C(t_2)) \, ,$$

i.e.

$$|B(t_1 + \tau) + (t_1 + \tau) \, w - B(t_1) - t_1 w| =$$
$$= |B(t_2 + \tau) + (t_2 + \tau) \, w - B(t_2) - t_2 w| \, .$$

Using (8.2) and (8.3) we obtain

$$|v_1 + \tau w| = |v_2 + \tau w|$$

or

$$(v_1 + \tau w)(v_1 + \tau w) = (v_2 + \tau w)(v_2 + \tau w) \, .$$

Therefore

(8.5) $$v_1 \, . \, v_1 + 2\tau \, v_1 \, . \, w = v_2 \, . \, v_2 + 2\tau \, v_2 \, . \, w \, .$$

As $\tau > 0$, (8.5) and (8.4) imply $v_1 w = v_2 w$ and because w has been an arbitrary vector, we obtain $v_1 = v_2$.

II. Let $t_0 \in \mathscr{T}_1$, $\tau > 0$, $t_0 + \tau \in \mathscr{T}_1$ and let

(8.6) $$B(t_0 + \tau) = B(t_0) + u \, .$$

Let n be a natural number, p an integer and let

$$t_0 + \frac{p}{2^n} \tau \in \mathscr{T}_1 \, .$$

Then

(8.7) $$B\left(t_0 + \frac{p}{2^n} \tau\right) = B(t_0) + \frac{p}{2^n} u \, .$$

Indeed, denote

$$B\left(t_0 + \frac{\tau}{2^n}\right) - B(t_0) = u_n \, .$$

Then we deduce from I that for every integer q such that

$$t_0 + \frac{q - 1}{2^n} \tau \in \mathscr{T}_1 \, , \quad t_0 + \frac{q}{2^n} \tau \in \mathscr{T}_1$$

the following identity holds:

$$B\left(t_0 + \frac{q}{2^n}\tau\right) - B\left(t_0 + \frac{q-1}{2^n}\tau\right) = \boldsymbol{u}_n\,. \tag{8.8}$$

If $p > 0$, we sum (8.8) from $q = 1$ to $q = p$ obtaining

$$B\left(t_0 + \frac{p}{2^n}\tau\right) - B(t_0) = p\boldsymbol{u}_n\,; \tag{8.9}$$

if $p < 0$, we sum (8.8) from $q = p + 1$ to $q = 0$ obtaining

$$B(t_0) - B\left(t_0 + \frac{p}{2^n}\tau\right) = |p|\,\boldsymbol{u}_n$$

and (8.9) holds as well. It is evident that (8.9) also holds for $p = 0$. If we put $p = 2^n$ in (8.9), we obtain with regard to (8.2) that $\boldsymbol{u} = 2^n\boldsymbol{u}_n$ and therefore (8.9) implies (8.7).

III. Let $t_0 \in \mathscr{T}_1$, $\tau > 0$, $t_0 + \tau \in \mathscr{T}_1$ and let (8.6) hold. If r is a real number such that $t_0 + r\tau \in \mathscr{T}_1$, then

$$B(t_0 + r\tau) = B(t_0) + r\boldsymbol{u}\,. \tag{8.10}$$

Indeed, if r is of the form $p/2^n$, (8.10) is implied by (8.7) and Axiom IV.3 implies that (8.10) is true for all r such that $t_0 + r\tau \in \mathscr{T}_1$.

IV. Put $Q = B(t_0) - (t_0/\tau)\,\boldsymbol{u}$, $\boldsymbol{v} = \boldsymbol{u}/\tau$. Then, if $t \in \mathscr{T}_1$, (8.10) implies

$$B(t) = B\left(t_0 + \frac{t - t_0}{\tau}\tau\right) = B(t_0) + \frac{t - t_0}{\tau}\boldsymbol{u} = Q + t\boldsymbol{v}$$

and therefore (8.1) is true.

If we now use all axioms of the groups I, II, III and IV and the supposition E, we deduce from Theorems 7.1 and 8.1:

Theorem 8.2. *Let us have an open interval $\mathscr{T}_1 \subset \mathscr{T}$, a real number $m > 0$ and points $C(t) \in E_3$ for all $t \in \mathscr{T}_1$. Suppose that the motion of the material point $\{C(t), m\}$ in $\mathscr{T}_1$ can be interpreted by the operation of a system of forces $\{\boldsymbol{G}_j(t)\}_{j\in J}$, where the set J is finite. Let the map $t \mapsto \sum_{j\in J} \boldsymbol{G}_j(t)$ be continuous on $\mathscr{T}_1$ and let $t_0 \in \mathscr{T}_1$. Then there exists a point $Q \in E_3$ and a vector $\boldsymbol{v} \in V_3$ such that*

$$C(t) = Q + t\boldsymbol{v} + \frac{1}{m}\int_{t_0}^{t}\left(\int_{t_0}^{\sigma_2}\sum_{j\in J}\boldsymbol{G}_j(\sigma_1)\,\mathrm{d}\sigma_1\right)\mathrm{d}\sigma_2 \quad \text{for all} \quad t \in \mathscr{T}_1\,. \tag{8.11}$$

The function C is therefore twice differentiable and

$$m\ddot{C}(t) = \sum_{j\in J}\boldsymbol{G}_j(t) \quad \text{for all} \quad t \in \mathscr{T}_1\,. \tag{8.12}$$

9. GRAVITATIONAL LAW. EQUATIONS OF THE MOTION OF PARTICLES

We shall already use all the axioms. As we have seen in Chap. 6, the axioms of the groups V and VI imply the supposition E.

From Axioms VI.3, V.2, V.3 and V.4 we deduce

Theorem 9.1. *If* α, β *are two different particles and* $t \in \mathscr{T}$, *then* $\mathsf{P}_\alpha(t) \neq \mathsf{P}_\beta(t)$. *There exists one and only one vector* $\boldsymbol{F}_{\alpha\beta}(t)$ *such that the particle* β *effects the particle* α *at the instant* t *by the force* $\boldsymbol{F}_{\alpha\beta}(t)$. *There exists a real number* $\varkappa > 0$ (*independent of* α, β, t) *such that*

$$\boldsymbol{F}_{\alpha\beta}(t) = \frac{\varkappa m_\alpha m_\beta}{|\Delta_{\mathsf{P}_\alpha}(\mathsf{P}_\beta, t)|^3}\, \Delta_{\mathsf{P}_\alpha}(\mathsf{P}_\beta; t) \tag{9.1}$$

or, if we have a cartesian reference system with

$$\mathsf{P}_\alpha(t) = [x_1, x_2, x_3]\,, \quad \mathsf{P}_\beta(t) = [y_1, y_2, y_3]: \tag{9.2}$$

$$\boldsymbol{F}_{\alpha\beta}(t) = \left(\varkappa m_\alpha m_\beta \frac{y_1 - x_1}{r^3}\,, \ \varkappa m_\alpha m_\beta \frac{y_2 - x_2}{r^3}\,, \ \varkappa m_\alpha m_\beta \frac{y_3 - x_3}{r^3}\right),$$

where

$$r = ((y_1 - x_1)^2 + (y_2 - r_2)^2 + (y_3 - x_3)^2)^{1/2}\,. \tag{9.3}$$

Now we can prove

Theorem 9.2. *There exists a natural number* n *such that the system includes exactly* n *particles* $\alpha_1, \ldots, \alpha_n$. *If in a cartesian reference system* $\mathscr{S}$,

$$\mathsf{P}_{\alpha_j}(t) = [x_{j1}(t), x_{j2}(t), x_{j3}(t)] \quad \textit{for} \quad t \in \mathscr{T}\,, \tag{9.4}$$

then

(a) *the functions* x_{jh} *are continuous and twice differentiable in* $\mathscr{T}$;

(b) *there exists a positive real number* $\varkappa$ *such that the system of differential equations*

$$\ddot{x}_{ih} = \varkappa \sum_{\substack{j=1 \\ j \neq i}}^{n} m_j \frac{x_{jh} - x_{ih}}{\left(\sum_{l=1}^{3} (x_{jl} - x_{il})^2\right)^{3/2}}\,, \qquad i = 1, 2, \ldots, n\,; \quad h = 1, 2, 3\,; \tag{9.5}$$

is fulfilled in $\mathscr{T}$.[5])

Proof. By Axioms VI.1 and VI.2, the set $\mathscr{M}$ of all particles contains n particles $\alpha_1, \ldots, \alpha_n$. By Theorem 9.1, for two different particles α_i, α_j and for every $t \in \mathscr{T}$ there exists one and only one vector $\boldsymbol{F}_{ij}(t)$ such that $\mathscr{G}(\alpha_i, \alpha_j, t, \boldsymbol{F}_{ij}(t))$. Axiom V.1 implies $\mathscr{F}(\{\mathsf{P}_{\alpha_i}(t), m\}, \{\boldsymbol{F}_j(t)\}_{1 \leq j \leq n, j \neq i}, \mathscr{T})$ for $i = 1, 2, \ldots, n$. By Axiom V.5, the func-

[5]) If $n = 1$, then the sum means the function identically equal to zero.

tions $t \mapsto \Delta_{\mathsf{P}_{\alpha_i}}(\mathsf{P}_{\alpha_j}; t)$ are continuous on $\mathscr{T}$, therefore, by (9.1), the functions $\boldsymbol{F}_{ij}$ are also continuous. Thus Theorem 8.2 implies

$$m_i \, \ddot{\mathsf{P}}_{\alpha_i}(t) = \sum_{\substack{j=1 \\ j \neq i}}^{n} \boldsymbol{F}_{ij}(t) \quad \text{for} \quad t \in \mathscr{T} \, ; \quad i = 1, 2, \ldots, n \, ;$$

which together with (9.2), (9.3) and (9.4) yields (9.5).

10. INDEPENDENCE OF THE SYSTEM OF AXIOMS

To prove the independence of a certain axiom (A) of all other axioms, we shall construct a model fulfilling all axioms except (A), but not fulfilling the axiom (A). However, we can imagine the following situation: If we have such a model (M1), we can construct another model (M2) fulfilling all axioms which has the same number of particles with the same masses, the same interval $\mathscr{T}$ as the set of time instants and the same functions P_α (i.e. the same trajectories of particles). Then both the models coincide in quantities which can be experimentally measured, differing only in notions which are fictive, i.e. in notions which are introduced only for the easier description of the system (e.g., the two models can differ in the interpretation of the relation $\mathscr{F}(\{B(t), m\}, \{\boldsymbol{F}_j(t)\}_{j \in J}, \mathscr{T}_1))$. Therefore, we introduce a stronger notion of the physical independence.

We will say that the axiom (A) is physically independent of all other axioms, if there exists a model (M1) fulfilling all axioms except (A), not fulfilling the axiom (A), and if every model (M2) which fulfils all axioms (including the axiom (A)), with the same number of particles as the model (M1), with the same masses of the corresponding particles and with the same interval $\mathscr{T}$ of time instants, has at least one function P_α different from the corresponding function P_α of the model (M1).

As we have proved that the functions P_α have to satisfy the system of equations (9.5) if the model fufils all axioms, it is sufficient for the proof of the physical independence of the axiom (A) to construct a model fulfilling all axioms except (A), in which the functions P_α do not satisfy the system (9.5).

11. INDEPENDENCE OF THE AXIOM I.1

Let us have an E_3 with the metric ϱ and choose a cartesian reference system $\mathscr{S}$ in it. If $\boldsymbol{U}, \boldsymbol{V}, \boldsymbol{W}$ are vectors in E_3, we shall define that the force $\boldsymbol{W}$ is the resultant of forces $\boldsymbol{U}, \boldsymbol{V}$ if and only if (4.1) holds.

Let us choose a natural number $n > 1$ and n real positive numbers $m_1, \ldots, m_n$. Our model will have n particles $\alpha_1, \ldots, \alpha_n$ with masses $m_1, \ldots, m_n$. Further, let us choose $6n$ real numbers (4.2) such that $a_{i3} = v_{i3} = 0$ for $i = 1, \ldots, n$ and that (4.3) holds; moreover, choose a real number $\varkappa > 0$ and a real number $\bar{t}$. Let us consider the system of differential equations

(11.1a)
$$m_i\ddot{x}_{i1} = \sum_{\substack{h=1\\h\neq i}}^{n} \varkappa m_i m_h \left(\frac{x_{h1} - x_{i1}}{\left(\sum_{l=1}^{3}(x_{hl} - x_{il})^2\right)^{3/2}} + \right.$$
$$\left. + \frac{(\dot{x}_{h2}-\dot{x}_{i2})\left(\sum_{l=1}^{3}(x_{hl}-x_{il})^2\right)^{3/2} - 3(x_{h2}-x_{i2})\left(\sum_{l=1}^{3}(x_{hl}-x_{il})^2\right)^{1/2}\sum_{l=1}^{3}(x_{hl}-x_{il})(\dot{x}_{hl}-\dot{x}_{il})}{\left(\sum_{l=1}^{3}(x_{hl}-x_{il})^2\right)^3}\right),$$

(11.1b)
$$m_i\ddot{x}_{i2} = \sum_{\substack{h=1\\h\neq i}}^{n} \varkappa m_i m_h \left(\frac{x_{h2} - x_{i2}}{\left(\sum_{l=1}^{3}(x_{hl} - x_{il})^2\right)^{3/2}} - \right.$$
$$\left. - \frac{(\dot{x}_{h1}-\dot{x}_{i1})\left(\sum_{l=1}^{3}(x_{hl}-x_{il})^2\right)^{3/2} - 3(x_{h1}-x_{i1})\left(\sum_{l=1}^{3}(x_{hl}-x_{il})^2\right)^{1/2}\sum_{l=1}^{3}(x_{hl}-x_{il})(\dot{x}_{hl}-\dot{x}_{il})}{\left(\sum_{l=1}^{3}(x_{hl}-x_{il})^2\right)^3)}\right),$$

(11.1c)
$$m_i\ddot{x}_{i3} = 0\,.$$

This system has a solution in an open interval $\mathscr{T}$ containing the number $\bar{t}$, and this solution is formed by functions $x_{1r}, \ldots, x_{nr}$; $r = 1, 2, 3$; fulfilling (4.5) and (4.6). We will choose the interval $\mathscr{T}$ as the set of time instants. We define the position of the particle α_i at an instant $t \in \mathscr{T}$ to be the point of E_3 whose coordinates in the reference system $\mathscr{S}$ are $[x_{i1}(t), x_{i2}(t), x_{i3}(t)]$.

If $\boldsymbol{F}$ is a vector whose coordinates in the system $\mathscr{S}$ are (f_1, f_2, f_3) and if $t \in \mathscr{T}$, we define that $\mathscr{G}(\alpha_i, \alpha_h, t, \boldsymbol{F})$ if and only if $i \neq h$ and (4.7) holds.

If $Y(t) = [y_1(t), y_2(t), y_3(t)]$, $\boldsymbol{F}_j(t) = (f_{j1}(t), f_{j2}(t), f_{j3}(t))$ in the reference system $\mathscr{S}$, we say that $\mathscr{F}(\{Y(t), m\}, \{\boldsymbol{F}_j(t)\}_{j\in J}, \mathscr{T}_1)$, $\mathscr{T}_1 \subset \mathscr{T}$ being an open interval, if and only if the following conditions are fulfilled:

(11.2a) the set J is finite;

(11.2b) the functions $\sum_{j\in J} f_{jl}$; $l = 1, 2, 3$; are continuous on $\mathscr{T}_1$;

(11.2c) the functions y_1, y_2, y_3 have continuous derivatives of the first order on $\mathscr{T}_1$;

(11.2d) there exists a number $t_0 \in \mathscr{T}_1$ and real numbers q_1, q_2, q_3 such that for all $t \in \mathscr{T}_1$ the following identities are valid:

$$\dot{y}_1(t) = q_1 + \frac{1}{m}\int_{t_0}^{t} \sum_{j\in J} f_{j1}(\sigma)\,\mathrm{d}\sigma + \frac{1}{m}\sum_{j\in J}(f_{j2}(t) - f_{j2}(t_0))\,,$$
$$\dot{y}_2(t) = q_2 + \frac{1}{m}\int_{t_0}^{t} \sum_{j\in J} f_{j2}(\sigma)\,\mathrm{d}\sigma - \frac{1}{m}\sum_{j\in J}(f_{j1}(t) - f_{j1}(t_0))\,,$$
$$\dot{y}_3(t) = q_3 + \frac{1}{m}\int_{t_0}^{t} \sum_{j\in J} f_{j3}(\sigma)\,\mathrm{d}\sigma\,.$$

Theorem 11.1. *The above described model fulfils all the axioms* I.1, ..., VI.3 *with the exception of Axiom* I.1. *The functions* P_α *do not generally satisfy the system* (9.5) *and therefore Axiom* I.1 *is physically independent of all the other axioms.*

Proof. We shall prove only the validity of Axioms I.2, II.3 and III.1. The proof of the other axioms can be left to the reader.

Axiom I.2. Suppose that $\mathscr{F}(\{B(t), m\}, \{\boldsymbol{F}_j(t)\}_{j\in J}, \mathscr{T}_1)$, $\mathscr{F}(\{C(t), m\}, \{\boldsymbol{F}_j(t)\}_{j\in J} \cup \cup \{\boldsymbol{G}(t)\}, \mathscr{T}_1)$ and let

$$B(t) = [y_1(t), y_2(t), y_3(t)]\,, \quad C(t) = [z_1(t), z_2(t), z_3(t)]\,,$$

$$\boldsymbol{G}(t) = g(t)\,\boldsymbol{u}\,, \quad \boldsymbol{u} = (u_1, u_2, u_3)\,, \quad |\boldsymbol{u}| = 1\,, \quad g(t) \geqq 0\,.$$

Then there exist numbers $t_0 \in \mathscr{T}_1$, $\tau_0 \in \mathscr{T}_1$ and real numbers $q_1, q_2, q_3, p_1, p_2, p_3$ such that (11.2) and

$$\dot{z}_1(t) = p_1 + \frac{1}{m}\int_{\tau_0}^{t} (g(\sigma)\,u_1 + \sum_{j\in J} f_{j1}(\sigma))\,\mathrm{d}\sigma +$$

$$+ \frac{1}{m}(g(t)\,u_2 - g(\tau_0)\,u_2 + \sum_{j\in J}(f_{j2}(t) - f_{j2}(\tau_0)))\,,$$

$$\dot{z}_2(t) = p_2 + \frac{1}{m}\int_{\tau_0}^{t} (g(\sigma)\,u_2 + \sum_{j\in J} f_{j2}(\sigma))\,\mathrm{d}\sigma -$$

$$- \frac{1}{m}(g(t)\,u_1 - g(\tau_0)\,u_1) + \sum_{j\in J}(f_{j1}(t) - f_{j1}(\tau_0)))\,,$$

$$\dot{z}_3(t) = p_3 + \frac{1}{m}\int_{\tau_0}^{t} (g(\sigma)\,u_3 + \sum_{j\in J} f_{j3}(\sigma))\,\mathrm{d}\sigma\,.$$

If we put

$$\boldsymbol{v} = \Bigg(p_1 - q_1 - \frac{1}{m}\int_{t_0}^{\tau_0} \sum_{j\in J} f_{j1}(\sigma)\,\mathrm{d}\sigma - \frac{1}{m}\sum_{j\in J}(f_{j2}(\tau_0) - f_{j2}(t_0))\,,$$

$$p_2 - q_2 - \frac{1}{m}\int_{t_0}^{\tau_0} \sum_{j\in J} f_{j2}(\sigma)\,\mathrm{d}\sigma + \frac{1}{m}\sum_{j\in J}(f_{j1}(\tau_0) - f_{j1}(t_0),$$

$$p_3 - q_3 - \frac{1}{m}\int_{t_0}^{\tau_0} \sum_{j\in J} f_{j3}(\sigma)\,\mathrm{d}\sigma\Bigg)\,,$$

$$\boldsymbol{w} = (u_2, -u_1, 0)\,,$$

we obtain

$$\dot{C}(t) - \dot{B}(t) = \boldsymbol{v} + \frac{1}{m}\int_{\tau_0}^{t} g(\sigma)\,\mathrm{d}\sigma\,\boldsymbol{u} + \frac{1}{m}(g(t) - g(\tau_0))\,\boldsymbol{w} \quad \text{for all} \quad t \in \mathscr{T}_1\,.$$

Hence by (2.1) we obtain

$$\Delta_B(C; t) = \Delta_B(C; \tau_0) + \boldsymbol{v}(t - \tau_0) + \frac{1}{m}\int_{\tau_0}^{t}\left(\int_{\tau_0}^{\sigma_2} g(\sigma_1)\,\mathrm{d}\sigma_1\right)\mathrm{d}\sigma_2\, \boldsymbol{u} + $$

$$+ \frac{1}{m}\int_{\tau_0}^{t}(g(\sigma) - g(\tau_0))\,\mathrm{d}\sigma\, \boldsymbol{w} \quad \text{for all} \quad t \in \mathscr{T}_1 .$$

If now $t \in \mathscr{T}_1$, $\tau > 0$, $t + \tau \in \mathscr{T}_1$, then (2.2) implies

$$\mathbf{V}_B(C; t, \tau) = \boldsymbol{v} + \frac{1}{m}\frac{1}{\tau}\int_{t}^{t+\tau}\left(\int_{\tau_0}^{\sigma_2} g(\sigma_1)\,\mathrm{d}\sigma_1\right)\mathrm{d}\sigma_2\, \boldsymbol{u} + \frac{1}{m}\frac{1}{\tau}\int_{t}^{t+\tau}(g(\sigma) - g(\tau_0))\,\mathrm{d}\sigma\, \boldsymbol{w} .$$

If $t_1 \in \mathscr{T}_1$, $\tau > 0$, $t_1 < t_2$, $t_2 + \tau \in \mathscr{T}_1$, then (2.3) with the above identity gives

(11.3) $$\mathbf{A}_B(C; t_1, t_2, \tau) =$$

$$= \boldsymbol{u}\frac{1}{m}\frac{1}{t_2 - t_1}\left(\frac{1}{\tau}\int_{t_2}^{t_2+\tau}\left(\int_{\tau_0}^{\sigma_2} g(\sigma_1)\,\mathrm{d}\sigma_1\right)\mathrm{d}\sigma_2 - \frac{1}{\tau}\int_{t_1}^{t_1+\tau}\left(\int_{\tau_0}^{\sigma_2} g(\sigma_1)\,\mathrm{d}\sigma_1\right)\mathrm{d}\sigma_2\right) +$$

$$+ \boldsymbol{w}\frac{1}{m}\frac{1}{t_2 - t_1}\left(\frac{1}{\tau}\int_{t_2}^{t_2+\tau}(g(\sigma) - g(\tau_0))\,\mathrm{d}\sigma - \frac{1}{\tau}\int_{t_1}^{t_1+\tau}(g(\sigma) - g(\tau_0))\,\mathrm{d}\sigma\right).$$

If there exists a real number $k \geqq 0$ such that $\mathbf{A}_B(C; t_1, t_2, \tau) = k\boldsymbol{u}$, then either the coefficient in (11.3) by $\boldsymbol{w}$ must be zero or $\boldsymbol{w}$ must be linearly dependent on $\boldsymbol{u}$. In the first case (11.3) becomes (4.12). In the latter case we have $\boldsymbol{u} = (0, 0, 1)$, $\boldsymbol{w} = (0, 0, 0)$ and (11.3) becomes (4.12) again. Therefore, in both cases

$$p_1 \leqq |\boldsymbol{G}(t)| \leqq p_2 \quad \text{for all} \quad t \in \langle t_1, t_2 + \tau\rangle$$

implies

$$p_1 \leqq m|\mathbf{A}_B(C; t_1, t_2, \tau)| \leqq p_2 .$$

as in Chap. 4. Consequently, Axiom I.2 is fulfilled.

Axiom II.3. Let $\mathscr{T}_1 \subset \mathscr{T}$ be an open interval and let

$$B(t) = [y_1(t), y_2(t), y_3(t)] \in E_3 , \quad C(t) = [z_1(t), z_2(t), z_3(t)] \in E_3 \quad \text{for} \quad t \in \mathscr{T}_1 .$$

Suppose that

(11.4) $$\mathscr{F}(\{B(t), m\}, \{\boldsymbol{F}_j(t)\}_{j\in J}, \mathscr{T}_1) .$$

Let $\boldsymbol{G} = (g_1, g_2, g_3)$ be a vector and let

$$m\, \mathbf{A}_B(C; t_1, t_2, \tau) = \boldsymbol{G} \quad \text{for every} \quad t_1 \in \mathscr{T}_1,\ t_1 < t_2,\ \tau > 0,\ t_2 + \tau \in \mathscr{T}_1 .$$

Let the function $t \mapsto \Delta_B(C; t)$ be continuous on $\mathscr{T}_1$. Then the function $t \mapsto |\Delta_B(C; t)|$ is bounded on every compact interval $K_1 \in \mathscr{T}_1$ and Theorem 3.2 thus implies that, if $t_0 \in \mathscr{T}_1$, there exists a vector $\boldsymbol{v} = (v_1, v_2, v_3)$ such that

$$\Delta_B(C; t) = \Delta_B(C; t_0) + (t - t_0)\,\boldsymbol{v} + \frac{1}{2m}(t - t_0)^2\,\boldsymbol{G} \quad \text{for all} \quad t \in \mathscr{T}_1\,,$$

or, by (2.1),

(11.5)

$$C(t) = B(t) + \Delta_B(C; t_0) + (t - t_0)\,\boldsymbol{v} + \frac{1}{2m}(t - t_0)^2\,\boldsymbol{G} \quad \text{for all} \quad t \in \mathscr{T}_1\,.$$

Now, (11.4) and (11.2c) imply the existence of $\dot{B}$ and therefore (11.5) implies

$$\dot{C}(t) = \dot{B}(t) + \boldsymbol{v} + \frac{1}{m}(t - t_0)\,\boldsymbol{G} \quad \text{for all} \quad t \in \mathscr{T}_1\,,$$

or, $\boldsymbol{G}$ being constant,

$$\text{(11.6)} \qquad \dot{C}(t) = \dot{B}(t) + \boldsymbol{v} + \frac{1}{m}\int_{t_0}^{t} \boldsymbol{G}\,\mathrm{d}\sigma \quad \text{for all} \quad t \in \mathscr{T}_1\,, \quad \text{if} \quad t_0 \in \mathscr{T}_1\,.$$

Therefore (11.2d) and (11.6) imply that there exists a number $t_0 \in \mathscr{T}_1$ and real numbers q_1, q_2, q_3 such that for all $t \in \mathscr{T}_1$,

$$\dot{z}_1(t) = (q_1 + v_1) + \frac{1}{m}\int_{t_0}^{t}(g_1 + \sum_{j\in J} f_{j1}(\sigma))\,\mathrm{d}\sigma + \frac{1}{m}(g_2 + \sum_{j\in J} f_{j2}(t) - g_2 - \sum_{j\in J} f_{j2}(t_0)),$$

$$\dot{z}_2(t) = (q_2 + v_2) + \frac{1}{m}\int_{t_0}^{t}(g_2 + \sum_{j\in J} f_{j2}(\sigma))\,\mathrm{d}\sigma - \frac{1}{m}(g_1 + \sum_{j\in J} f_{j1}(t) - g_1 - \sum_{j\in J} f_{j1}(t_0)),$$

$$\dot{z}_3(t) = (q_3 + v_3) + \frac{1}{m}\int_{t_0}^{t}(g_3 + \sum_{j\in J} f_{j3}(\sigma))\,\mathrm{d}\sigma\,.$$

Therefore, by (11.2), $\mathscr{F}(\{C(t), m\}, \{\boldsymbol{F}_j(t)\}_{j\in J} \cup \{\boldsymbol{G}\}, \mathscr{T}_1)$ holds. Axiom II.3 is thus verified.

Axiom III.1. Let $\mathscr{T}_1 \subset \mathscr{T}$ be an open interval and suppose that there exist (for all $t \in \mathscr{T}_1$) points $C(t)$ such that $\mathscr{F}(\{C(t), m\}, \{\boldsymbol{F}_j(t)\}_{j\in J} \cup \{\boldsymbol{G}(t)\}, \mathscr{T}_1)$. Then, by (11.2a), the set J is finite, and, by (11.2b), the map $t \mapsto \boldsymbol{G}(t) + \sum_{j\in J} \boldsymbol{F}_j(t)$ is continuous on $\mathscr{T}_1$. If $\boldsymbol{G}(t) = g(t)\,\boldsymbol{u}$, where $|\boldsymbol{u}| = 1$ and g is continuous on $\mathscr{T}_1$, then the map $t \mapsto \sum_{j\in J} \boldsymbol{F}_j(t)$ is continuous on $\mathscr{T}_1$ as well and we can define the functions y_1, y_2, y_3 by (11.2d). If we now put $B(t) = [y_1(t), y_2(t), y_3(t)]$, we easily see that $\mathscr{F}(\{B(t), m\}, \{\boldsymbol{F}_j(t)\}_{j\in J}, \mathscr{T}_1)$. Hence Axiom III.1 is fulfilled.

12. INDEPENDENCE OF AXIOM I.2

Let us have an E_3 with the metric ϱ and choose some cartesian reference system $\mathscr{S}$ in it. If $\boldsymbol{U}, \boldsymbol{V}, \boldsymbol{W}$ are vectors in E_3, we shall define that the force $\boldsymbol{W}$ is the resultant of forces $\boldsymbol{U}, \boldsymbol{V}$ if and only if (4.1) holds.

Let us choose a natural number $n > 1$ and n real positive numbers $m_1, \ldots, m_n$. Our model will have n particles $\alpha_1, \ldots, \alpha_n$ with masses $m_1, \ldots, m_n$. Further, let us choose $6n$ real numbers (4.2) such that (4.3) holds; moreover, choose a real number $\varkappa > 0$ and a real number $\bar{t}$. Let us consider the system of differential equations

$$(12.1) \qquad m_i \ddot{x}_{ir} = \sum_{\substack{h=1 \\ h \neq i}}^{n} \varkappa m_i m_h \left(\frac{x_{hr} - x_{ir} + \dot{x}_{hr} - \dot{x}_{ir}}{\left(\sum_{l=1}^{3} (x_{hl} - x_{il})^2\right)^{3/2}} - \right.$$

$$\left. - \frac{3(x_{hr} - x_{ir}) \sum_{l=1}^{3} (x_{hl} - x_{il})(\dot{x}_{hl} - \dot{x}_{il})}{\left(\sum_{l=1}^{3} (x_{hl} - x_{il})^2\right)^{5/2}} \right); \quad i = 1, \ldots, n \, ; \quad r = 1, 2, 3 \, .$$

This system has a solution in an open interval $\mathcal{T}$ containing the number $\bar{t}$, and this solution is formed by functions $x_{1r}, \ldots, x_{nr}$; $r = 1, 2, 3$; fulfilling (4.5) and (4.6). We will choose the interval $\mathcal{T}$ as the set of time instants. We define the position of the particle α_i at an instant $t \in \mathcal{T}$ to be the point of E_3 whose coordinates in the reference system $\mathcal{S}$ are $[x_{i1}(t), x_{i2}(t), x_{i3}(t)]$.

If $\boldsymbol{F}$ is a vector whose coordinates in the system $\mathcal{S}$ are (f_1, f_2, f_3) and if $t \in \mathcal{T}$, we define that $\mathcal{G}(\alpha_i, \alpha_h, t, \boldsymbol{F})$ if and only if $i \neq h$ and (4.7) holds.

If $Y(t) = [y_1(t), y_2(t), y_3(t)]$, $\boldsymbol{F}_j(t) = (f_{j1}(t), f_{j2}(t), f_{j3}(t))$ in the reference system $\mathcal{S}$, we will say that $\mathcal{F}(\{Y(t), m\}, \{\boldsymbol{F}_j(t)\}_{j \in J}, \mathcal{T}_1)$, $\mathcal{T}_1 \subset \mathcal{T}$ being an open interval, if and only if the following conditions are fulfilled:

(12.2a) the set J is finite;

(12.2b) the functions $\sum_{j \in J} f_{jl}$; $l = 1, 2, 3$; are continuous on $\mathcal{T}_1$;

(12.2c) the functions y_1, y_2, y_3 have continuous derivatives of the first order on $\mathcal{T}_1$;

(12.2d) there exists a number $t_0 \in \mathcal{T}_1$ and real numbers q_1, q_2, q_3 such that for all $t \in \mathcal{T}_1$,

$$\dot{y}_r(t) = q_r + \frac{1}{m} \int_{t_0}^{t} \sum_{j \in J} f_{jr}(\sigma) \, d\sigma + \frac{1}{m} \sum_{j \in J} (f_{jr}(t) - f_{jr}(t_0)) \, ; \quad r = 1, 2, 3 \, .$$

Theorem 12.1. *The above described model fulfils all the axioms* I.1, …, VI.3 *with the exception of Axiom* I.2. *The functions* P_α *do not generally satisfy the system* (9.5) *and therefore Axiom* I.2 *is physically independent of all the other axioms.*

Proof is analogous to that of Theorem 11.1 and we leave it to the reader.

Let us have an E_3 and a cartesian reference system $\mathscr{S}$ in it. If $\boldsymbol{U}, \boldsymbol{V}, \boldsymbol{W}$ are vectors in E_3, we shall define that $\boldsymbol{W} = \boldsymbol{U} \oplus \boldsymbol{V}$ if and only if (4.1) holds.

Our model will have 2 particles α_1, α_2, the mass of each particle will be 1. The set of time instants will be an open interval $\mathscr{T}$ containing the number 0 in which there exists a solution of the system of differential equations

$$(13.1) \qquad \ddot{x}_1 = (x_2 - x_1)^{-2}, \quad \ddot{x}_2 = -(x_2 - x_1)^{-2}$$

satisfying the initial conditions $x_1(0) = -1$, $x_2(0) = 1$ and such that

$$(13.2) \qquad x_1(t) \neq x_2(t) \quad \text{for} \quad t \in \mathscr{T}.$$

Let λ be a discontinuous real function such that

$$(13.3) \qquad \lambda(t_1 + t_2) = \lambda(t_1) + \lambda(t_2) \quad \text{for all real} \quad t_1, t_2$$

(see [11]). We set (in the reference system $\mathscr{S}$)

$$(13.4) \qquad \mathsf{P}_{\alpha_i}(t) = [x_i(t) + \lambda(t), 0, 0] \quad \text{for} \quad i = 1, 2; \quad t \in \mathscr{T}.$$

We say that $\mathscr{G}(\alpha_i, \alpha_h, t, \boldsymbol{F})$ if and only if $i \neq h$ and

$$(13.5) \qquad \boldsymbol{F} = \left(\frac{x_h(t) - x_i(t)}{|x_h(t) - x_i(t)|^3}, 0, 0 \right) \quad \text{(in the reference system } \mathscr{S}\text{)}.$$

If $Y(t) = [y_1(t), y_2(t), y_3(t)]$, $\boldsymbol{F}_j(t) = (f_{j1}(t), f_{j2}(t), f_{j3}(t))$ in the reference system $\mathscr{S}$, we say that $\mathscr{F}(\{Y(t), m\}, \{\boldsymbol{F}_j(t)\}_{j \in J}, \mathscr{T}_1)$, $\mathscr{T}_1 \subset \mathscr{T}$ being an open interval, if and only if the following conditions are fulfilled:

(13.6a) the set J is finite;

(13.6b) the functions $\sum_{j \in J} f_{jl}$; $l = 1, 2, 3$; are continuous on $\mathscr{T}_1$;

(13.6c) there exists a number $t_0 \in \mathscr{T}_1$ and real numbers $a_1, a_2, a_3, q_1, q_2, q_3$ such that for all $t \in \mathscr{T}_1$,

$$y_l(t) = a_l + q_l t + \varepsilon_l \lambda(t) + \frac{1}{m} \int_{t_0}^{t} \left(\int_{t_0}^{\sigma_1} \sum_{j \in J} f_{jl}(\sigma_2) \, d\sigma_2 \right) d\sigma_1; \quad l = 1, 2, 3;$$

where

$$\varepsilon_l = 0 \quad \text{if} \quad \sum_{j \in J} f_{jl} \quad \text{is constant on} \quad \mathscr{T}_1,$$

$$\varepsilon_l = 1 \quad \text{if} \quad \sum_{j \in J} f_{jl} \quad \text{is not constant on} \quad \mathscr{T}_1.$$

Theorem 13.1. *The above described model fulfils all the axioms* I.1, ..., VI.3 *with the exception of Axiom* I.3. *As the functions* P_α *do not satisfy the system* (9.5), *Axiom* I.3 *is physically independent of all the other axioms.*

Proof. Axioms I.1 and I.2. Suppose that (4.9), (4.10) and (4.11) hold, where

(13.7) $$B(t) = [y_1(t), y_2(t), y_3(t)]\,, \quad C(t) = [z_1(t), z_2(t), z_3(t)]\,,$$

(13.8) $$\boldsymbol{F}_j(t) = (f_{j1}(t), f_{j2}(t), f_{j3}(t))\,, \quad \boldsymbol{u} = (u_1, u_2, u_3)\,.$$

Then the set J is finite, the functions $\sum_{j\in J} f_{jl}$ and $gu_l + \sum_{j\in J} f_{jl}$ are continuous and we have (13.6c) and

(13.9)
$$z_l(t) = \bar{a}_l + \bar{q}_l t + \bar{\varepsilon}_l\,\lambda(t) + \frac{1}{m}\int_{\tau_0}^{t}\left(\int_{\tau_0}^{\sigma_1}(g(\sigma_2)\,u_l + \sum_{j\in J} f_{jl}(\sigma_2)\,\mathrm{d}\sigma_3)\right)\mathrm{d}\sigma_1$$

(where $\tau_0 \in \mathscr{T}_1$, $\bar{a}_l$, $\bar{q}_l$ are real numbers and either $\bar{\varepsilon}_l = 0$ or $\bar{\varepsilon}_l = 1$) for $l = 1, 2, 3$ and all $t \in \mathscr{T}_1$. From (13.9) and (13.6c) we deduce that

$$z_l(t) - y_l(t) = \tilde{a}_l + \tilde{q}_l t + \tilde{\varepsilon}_l\,\lambda(t) + \frac{1}{m}\int_{t_0}^{t}\left(\int_{t_0}^{\sigma_1} g(\sigma_2)\,\mathrm{d}\sigma_2\right)\mathrm{d}\sigma_1\, u_l\,,$$

where

$$\tilde{a}_l = \bar{a}_l - a_l + \frac{1}{m}\int_{\tau_0}^{t_0}\left(\int_{\tau_0}^{\sigma_1}(g(\sigma_2)\,u_l + \sum_{j\in J} f_{jl}(\sigma_2))\,\mathrm{d}\sigma_2\right)\mathrm{d}\sigma_2 -$$
$$- \frac{\tau_0}{m}\int_{\tau_0}^{t_0}(g(\sigma_2)\,u_l + \sum_{j\in J} f_{jl}(\sigma_2)\,\mathrm{d}\sigma_2\,,$$
$$\tilde{q}_l = \bar{q}_l - q_l + \frac{1}{m}\int_{\tau_0}^{t_0}(g(\sigma_2)\,u_l + \sum_{j\in J} f_{jl}(\sigma_2))\,\mathrm{d}\sigma_2\,,$$
$$\tilde{\varepsilon}_l = \bar{\varepsilon}_l - \varepsilon_l\,.$$

If we put

$$\tilde{A} = (\tilde{a}_1, \tilde{a}_2, \tilde{a}_3)\,, \quad \tilde{\boldsymbol{Q}} = (\tilde{q}_1, \tilde{q}_2, \tilde{q}_3)\,, \quad \tilde{\boldsymbol{E}} = (\tilde{\varepsilon}_1, \tilde{\varepsilon}_2, \tilde{\varepsilon}_3)\,,$$

we obtain by (2.1) for all $t \in \mathscr{T}_1$:

$$\Delta_B(C; t) = \tilde{A} + \tilde{\boldsymbol{Q}}t + \tilde{\boldsymbol{E}}\,\lambda(t) + \frac{1}{m}\int_{t_0}^{t}\left(\int_{t_0}^{\sigma_1} g(\sigma_2)\,\mathrm{d}\tau_2\right)\mathrm{d}\sigma_1\,\boldsymbol{u}\,.$$

By applying (2.2) and (13.3) we now obtain (if $t \in \mathscr{T}_1$, $\tau > 0$, $t + \tau \in \mathscr{T}_1$)

$$\mathbf{V}_B(C; t, \tau) = \tilde{\boldsymbol{Q}} + \tilde{\boldsymbol{E}}\frac{\lambda(t+\tau) - \lambda(t)}{\tau} + \frac{1}{m\tau}\left(\int_{t_0}^{t+\tau}\left(\int_{t_0}^{\sigma_1} g(\sigma_2)\,\mathrm{d}\sigma_2\right)\mathrm{d}\sigma_1 -\right.$$
$$\left.- \int_{t_0}^{t}\left(\int_{t_0}^{\sigma_1} g(\sigma_2)\,\mathrm{d}\sigma_2\right)\mathrm{d}\sigma_1\right)\boldsymbol{u} = \tilde{\boldsymbol{Q}} + \frac{\lambda(\tau)}{\tau}\tilde{\boldsymbol{E}} + \frac{1}{m\tau}\int_{t}^{t+\tau}\left(\int_{t_0}^{\sigma_1} g(\sigma_2)\,\mathrm{d}\sigma_2\right)\mathrm{d}\sigma_1\,\boldsymbol{u}$$

and by (2.3), provided $t_1 \in \mathscr{T}_1$, $t_1 < t_2$, $t_2 + \tau \in \mathscr{T}_1$, we have

$$\mathbf{A}_B(C; t_1, t_2, \tau) = \frac{1}{m}\frac{1}{t_2 - t_1}\left\{\frac{1}{\tau}\int_{t_2}^{t_2+\tau}\left(\int_{t_0}^{\sigma_1} g(\sigma_2)\,\mathrm{d}\sigma_2\right)\mathrm{d}\sigma_1 -\right.$$
$$\left.- \frac{1}{\tau}\int_{t_1}^{t_1+\tau}\left(\int_{t_0}^{\sigma_1} g(\sigma_2)\,\mathrm{d}\sigma_2\right)\mathrm{d}\sigma_1\,\boldsymbol{u}\,.\right.$$

Hence we easily obtain (4.12) and we can verify Axioms I.1 and I.2 as in Chap. 4.

Axiom II.3. Let $\mathscr{T}_1 \subset \mathscr{T}$ be an open interval and let us have (13.7) for all $t \in \mathscr{T}_1$. Let $\boldsymbol{G} = (g_1, g_2, g_3)$ be a vector and let

$$m\, \mathbf{A}_B(C; t_1, t_2, \tau) = \boldsymbol{G} \quad \text{for every} \quad t_1 \in \mathscr{T}_1,\ t_1 < t_2,\ \tau > 0,\ t_2 + \tau \in \mathscr{T}_1\,.$$

If the function $t \mapsto \Delta_B(C; t)$ is continuous on $\mathscr{T}_1$, we obtain as in Chap. 11 that, if $t_0 \in \mathscr{T}_1$, there exists a vector $\boldsymbol{v} = (v_1, v_2, v_3)$ such that (11.5) holds. The vector $\boldsymbol{G}$ being constant, we can write (11.5) with regard to (13.7) in the form

(13.10)

$$z_l(t) = y_l(t) + (z_l(t_0) - y_l(t_0) - v_l t_0) + v_l t + \frac{1}{m}\int_{t_0}^{t}\left(\int_{t_0}^{\sigma_1} g_l \,\mathrm{d}\sigma_2\right)\mathrm{d}\sigma_1\ ;\ l = 1, 2, 3\,.$$

If now $\mathscr{F}(\{B(t), m\}, \{\boldsymbol{F}_j(t)\}_{j\in J}, \mathscr{T}_1)$ with $\boldsymbol{F}_j(t) = (f_{j1}(t), f_{j2}(t), f_{j3}(t))$, then (13.6a, b, c) holds. From (13.10) and (13.6c) we obtain

$$(13.11) \qquad z_l(t) = (a_l + z_l(t_0) - y_l(t_0) - v_l t_0) + (q_l + v_l)\, t + \varepsilon_l\, \lambda(t) +$$
$$+ \frac{1}{m}\int_{t_0}^{t}\left(\int_{t_0}^{\sigma_1} (g_l + \sum_{j\in J} f_{jl}(\sigma_2))\,\mathrm{d}\sigma_2\right)\mathrm{d}\sigma_1 \quad \text{for all} \quad t \in \mathscr{T}_1\ ; \quad l = 1, 2, 3\,.$$

The functions $t \mapsto \sum_{j\in J} f_{jl}(t)$ and $t \mapsto g_l + \sum_{j\in J} f_{jl}(t)$ are either both constant or both non-constant in $\mathscr{T}_1$ and therefore, if we compare (13.11) and (13.6), we obtain $\mathscr{F}(\{C(t), m\}, \{\boldsymbol{F}_j(t)\}_{j\in J} \cup \{\boldsymbol{G}\}, \mathscr{T}_1)$. Axiom II.3 is thus verified

We leave the verification of the other axioms to the reader.

14. INDEPENDENCE OF AXIOM II.1

Let us have an E_3 and a cartesian reference system $\mathscr{S}$ in it. If $\boldsymbol{U}, \boldsymbol{V}, \boldsymbol{W}$ are vectors in E_3, we shall define that $\boldsymbol{W} = \boldsymbol{U} \oplus \boldsymbol{V}$ if and only if (4.1) holds.

Our model will have 3 particles $\alpha_1, \alpha_2, \alpha_3$, the mass of each particle will be 1. The set $\mathscr{T}$ of time instants will be the set of all real numbers. The positions of the particles will have in the system $\mathscr{S}$ the coordinates

$$(14.1) \qquad \mathsf{P}_{\alpha_1}(t) = [0, 0, 0]\,, \quad \mathsf{P}_{\alpha_2}(t) = [1, t, 0]\,, \quad \mathsf{P}_{\alpha_3}(t) = [-1, 2t, 0]\,.$$

Define the vectors

$$(14.2) \quad \boldsymbol{G}(\alpha_1, \alpha_2, t) = -\boldsymbol{G}(\alpha_2, \alpha_1, t) = ((1 + t^2)^{-3/2},\ t(1 + t^2)^{-3/2}, 0)\,,$$

$$(14.2b) \quad \boldsymbol{G}(\alpha_1, \alpha_3, t) = -\boldsymbol{G}(\alpha_3, \alpha_1, t) = (-(1 + 4t^2)^{-3/2},\ 2t(1 + 4t^2)^{-3/2}, 0)\,,$$

$$(14.2c) \quad \boldsymbol{G}(\alpha_2, \alpha_3, t) = -\boldsymbol{G}(\alpha_3, \alpha_2, t) = (-2(4 + t^2)^{-3/2},\ t(4 + t^2)^{-3/2}, 0)\,.$$

We will say that $\mathscr{G}(\alpha_i, \alpha_j, t, \boldsymbol{F})$ if and only if $i \neq j$ and $\boldsymbol{F} = \boldsymbol{G}(\alpha_i, \alpha_j, t)$.

Let $\mathscr{T}_1 \subset \mathscr{T}$ be an open interval and for $t \in \mathscr{T}_1$ let us have material points

$\{Y(t), m\}$ and a system of vectors $\{F_j(t)\}_{j \in J}$. Let $J = J_1 \cup J_2$. Suppose that the vectors $F_j(t)$ have a constant direction for $t \in \mathscr{T}_1$ if $j \in J_1$ and that $F_j(t)$ have not a constant direction for $t \in \mathscr{T}_1$ if $j \in J_2$. Then we shall define that $\mathscr{F}(\{Y(t), m\}, \{F_j(t)\}_{j \in J}, \mathscr{T}_1)$ if and only if the following conditions are fulfilled:

(14.3a) the set J is finite;

(14.3b) the functions $t \mapsto F_j(t)$ are continuous on $\mathscr{T}_1$;

(14.3c) the function $t \mapsto Y(t)$ has a continuous derivative of the second order on $\mathscr{T}_1$;

(14.3d) $$m\ddot{Y} = \sum_{j \in J_1} F_j \quad \text{on} \quad \mathscr{T}_1 .$$

Theorem 14.1. *The above described model fulfils all the axioms* I.1, ..., VI.3 *with the exception of Axiom* II.1. *As the functions* P_α *do not satisfy the system* (9.5), *Axiom* II.1 *is physically independent of all the other axioms.*

Proof. The relations $W = U \oplus V$ and $\mathscr{G}(\alpha_i, \alpha_j, t, F)$ are defined in the same way as in Chap. 4. The relation $\mathscr{F}(\{Y(t), m\}, \{F_j(t)\}_{j \in J}, \mathscr{T}_1)$ is defined analogously as in Chap. 4, only vectors with non-constant directions are considered as null vectors in Chap. 14. All axioms with the exception of Axioms II.1 and V.1 can be therefore verified as in Chap. 4. Moreover, Axiom V.1 is implied by the fact that the forces $G(\alpha_i, \alpha_j, t)$ have not constant directions in $\mathscr{T}_1$ and that $\ddot{\mathsf{P}}_{\alpha_i}(t) = 0$ in $\mathscr{T}_1$ for $i = 1, 2, 3$.

15. INDEPENDENCE OF AXIOM II.2

Let us have an E_3 and a cartesian reference system $\mathscr{S}$ in it. We shall define that the relation $W = U \oplus V$ is fulfilled for no three vectors W, U, V.

Our model will have 3 particles $\alpha_1, \alpha_2, \alpha_3$, the mass of each particle will be 1. The set $\mathscr{T}$ of time instants will be the set of all real numbers. The position of the particles will be defined by (14.1). The relation $\mathscr{G}(\alpha_i, \alpha_j, t, F)$ will be defined as in Chap. 14. The relation $\mathscr{F}(\{Y(t), m\}, \{F_j(t)\}_{j \in J}, \mathscr{T}_1)$ will be defined by (14.3) as in Chap. 14.

Theorem 15.1. *The above described model fulfils all the axioms* I.1, ..., VI.3 *with the exception of Axiom* II.2. *As the functions* P_α *do not satisfy the system* (9.5), *Axiom* II.2 *is physically independent of all the other axioms.*

Proof. Axiom II.1 is fulfilled trivially, Axiom II.2 is not fulfilled. The other axioms can be proved as in Chap. 14, because in these axioms the relation $W = U \oplus V$ does not occur.

16. INDEPENDENCE OF AXIOM II.3

Let us have an E_3 and a cartesian reference system $\mathscr{S}$ in it.

Our model will have 3 particles $\alpha_1, \alpha_2, \alpha_3$, the mass of each particle will be 1. The set $\mathscr{T}$ of time instants will be the set of all real numbers. The position of the particles will be defined by (14.1). The relation $\mathscr{G}(\alpha_i, \alpha_j, t, \boldsymbol{F})$ will be defined as in Chap. 14.

The relation $\boldsymbol{W} = \boldsymbol{U} \oplus \boldsymbol{V}$ will be defined in a rather more complicated way. First of all, denote by $\mathfrak{M}$ the set of all non ordered pairs $(\boldsymbol{U}_1, \boldsymbol{U}_2)$, where $\boldsymbol{U}_1, \boldsymbol{U}_2$ are unit vectors of E_3 (the case $\boldsymbol{U}_1 = \boldsymbol{U}_2$ is not excluded). Then denote

(16.1) $$\boldsymbol{u}(\alpha_i, \alpha_j, t) = \boldsymbol{G}(\alpha_i, \alpha_j, t)/|\boldsymbol{G}(\alpha_i, \alpha_j, t)| ,$$

where $i, j = 1, 2, 3$; $i \neq j$; and $\boldsymbol{G}(\alpha_i, \alpha_j, t)$ are defiued by (14.2). Then define the set of non ordered pairs of unit vectors

(16.2a) $$\mathfrak{M}_1 = \{\boldsymbol{u}(\alpha_1, \alpha_2, t), \ \boldsymbol{u}(\alpha_1, \alpha_3, t)\}_{t \in \mathscr{T}} ,$$

(16.2b) $$\mathfrak{M}_2 = \{\boldsymbol{u}(\alpha_2, \alpha_1, t), \ \boldsymbol{u}(\alpha_2, \alpha_3, t)\}_{t \in \mathscr{T}} ,$$

(16.2c) $$\mathfrak{M}_3 = \{\boldsymbol{u}(\alpha_3, \alpha_1, t), \ \boldsymbol{u}(\alpha_3, \alpha_2, t)\}_{t \in \mathscr{T}} ,$$

(16.2d) $$\mathfrak{M}_4 = \mathfrak{M} \setminus (\mathfrak{M}_1 \cup \mathfrak{M}_2 \cup \mathfrak{M}_3) .$$

The sets $\mathfrak{M}_1, \mathfrak{M}_2, \mathfrak{M}_3, \mathfrak{M}_4$ are pairewise disjoint, each of them contains a continuum of elements and

(16.3) $$\mathfrak{M} = \bigcup_{i=1}^{4} \mathfrak{M}_i .$$

Then, denote by $\mathfrak{N}$ the set of all unit vectors of E_3 and by $\mathfrak{N}_0$ the set

(16.4) $$\begin{aligned}\mathfrak{N}_0 = & \{\boldsymbol{u}(\alpha_1, \alpha_2, t)\}_{t \in \mathscr{T}} \cup \{\boldsymbol{u}(\alpha_1, \alpha_3, t)\}_{t \in \mathscr{T}} \cup \\ & \cup \{\boldsymbol{u}(\alpha_2, \alpha_1, t)\}_{t \in \mathscr{T}} \cup \{\boldsymbol{u}(\alpha_2, \alpha_3, t)\}_{t \in \mathscr{T}} \cup \\ & \cup \{\boldsymbol{u}(\alpha_3, \alpha_1, t\}_{t \in \mathscr{T}} \cup \{\boldsymbol{u}(\alpha_3, \alpha_2, t)\}_{t \in \mathscr{T}} .\end{aligned}$$

Now, choose pairewise disjoint sets $\mathfrak{N}_1, \mathfrak{N}_2, \mathfrak{N}_3, \mathfrak{N}_4, \mathfrak{N}_5$, each of them consisting of a continuum of unit vectors, such that

(16.5) $$\mathfrak{N} \setminus \mathfrak{N}_0 = \bigcup_{i=1}^{5} \mathfrak{N}_i .$$

Then define one-to-one maps

(16.6) $$\varphi_i : \mathfrak{M}_i \to \mathfrak{N}_i ; \quad i = 1, 2, 3, 4 ;$$

(16.7) $$\psi \ : \mathfrak{N} \ \to \mathfrak{N}_5 .$$

Now, if $\boldsymbol{U}, \boldsymbol{V}$ are vectors of E_3, we will define a vector $\boldsymbol{W}$ satisfying the relation $\boldsymbol{W} = \boldsymbol{U} \oplus \boldsymbol{V}$. The vector $\boldsymbol{W}$ will be defined in the following way:

(16.8a) if $\boldsymbol{U} = \boldsymbol{V} = \boldsymbol{0}$, then $\boldsymbol{W} = \boldsymbol{0}$;

(16.8b) if $\boldsymbol{U} = \boldsymbol{0}$, $\boldsymbol{V} \neq \boldsymbol{0}$, then $\boldsymbol{W} = |\boldsymbol{V}| \cdot \psi(\boldsymbol{V}/|\boldsymbol{V}|)$;

(16.8c) if $\boldsymbol{U} \neq \boldsymbol{0}$, $\boldsymbol{V} = \boldsymbol{0}$, then $\boldsymbol{W} = |\boldsymbol{U}| \cdot \psi(\boldsymbol{U}/|\boldsymbol{U}|)$;

(16.8d) if $\boldsymbol{U} \neq \boldsymbol{0}$, $\boldsymbol{V} \neq \boldsymbol{0}$, $(\boldsymbol{U}, \boldsymbol{V}) \in \mathfrak{M}_\iota$ then

$$\boldsymbol{W} = (|\boldsymbol{U}| + |\boldsymbol{V}|) \cdot \varphi_i(\boldsymbol{U}/|\boldsymbol{U}|, \boldsymbol{V}/|\boldsymbol{V}|) .$$

The relation $W = U \oplus V$ has the following properties (we denote by dir v a unit vector u such that $v = |v| . u$):

(16.9a) *To each pair of vectors U, V there exists one and only one vector W such that $W = U \oplus V$.*

(16.9b) *If W is a vector, then there exist vectors U, V such that $W = U \oplus V$ if and only if $W = 0$ or* $\text{dir}\, W \in \mathfrak{N} \setminus \mathfrak{N}_0$.

(16.9c) *Let $W = U_1 \oplus V_1$ and $W = U_2 \oplus V_2$. Let $U_1 \neq 0$, $V_1 \neq 0$. Then also $U_2 \neq 0$, $V_2 \neq 0$. Moreover,*

$$\text{dir}\, U_1 = \text{dir}\, U_2\,, \quad \text{dir}\, V_1 = \text{dir}\, V_2$$

or

$$\text{dir}\, U_1 = \text{dir}\, V_2\,, \quad \text{dir}\, V_1 = \text{dir}\, U_2\,.$$

Indeed, if $U_1 \neq 0$, $V_1 \neq 0$, then $\{\text{dir}\, U_1, \text{dir}\, V_1\} \in \mathfrak{M}_i$, $1 \leqq i \leqq 4$, therefore, by (16.8d) and (16.6), $W \neq 0$ and $\text{dir}\, W \in \mathfrak{N}_i$. On the other hand, if $U_2 = 0$ or $V_2 = 0$, then, by (16.8a, b, c) and (16.7), $W = 0$ or $\text{dir}\, W \in \mathfrak{N}_5$. But $\mathfrak{N}_i \cap \mathfrak{N}_5 = \emptyset$ for $i = 1, 2, 3, 4$. As φ_i are one-to-one maps $\mathfrak{M}_i \to \mathfrak{N}_i$, we obtain the second assertion of (16.9c).

(16.9d) *Let $\mathcal{T}_1 \subset \mathcal{T}$ be an open interval. For every $t \in \mathcal{T}_1$ let us have vectors $H(t), U(t), V(t)$. Let $H(t) = U(t) \oplus V(t)$ and $H(t) = G(\alpha_i, \alpha_j, t) \oplus G(\alpha_i, \alpha_k, t)$ for all $t \in \mathcal{T}_1$, where i, j, k is a permutation of the numbers* 1, 2, 3. *Then neither* dir $U(t)$ *nor* dir $V(t)$ *are constant in $\mathcal{T}_1$.*

Indeed, by (14.2), $G(\alpha_i, \alpha_j, t) \neq 0$, $G(\alpha_i, \alpha_k, t) \neq 0$ for all $t \in \mathcal{T}_1$, therefore, by (16.9c), $U(t) \neq 0$, $V(t) \neq 0$ for all $t \in \mathcal{T}_1$ as well. (14.2) implies also that there exist numbers $t_1 \in \mathcal{T}_1$, $t_2 \in \mathcal{T}_1$ such that $\text{dir}\, G(\alpha_i, \alpha_j, t_1) \neq \text{dir}\, G(\alpha_i, \alpha_k, t_2)$. It is also $\text{dir}\, G(\alpha_i, \alpha_j, t_1) \neq \text{dir}\, G(\alpha_i, \alpha_j, t_2)$, $\text{dir}\, G(\alpha_i, \alpha_k, t_2) \neq \text{dir}\, G\alpha_i, \alpha_k, t_2)$. From (16.9c) we now obtain that $U(t_1) \neq U(t_2)$ and $V(t_1) \neq V(t_2)$.

(16.9e) *Let $\mathcal{T}_1 \subset \mathcal{T}$ be an open interval. For every $t \in \mathcal{T}_1$ let us have vectors $H(t)$ such that $H(t) = G(\alpha_i, \alpha_j, t) \oplus G(\alpha_i, \alpha_k, t)$, where i, j, k is a permutation of the numbers* 1, 2, 3. *Then* dir $H(t)$ *is not constant in $\mathcal{T}_1$.*

This is implied by (14.2) and by the fact that φ_i are one-to-one maps.

If now $\mathcal{T}_1 \subset \mathcal{T}$ is an open interval, we shall define that $\mathcal{F}(\{Y(t), m\}, \{F_j(t)\}_{j\in J}, \mathcal{T}_1)$ if the following conditions are fulfilled:

(A) There exists a point $B \in E_3$ and a vector u such that

$$Y(t) = B + tu\,. \tag{16.10}$$

(B) The system $\{F_j(t)\}_{j\in J}$ is one of the following systems:

(16.11a) empty system;

(16.11b) the system $\{0\}$;

(16.11c) the system $\{H(t)\}$, where $H(t) = G(\alpha_i, \alpha_j, t) \oplus G(\alpha_i, \alpha_k, t)$, i, j, k being a permutation of the numbers 1, 2, 3;

(16.11d) the system $\{U(t)\} \cup \{V(t)\}$, $U(t)$, $V(t)$ being such vectors that for all $t \in \mathcal{T}_1$ there exist vectors $H(t)$ such that

$$H(t) = U(t) \oplus V(t),$$

$$H(t) = G(\alpha_i, \alpha_j, t) \oplus G(\alpha_i, \alpha_k, t),$$

where $i. j, k$ is a permutation of the numbers 1, 2. 3.

Theorem 16.1. *The above described model fulfils all the axioms* I.1, ..., VI.3 *with the exception of Axiom* II.3. *As the function* P_α *do not satisfy the system* (9.5), *Axiom* II.3 *is physically independent of all the other axioms.*

Proof. Axioms I.1, I.2 and I.3. Suppose that $\mathcal{F}(\{B(t), m\}, \{F_j(t)\}_{j\in J}, \mathcal{T}_1)$, $\mathcal{F}(\{C(t), m\}, \{F_j(t)\}_{j\in J} \cup \{G(t)\}, \mathcal{T}_1)$. Let the direction of all the vectors $G(t)$, $t \in \mathcal{T}_1$, be constant. Then, by (16.9d) and (16.9e), neither (16.11c) nor (16.11d) occurs for the system $\{F_j(t)\}_{j\in J} \cup \{G(t)\}$, therefore, as we can easily see from (16.11),

$$J = \emptyset, \quad G(t) = 0. \tag{16.12}$$

By (16.10), there exist points B_0, C_0 and vectors u, v such that

$$B(t) = B_0 + tu, \quad C(t) = C_0 + tv. \tag{16.13}$$

From (16.12) and (16.13) we obtain the validity of Axioms I.1–I.3.

Axiom II.1 is implied by (16.8), (16.11) and (16.10).

Axiom II.2 is implied by (16.9a).

Axiom III.1. If $\mathcal{F}(\{C(t), m\}, \{F_j(t)\}_{j\in J} \cup \{G(t)\}, (\mathcal{T}_1)$ and the vectors $G(t)$ have the same direction for all $t \in \mathcal{T}_1$, then, by (16.9d), (16.9e) and (16.11), we have (16.12) and we can define $B(t)$ by (16.13).

The verification of the other axioms can be left to the reader.

17. INDEPENDENCE OF AXIOM III.1

Let us have an E_3 and a cartesian reference system $\mathcal{S}$ in it. If U, V, W are vectors in E_3, we shall define that $W = U \oplus V$ if and only if (4.1) takes place.

Our model will have 3 particles $\alpha_1, \alpha_2, \alpha_3$, the mass of each particle will be 1. The set $\mathcal{T}$ of time instants will be the set of all real numbers. The position of the particles will be define by (14.1).

Define the vectors $G(\alpha_i, \alpha_j, t)$ by (14.2). We will say that $\mathcal{G}(\alpha_i, \alpha_j, t, F)$ if and only if $i \neq j$ and $F = G(\alpha_i, \alpha_j, t)$.

Define vectors

$$H_1(t) = G(\alpha_1, \alpha_2, t) + G(\alpha_1, \alpha_3, t), \tag{17.1a}$$

$$H_2(t) = G(\alpha_2, \alpha_1, t) + G(\alpha_2, \alpha_3, t), \tag{17.1b}$$

$$H_3(t) = G(\alpha_3, \alpha_1, t) + G(\alpha_3, \alpha_2, t). \tag{17.1c}$$

If $\mathscr{T}_1 \subset \mathscr{T}$ is an open interval, we shall say that $\mathscr{F}(\{Y(t), m\}, \{\boldsymbol{F}_j(t)\}_{j \in J}, \mathscr{T}_1)$ if and only if the following conditions are fulfilled:

(17.2a) the set J is finite;

(17.2b) the function $t \mapsto Y(t)$ has a continuous derivative of the second order in $\mathscr{T}_1$;

(17.2c) $$\sum_{j \in J} \boldsymbol{F}_j(t) = \boldsymbol{H}_k(t) + \boldsymbol{u} \quad \text{for} \quad t \in \mathscr{T}_1 ,$$

where $k = 1$ or $k = 2$ or $k = 3$ and $\boldsymbol{u}$ is a constant vector;

(17.2d) $$m\ddot{Y} = \boldsymbol{u} \quad \text{in} \quad \mathscr{T}_1 .$$

Theorem 17.1. *The above described model fulfils all the axioms* I.1, ..., VI.3 *with the exception of Axiom* III.1. *As the functions* P_α *do not satisfy the system* (9.5), *Axiom* III.1 *is physically independent of all the other axioms.*

Proof. Axioms I.1, I.2 and I.3. Suppose that $\mathscr{F}(\{B(t), m\}, \{\boldsymbol{F}_j(t)\}_{j \in J}, \mathscr{T}_1)$, $\mathscr{F}(\{C(t), m\}, \{\boldsymbol{F}_j(t)\} \cup \{\boldsymbol{G}(t)\}, \mathscr{T}_1)$ and that $\boldsymbol{G}(t)$ have a constant direction in $\mathscr{T}_1$. Then by (17.2a), the set J is finite, and by (17.2c). there exist natural numbers m, n; $1 \leqq m \leqq 3$; $1 \leqq n \leqq 3$; and vectors $\boldsymbol{u}_1, \boldsymbol{u}_2$ such that

$$\sum_{j \in J} \boldsymbol{F}_j(t) = \boldsymbol{H}_m(t) + \boldsymbol{u}_1 \qquad \text{in} \quad \mathscr{T}_1 ,$$

$$\sum_{j \in J} \boldsymbol{F}_j(t) + \boldsymbol{G}(t) = \boldsymbol{H}_n(t) + \boldsymbol{u}_2 \quad \text{in} \quad \mathscr{T}_1 ,$$

therefore

$$\boldsymbol{G}(t) = \boldsymbol{H}_n(t) - \boldsymbol{H}_m(t) + \boldsymbol{u}_2 - \boldsymbol{u}_1 \quad \text{in} \quad \mathscr{T}_1 .$$

Because the function $t \mapsto \boldsymbol{H}_n(t) - \boldsymbol{H}_m(t)$ has not a constant direction in $\mathscr{T}_1$ for $m \neq n$, it must be $m = n$ and

$$\boldsymbol{G}(t) = \boldsymbol{u}_2 - \boldsymbol{u}_1 .$$

By (17.2d), we obtain

$$m\,\ddot{B}(t) = \boldsymbol{u}_1 , \quad m\,\ddot{C}(t) = \boldsymbol{u}_2 \quad \text{in} \quad \mathscr{T}_1 ,$$

therefore there exist points $B_0 \in E_3$, $C_0 \in E_3$ and vectors $\boldsymbol{v}_1 \in V_3$, $\boldsymbol{v}_2 \in V_3$ such that

$$m\,B(t) = \tfrac{1}{2}\boldsymbol{u}_1 t^2 + \boldsymbol{v}_1 t + B_0 , \quad m\,C(t) = \tfrac{1}{2}\boldsymbol{u}_2 t^2 + \boldsymbol{v}_2 t + C_0 ,$$

or, by (2.1),

$$m\,\Delta_B(C; t) = \tfrac{1}{2}(\boldsymbol{u}_2 - \boldsymbol{u}_1)\,t^2 + (\boldsymbol{v}_2 - \boldsymbol{v}_1)\,t + (C_0 - B_0) \quad \text{for all} \quad t \in \mathscr{T}_1 .$$

This equality implies Axiom I.3. By (2.2) and (2.3) we obtain

$$m\,\mathbf{A}_B(C; t_1, t_2, \tau) = \boldsymbol{u}_2 - \boldsymbol{u}_1 = \boldsymbol{G}(t) \quad \text{for} \quad t_1 \in \mathscr{T}_1,\ t_1 < t_2,\ \tau > 0,\ t_2 + \tau \in \mathscr{T}_1 .$$

This equality implies Axioms I.1 and I.2.

Axiom II.3. Suppose that

(17.3) $$\mathscr{F}(\{B(t), m\}, \{\boldsymbol{F}_j(t)\}_{j \in J}, \mathscr{T}_1)$$

and that the points $C(t)$ satisfy

(17.4) $m\,\mathbf{A}_B(C; t_1, t_2, \tau) = \boldsymbol{G}$ if $t_1 \in \mathscr{T}_1,\ t_1 < t_2,\ \tau > 0,\ t_2 + \tau \in \mathscr{T}_1$.

Suppose that the function $t \mapsto \Delta_B(C; t)$ is continuous on $\mathscr{T}_1$ and therefore bounded on every compact subinterval of $\mathscr{T}_1$. Then, by Theorem 3.2, there exists a vector $\boldsymbol{v}$ such that, if $t_0 \in \mathscr{T}_1$, we have

$$\Delta_B(C; t) = \Delta_B(C; t_0) + (t - t_0)\,\boldsymbol{v} + \frac{1}{2m}(t - t_0)^2\,\boldsymbol{G} \quad \text{for all} \quad t \in \mathscr{T}_1\,,$$

or, by (2.1),

(17.5)

$$C(t) = B(t) + \Delta_B(C; t_0) + (t - t_0)\,\boldsymbol{v} + \frac{1}{2m}(t - t_0)^2\,\boldsymbol{G} \quad \text{for all} \quad t \in \mathscr{T}_1\,.$$

Now, (17.3) and (17.2a, c) imply that the set J is finite and that there exists a natural number k; $1 \leqq k \leqq 3$; and a vector $\boldsymbol{u}$ such that

(17.6) $$\sum_{j \in J} \boldsymbol{F}_j(t) = \boldsymbol{H}_k(t) + \boldsymbol{u} \quad \text{for all} \quad t \in \mathscr{T}_1$$

(17.3) and (17.2d) imply that

(17.7) $$m\,\ddot{B}(t) = \boldsymbol{u} \quad \text{in} \quad \mathscr{T}_1\,.$$

We obtain from (17.5) and (17.7) that

$$m\,\ddot{C}(t) = \boldsymbol{u} + \boldsymbol{G}\,.$$

Because

$$\sum_{j \in J} \boldsymbol{F}_j(t) + \boldsymbol{G} = \boldsymbol{H}_k(t) + \boldsymbol{u} + \boldsymbol{G}$$

by (17.6), we deduce from (17.2) that $\mathscr{F}(\{C(t), m\}, \{\boldsymbol{F}_j(t)\}_{j \in J} \cup \{\boldsymbol{G}\}, \mathscr{T}_1)$. Axiom II.3 is therefore fulfilled.

Axioms IV.1, IV.2, IV.3. If $\mathfrak{M}$ is the empty system of vectors or if $\mathfrak{M} = \{\boldsymbol{0}\}$, then, by (17.2c), the relation $\mathscr{F}(\{Y(t), m\}, \mathfrak{M}, \mathscr{T}_1)$ is never fulfilled. Therefore, Axioms IV.1, IV.2 and IV.3 are fulfilled trivially.

The verification of the other axioms can be left to the reader.

18. INDEPENDENCE OF AXIOM IV.1

Let us have an E_3 with the metric ϱ and a cartesian reference system $\mathscr{S}$ in it. If $\boldsymbol{U}, \boldsymbol{V}, \boldsymbol{W}$ are vectors in E_3, we shall define that $\boldsymbol{W} = \boldsymbol{U} \oplus \boldsymbol{V}$ if and only if (4.1) holds.

Choose a natural number n. Our model will have n particles $\alpha_1, \ldots, \alpha_n$, the mass of each particle will be 1. Further, let us choose $6n$ real numbers (4.2) such that (4.3) holds. Moreover, choose a real number $\bar{t}$. Let us consider the system of differential equations

(18.1) $$\ddot{x}_{ir} = 1 + \sum_{\substack{h=1 \\ h \neq i}}^{n} \frac{x_{hr} - x_{ir}}{\left(\sum_{l=1}^{3} (x_{hl} - x_{il})^2\right)^{3/2}}\,; \quad i = 1, \ldots, n\,; \quad r = 1, 2, 3\,.$$

This system has a solution in an open interval $\mathscr{T}$ containing the number $\bar{t}$, and this solution is formed by functions $x_{1r}, \ldots, x_{nr}$; $r = 1, 2, 3$; fulfilling the initial conditions (4.5) and such that (4.6) holds. We will choose the interval $\mathscr{T}$ as the set of time instants. The position of the particle α_i at an instant $t \in \mathscr{T}$ is defined to be the point of E_3 whose coordinates in the reference system $\mathscr{S}$ are $[x_{i1}(t), x_{i2}(t), x_{i3}(t)]$.

If $\boldsymbol{F} = (f_1, f_2, f_3)$ is a vector, we define that $\mathscr{G}(\alpha_i, \alpha_h, t, \boldsymbol{F})$ if and only if $i \neq h$ and (4.7) holds (with $\varkappa = m_i = m_h = 1$).

Let $\mathscr{T}_1 \subset \mathscr{T}$ be an open interval and let $Y(t) = [y_1(t), y_2(t), y_3(t)] \in E_3$ for $t \in \mathscr{T}_1$, $\boldsymbol{F}_j(t) = (f_{j1}(t), f_{j2}(t), f_{j3}(t)) \in V_3$ for $j \in J$, $t \in \mathscr{T}_1$. We will say that $\mathscr{F}(\{Y(t), m\}, \{\boldsymbol{F}_j(t)\}_{j \in J}, \mathscr{T}_1)$ if and only if the following conditions are fulfilled:

(18.2a) the set J is finite;

(18.2b) the functions $\sum_{j \in J} f_{jl}$ for $l = 1, 2, 3$ are continuous on $\mathscr{T}_1$;

(18.2c) the functions y_1, y_2, y_3 have continuous derivatives of the second order on $\mathscr{T}_1$;

(18.2d) $$m\ddot{y}_l = 1 + \sum_{j \in J} f_{jl} \quad \text{on} \quad \mathscr{T}_1 \quad \text{for} \quad l = 1, 2, 3\,.$$

Theorem 18.1. *The above described model fulfils all the axioms* I.1, ..., VI.3 *with the exception of Axiom* IV.1. *As the functions* P_α *do not satisfy the system* (9.5), *Axiom* IV.1 *is physically independent of all the other axioms.*

Proof. Axioms I.1, I.2, I.3. Let (4.9), (4.10) and (4.11) be fulfilled with $B(t) = [b_1(t), b_2(t), b_3(t)]$, $C(t) = [c_1(t), c_2(t), c_3(t)]$, $\boldsymbol{F}_j(t) = (f_{j1}(t), f_{j2}(t), f_{j3}(t))$, $\boldsymbol{u} = (u_1, u_2, u_3)$. Then, by (18.2a), the set J is finite, by (18.2b) the functions $\sum_{j \in J} f_{jl}$, $gu_l + \sum_{j \in J} f_{jl}$ are continuous on $\mathscr{T}_1$ and, by (18.2c, d), we have

$$m\ddot{b}_l = 1 + \sum_{j \in J} f_{jl}\,, \quad m\ddot{c}_l = 1 + gu_l + \sum_{j \in J} f_{jl}\,; \quad l = 1, 2, 3\,; \quad \text{on} \quad \mathscr{T}_1$$

and therefore

$$m(\ddot{C} - \ddot{B}) = g\boldsymbol{u} \quad \text{on} \quad \mathscr{T}_1\,.$$

Hence we can deduce Axioms I.1, I.2, I.3 as in Chap. 4.

Axioms IV.2, IV.3. If $B(t) = [b_1(t), b_2(t), b_3(t)]$ and $\mathfrak{M}$ is the empty system of forces or $\mathfrak{M} = \{\boldsymbol{0}\}$, then, by (18.2d), $\mathscr{F}(\{B(t), m\}, \mathfrak{M}, \mathscr{T}_1)$ if and only if there exists a vector $\boldsymbol{v} = (v_1, v_2, v_3)$ such that for $t_0 \in \mathscr{T}_1$ we have

$$b_l(t) = \frac{1}{2m}(t - t_0)^2 + v_l(t - t_0) + b_l(t_0)\,; \quad l = 1, 2, 3\,; \quad \text{for} \quad t \in \mathscr{T}_1\,.$$

Hence we can easily deduce Axioms IV.2, IV.3.

The verification of the other axioms can be left to the reader.

Let us have an E_3 with the metric ϱ and a cartesian reference system $\mathcal{S}$ in it. If U, V, W are vectors in E_3, we shall define that $W = U \oplus V$ if and only if (4.1) holds.

Choose a natural number $n > 1$ and n real positive numbers $m_1, \ldots, m_n$. Our model will have n particles $\alpha_1, \ldots, \alpha_n$ with masses $m_1, \ldots, m_n$. Further, let us choose $6n$ real numbers (4.2) such that (4.3) holds. Moreover, choose a real number $\varkappa > 0$ and a real number $\bar{t}$. Let us consider the system of differential equations (4.4). This system has a solution in an open interval $\mathcal{T}$ containing the number $\bar{t}$, and this solution is formed by functions $x_{1r}, \ldots, x_{nr}$; $r = 1, 2, 3$; fulfilling the initial conditions (4.5) and such that (4.6) holds. $\mathcal{T}$ will be the set of time instants. The position of the particle α_i at an instant $t \in \mathcal{T}$ is defined to be the point of E_3 whose coordinates in the reference system $\mathcal{S}$ are

$$\mathsf{P}_{\alpha_i}(t) = [\cos t + x_{i1}(t),\ \sin t + x_{i2}(t),\ x_{i3}(t)]\,. \tag{19.1}$$

If $F = (f_1, f_2, f_3)$ is a vector, we define that $\mathcal{G}(\alpha_i, \alpha_h, t, F)$ if and only if $i \neq h$ and (4.7) holds.

Let $\mathcal{T}_1 \subset \mathcal{T}$ be an open interval and let $Y(t) = [y_1(t), y_2(t), y_3(t)] \in E_3$ for $t \in \mathcal{T}_1$, $F_j(t) = (f_{j1}(t), f_{j2}(t), f_{j3}(t)) \in V_3$ for $j \in J$, $t \in \mathcal{T}_1$. We will say that $\mathcal{F}(\{Y(t), m\}, \{F_j(t)\}_{j \in J}, \mathcal{T}_1)$ if and only if either

(19.2a) the set J is finite and non-empty;

(19.2b) the functions $\sum_{j \in J} f_{jl}$ for $l = 1, 2, 3$ are continuous on $\mathcal{T}_1$;

(19.2c) the functions y_1, y_2, y_3 have continuous derivatives of the second order on $\mathcal{T}_1$;

(19.2d) for all $t \in \mathcal{T}_1$,

$$\ddot{x}_1(t) = -\cos t + \frac{1}{m} \sum_{j \in J} f_{j1}(t)\,,$$

$$\ddot{x}_2(t) = -\sin t + \frac{1}{m} \sum_{j \in J} f_{j2}(t)\,,$$

$$\ddot{x}_3(t) = \frac{1}{m} \sum_{j \in J} f_{j3}(t)$$

or

(19.3a) the set J is empty;

(19.3b) $x(t) = (\cos t, \sin t, 0)$ for all $t \in \mathcal{T}_1$.

Theorem 19.1. *The above described model fulfils all the axioms* I.1, …, VI.3 *with the exception of Axiom* IV.2. *As the functions* P_α *do not satisfy the system* (9.5), *Axiom* IV.2 *is physically independent of all the other axioms.*

Proof. We shall verify only Axioms II.3 and IV.1. The other axioms can be verified in an analogous way as in the proof of Theorem 18.1.

Axiom II.3. Suppose that (4.9) holds, where $B(t) = [b_1(t), b_2(t), b_3(t)]$, $F_j(t) = (f_{j1}(t), f_{j2}(t), f_{j3}(t))$. Then, by (19.2a) or (19.3a), the set J is finite, and, by (19.2d) or (19.3b),

(19.4a) $$\ddot{b}_1(t) = -\cos t + \frac{1}{m}\sum_{j\in J} f_{j1}(t),$$

(19.4b) $$\ddot{b}_2(t) = -\sin t + \frac{1}{m}\sum_{j\in J} f_{j2}(t),$$

(19.4c) $$\ddot{b}_3(t) = \frac{1}{m}\sum_{j\in J} f_{j3}(t).$$

Let G be a vector such that (4.11) holds, where $u = (u_1, u_2, u_3)$, and let $C(t) = [c_1(t), c_2(t), c_3(t)] \in E_3$ satisfy

$$m\,\mathbf{A}_B(C; t_1, t_2, \tau) = G \quad \text{for every} \quad t_1 \in \mathscr{T}_1,\ t_1 < t_2,\ \tau > 0,\ t_2 + \tau \in \mathscr{T}_1 .$$

Let the function $t \mapsto \Delta_B(C; t)$ be continuous on $\mathscr{T}_1$. By Theorem 3.2, there exists a vector v such that (4.13) holds. Now, (19.4), (4.11) and (4.13) imply

$$\ddot{c}_1(t) = -\cos t + \frac{1}{m}\left(g(t)\,u_1 + \sum_{j\in J} f_{j1}(t)\right),$$

$$\ddot{c}_2(t) = -\sin t + \frac{1}{m}\left(g(t)\,u_2 + \sum_{j\in J} f_{j2}(t)\right),$$

$$\ddot{c}_3(t) = \frac{1}{m}\left(g(t)\,u_3 + \sum_{j\in J} f_{j3}(t)\right),$$

therefore, by (19.2), we have $\mathscr{F}(\{C(t), m\}, \{F_j(t)\}_{j\in J} \cup \{G\}, \mathscr{T}_1)$. Axiom II.3 is thus verified.

Axiom IV.1. Let $\mathfrak{M}$ be the empty system of forces and let $\mathscr{F}(\{B(t), m\}, \mathfrak{M}, \mathscr{T}_1)$, where $B(t) = [b_1(t), b_2(t), b_3(t)]$. Then, by (19.3b), we have

$$b_1(t) = \cos t, \quad b_2(t) = \sin t, \quad b_3(t) = 0$$

and therefore, if $t_1 \in \mathscr{T}_1$, $t_1 < t_2$, $\tau > 0$, $t_2 + \tau \in \mathscr{T}_1$, we have

$$\varrho(B(t_1 + \tau), B(t_1)) = ((\cos(t_1 + \tau) - \cos t_1)^2 + (\sin(t_1 + \tau) - \sin t_1)^2)^{1/2} =$$
$$= (\cos^2(t_1 + \tau) + \sin^2(t_1 + \tau) + \cos^2 t_1 +$$
$$+ \sin^2 t_1 - 2\cos(t_1 + \tau)\cos t_1 - 2\sin(t_1 + \tau)\sin t_1)^{1/2} = (2 - 2\cos\tau)^{1/2}$$

and similarly

$$\varrho(B(t_2 + \tau), B(t_2)) = (2 - 2\cos\tau)^{1/2}.$$

Hence Axiom IV.1 is fulfilled.

20. INDEPENDENCE OF AXIOM IV.3

Let us have an E_3 with the metric ϱ and a cartesian reference system $\mathscr{S}$ in it. If U, V, W are vectors in E_3, we shall define that $W = U \oplus V$ if and only if (4.1) holds.

As in Chap. 18, choose a natural number $n > 1$, n real positive numbers $m_1, \ldots, m_n$, a real number $\varkappa > 0$, a real number $\bar{t}$ and $6n$ real numbers (4.2) such that (4.3) holds. Our model will have n particles $\alpha_1, \ldots, \alpha_n$ with masses $m_1, \ldots, m_n$. The set of time instants will be an open interval $\mathscr{T}$ containing the number $\bar{t}$, in which the system (4.4) has as a solution formed by functions $x_{1r}, \ldots, x_{nr}$; $r = 1, 2, 3$; fulfilling the initial conditions (4.5) and such that (4.6) holds. The position of α_i at the instant $t \in \mathscr{T}$ will be the point

$$\mathsf{P}_\alpha(t) = [\lambda(t) + x_{i1}(t), \lambda(t) + x_{i2}(t), \lambda(t) + x_{i3}(t)],$$

where λ is a discontinuous real function such that (13.3) holds.

If $F = (f_1, f_2, f_3)$ is a vector, we define that $\mathscr{G}(\alpha_i, \alpha_h, t, F)$ if and only if $i \neq h$ and (4.7) is fulfilled.

Let $\mathscr{T}_1 \subset \mathscr{T}$ be an open interval and let $Y(t) = [y_1(t), y_2(t), y_3(t)] \in E_3$ for $t \in \mathscr{T}_1$, $F_j(t) = (f_{j1}(t), f_{j2}(t), f_{j3}(t)) \in V_3$ for $j \in J$, $t \in \mathscr{T}_1$. We will say that $\mathscr{F}(\{Y(t), m\}, \{F_j(t)\}_{j\in J}, \mathscr{T}_1)$ if and only if

(20.1a) the set J is finite;

(20.1b) the functions $\sum_{j\in J} f_{jl}$ for $l = 1, 2, 3$ are continuous on $\mathscr{T}_1$;

(20.1c) the functions $y_l - \lambda$; $l = 1, 2, 3$; have continuous derivatives of the second order on $\mathscr{T}_1$;

(20.1d) $$(y_l - \lambda)^{\cdot\cdot} = \frac{1}{m} \sum_{j\in J} f_{jl} \quad \text{in} \quad \mathscr{T}_1 \quad \text{for} \quad l = 1, 2, 3\,.$$

Theorem 20.1. *The above described model fulfils all the axioms* I.1, …, VI.3 *with the exception of Axiom* IV.3. *As the functions* P_α *do not satisfy the system* (9.5), *Axiom* IV.3 *is physically independent of all the other axioms.*

Proof is analogous to that of Theorems 4.1, 13.1 and 19.1 and can be left to the reader.

21. INDEPENDENCE OF AXIOM V.1

Let us have an E_3 with the metric ϱ and a cartesian reference system $\mathscr{S}$ in it. The interval $\mathscr{T}$ will be the set of all real numbers. The relations $W = U \oplus V$ and $\mathscr{F}(\{Y(t), m\}, \{F_j(t)\}_{j\in J}, \mathscr{T}_1)$ will be defined as in Chap. 4.

Our model will have 3 particles $\alpha_1, \alpha_2, \alpha_3$ with the positions

$$\mathsf{P}_{\alpha_1}(t) = [0, 0, 0]\,, \quad \mathsf{P}_{\alpha_2}(t) = [0, 1, 0]\,, \quad \mathsf{P}_{\alpha_3}(t) = [0, 0, 1]\,.$$

The mass of each particle will be 1. If F is a vector and $t \in \mathscr{T}$ we will say that $\mathscr{G}(\alpha_i, \alpha_j, t, \boldsymbol{F})$ if and only if

$$\boldsymbol{F} = (\mathsf{P}_{\alpha_j}(t) - \mathsf{P}_{\alpha_i}(t))/(\varrho(\mathsf{P}_{\alpha_i}(t), \mathsf{P}_{\alpha_j}(t)))^3 .$$

Theorem 21.1. *The above described model fulfils all the axioms* I.1, …, VI.3 *with the exception of Axiom* V.1. *As the functions* P_α *do not satisfy the system* (9.5), *Axiom* V.1 *is physically independent of all the other axioms.*

We leave the proof of this theorem (as well as those of the subsequent ones) to the reader.

22. INDEPENDENCE OF AXIOM V.2

The interval $\mathscr{T}$, the particles, their masses and positions and the relations $\boldsymbol{W} = \boldsymbol{U} \oplus \boldsymbol{V}$ and $\mathscr{F}(\{Y(t), m\}, \{\boldsymbol{F}_j(t)\}_{j\in J}, \mathscr{T}_1)$ will be defined as in Chap. 21. We will define that the relation $\mathscr{G}(\alpha_i, \alpha_j, t, \boldsymbol{F})$ is fulfilled for no vector $\boldsymbol{F}$.

Theorem 22.1. *The above described model fulfils all the axioms* I.1, …, VI.3 *with the exception of Axiom* V.2. *As the functions* P_α *do not satisfy the system* (9.5). *Axiom* V.2 *is physically independent of all the other axioms.*

23. INDEPENDENCE OF AXIOM V.3

Let us have an E_3 and a cartesian reference system $\mathscr{S}$ in it. If $\boldsymbol{U}, \boldsymbol{V}, \boldsymbol{W}$ are vectors in E_3, we shall define that $\boldsymbol{W} = \boldsymbol{U} \oplus \boldsymbol{V}$ if and only if (4.1) holds.

Our system will have n particles $\alpha_1, \ldots, \alpha_n$, where $n > 1$. The mass of each particle will be 1. Choose a real number $\bar{t}$ and $6n$ real numbers (4.2) such that $a_{i3} = v_{i3} = 0$ for $i = 1, \ldots, n$ and that (4.3) holds. Let us consider the system of differential equations

$$\ddot{x}_{i1} = \sum_{\substack{h=1 \\ h \neq 1}}^{n} \frac{x_{h2} - x_{i2}}{\left(\sum_{l=1}^{3} (x_{hl} - x_{il})^2\right)^{3/2}}, \tag{23.1a}$$

$$\ddot{x}_{i2} = \sum_{\substack{h=1 \\ h \neq i}}^{n} \frac{x_{i1} - x_{h1}}{\left(\sum_{l=1}^{3} (x_{hl} - x_{il})^2\right)^{3/2}}, \tag{23.1b}$$

$$\ddot{x}_{i3} = 0 . \tag{23.1c}$$

This system has a solution in an open interval $\mathscr{T}$ containing the number $\bar{t}$, and this solution is formed by functions $x_{1r}, \ldots, x_{nr}$; $r = 1, 2, 3$; fulfilling the initial conditions (4.5) and such that (4.6) holds. $\mathscr{T}$ will be the set of time instants. The position of the particle α_i at an instant $t \in \mathscr{T}$ is defined to be the point of E_3 whose coordinates in the reference system $\mathscr{S}$ are $[x_{i1}(t), x_{i2}(t), x_{i3}(t)]$.

If $F = (f_1, f_2, f_3)$ is a vector, we define that $\mathscr{G}(\alpha_i, \alpha_h, t, F)$ if and only if $i \neq h$ and

$$f_1 = \frac{x_{h2}(t) - x_{i2}(t)}{\left(\sum_{l=1}^{3} (x_{hl}(t) - x_{il}(t))^2\right)^{3/2}}, \quad f_2 = \frac{x_{i1}(t) - x_{h1}(t)}{\left(\sum_{l=1}^{3} (x_{hl}(t) - x_{il}(t))^2\right)^{3/2}}, \quad f_3 = 0.$$

The relation $\mathscr{F}(\{Y(t), m\}, \{F_j(t)\}_{j \in J}, \mathscr{T}_1)$ defined as in Chap. 4.

Theorem 23.1. *The above described model fulfils all the axioms* I.1, ..., VI.3 *with the exception of Axiom* V.3. *As the functions* P_α *do not satisfy the system* (9.5), *Axiom* V.3 *is physically independent of all the other axioms.*

24. INDEPENDENCE OF AXIOM V.4

Let us define a model as in Chap. 4, but instead of the system (4.4) let us use the system

$$m_i \ddot{x}_{ir} = \sum_{\substack{h=1 \\ h \neq i}}^{n} \varkappa \frac{m_i m_h (x_{hr} - x_{ir})}{\left(\sum_{l=1}^{3} (x_{hl} - x_{il})^2\right)^2}; \quad i = 1, \ldots, n; \quad r = 1, 2, 3;$$

and instead of the relation (4.7) the relation

$$f_r = \varkappa m_i m_h \frac{x_{hr}(t) - x_{ir}(t)}{\left(\sum_{l=1}^{3} (x_{hl}(t) - x_{il}(t))^2\right)^2}; \quad r = 1, 2, 3.$$

Theorem 24.1. *The above described model fulfils all the axioms* I.1, ..., VI.3 *with the exception of Axiom* V.4. *As the functions* P_α *do not satisfy the system* (9.5), *Axiom* V.4 *is physically independent of all the other axioms.*

25. INDEPENDENCE OF AXIOM V.5

Let us have an E_3 and a cartesian reference system $\mathscr{S}$ in it. The interval $\mathscr{T}$ will be the set of all real numbers. If U, V, W are vectors in E_3, we shall define that $W = U \oplus V$ if and only if (4.1) holds.

Define a real function p by

(25.1a) $p(t) = 1$ if $t = 0$ or t is irrational;

(25.1b) $p(t) = 1/q$ if $t = p/q$, where q is natural, p integer. $p \neq 0$, and the greatest common factor of p, q is 1.

For all $t \in \mathscr{T}$, define vectors

$$R(t) = (p(t), 0, 0), \tag{25.2}$$

$$H(t) = (\tfrac{1}{4}(p(t))^{-2}, 0, 0). \tag{25.3}$$

Our model will have 2 particles α, β, the mass of each particle will be 1. The position of the particles will be defined by

$$\mathsf{P}_\alpha(t) = [-p(t), 0, 0], \quad \mathsf{P}_\beta(t) = [p(t), 0, 0] \quad \text{for all} \quad t \in \mathscr{T}. \tag{25.4}$$

We will define that $\mathscr{G}(\alpha, \beta, t, F)$ or $\mathscr{G}(\beta, \alpha, t, F)$ if and only if $F = H(t)$ or $F = -H(t)$, respectively.

If $\mathscr{T}_1 \subset \mathscr{T}$ is an open interval, we will define that $\mathscr{F}(\{Y(t), m\}, \{F_j(t)\}_{j \in J}, \mathscr{T}_1)$ if and only if

(25.5a) the set J is finite

and there exists a number ε, $\varepsilon = 0$ or $\varepsilon = 1$ or $\varepsilon = -1$ such that

(25.5b) the function $t \mapsto \varepsilon\, H(t) + \sum_{j \in J} F_j(t)$ is continuous in $\mathscr{T}_1$;

(25.5c) the function $t \mapsto Y(t) + \varepsilon\, R(t)$ has a continuous derivative of the second order;

(25.5d) $(Y(t) + \varepsilon\, R(t))^{\cdot\cdot} = \varepsilon\, H(t) + \sum_{j \in J} F_j(t)$ for all $t \in \mathscr{T}_1$.

Theorem 25.1. *The above described model fulfils all the axioms* I.1, ..., VI.3 *with the exception of Axiom* V.5. *As the functions* P_α *do not satisfy the system* (9.5), *Axiom* V.5 *is physically independent of all the other axioms.*

Proof. Axioms I.1, I.2, I.3. Suppose that (4.9), (4.10) and (4.11) hold. Then, by (25.5), the set J is finite and there exist numbers ε_1, ε_2, equal to 0, 1 or -1, such that the functions $\varepsilon_1 H + \sum_{j \in J} F_j$, $\varepsilon_2 H + G + \sum_{j \in J} F_j$ are both continuous on $\mathscr{T}_1$. This is possible only if the function $(\varepsilon_2 - \varepsilon_1) H + G$ is continuous on $\mathscr{T}_1$. By (4.11), (25.3) and (25.1) this can occur only in the two following cases:

(a) $\varepsilon_1 = \varepsilon_2$;

(b) $G(t)$, $H(t)$ are linearly dependent for all $t \in \mathscr{T}_1$.

Case (a). In this case (25.5d) implies

$$(B(t) + \varepsilon_1 R(t))^{\cdot\cdot} = \varepsilon_1 H(t) + \sum_{j \in J} F_j(t) \quad \text{in} \quad \mathscr{T}_1,$$

$$(C(t) + \varepsilon_1 R(t))^{\cdot\cdot} = \varepsilon_1 H(t) + G(t) + \sum_{j \in J} F_j(t) \quad \text{in} \quad \mathscr{T}_1$$

and therefore

$$(C - B)^{\cdot\cdot} = G \quad \text{on} \quad \mathscr{T}_1$$

and Axioms I.1, I.2 and I.3 can be verified as in Chap. 4.

Case (b). In this case (25.5d) implies

$$(25.6)\qquad (C - B + (\varepsilon_2 - \varepsilon_1) R)^{\cdot\cdot} = G + (\varepsilon_2 - \varepsilon_1) H \quad \text{in} \quad \mathcal{T}_1 .$$

Because $G(t)$ and $H(t)$ are linearly dependent for all $t \in \mathcal{T}_1$, we deduce from (4.11), (25.2) and (25.3) that

$$R(t) = \eta\, p(t)\, u\,, \quad H(t) = \tfrac{1}{4}\eta(p(t))^{-2}\, u\,,$$

where $\eta = 1$ or $\eta = -1$. Now, (25.6) yields

$$(C - B + \eta(\varepsilon_2 - \varepsilon_1)\, p u)^{\cdot\cdot} = (g + \tfrac{1}{4}\eta(\varepsilon_2 - \varepsilon_1)\, p^{-2})\, u \quad \text{in} \quad \mathcal{T}_1\,;$$

if we choose a number $t_0 \in \mathcal{T}_1$, we obtain that there exist vectors v_1, v_2 such that

$$C(t) - B(t) = v_1 + t v_2 + \eta(\varepsilon_1 - \varepsilon_2)\, p(t)\, u +$$
$$+ \int_{t_0}^{t} \left(\int_{t_0}^{\sigma_2} (g(\sigma_1) + \tfrac{1}{4}\eta(\varepsilon_2 - \varepsilon_1)(p(\sigma_1))^{-2}) \,\mathrm{d}\sigma_1 \right) \mathrm{d}\sigma_2\, u \quad \text{for all} \quad t \in \mathcal{T}_1 .$$

Hence we easily deduce Axiom I.1.

If now $t_1 \in \mathcal{T}_1$, $\tau > 0$, $t_1 < t_2$, $t_2 + \tau \in \mathcal{T}_1$, then, because the function $(\varepsilon_2 - \varepsilon_1) \,.\, H + G$ is continuous on $\mathcal{T}_1$, the function $t \mapsto |G(t)|$ can be bounded on the interval $\langle t_1, t_2 + \tau \rangle$ only if $\varepsilon_1 = \varepsilon_2$. We can therefore verify Axiom I.2 as in the case (a). Similarly, the function $t \mapsto |G(t)|$ can be bounded on every compact subinterval of $\mathcal{T}_1$ only if $\varepsilon_1 = \varepsilon_2$, and Axiom I.3 can be therefore proved as in the case (a).

Axioms IV.1, IV.2, IV.3. If $\mathcal{F}(\{Y(t); m\}, \mathfrak{M}, \mathcal{T}_1)$, where $\mathfrak{M} = \{0\}$ or $\mathfrak{M}$ is the empty system of forces, then, by (25.5b), the function $t \mapsto \varepsilon\, H(t)$ must be continuous on $\mathcal{T}_1$; but this can occur only if $\varepsilon = 0$. Therefore, by (25.5d), we obtain $\ddot{Y} = 0$ on $\mathcal{T}_1$ and from this equation we can easily deduce Axioms IV.1, IV.2 and IV.3.
We leave the verification of the other axioms to the reader.

26. INDEPENDENCE OF AXIOM VI.1

Let us have an E_3. The interval $\mathcal{T}$ will be the set of all real numbers. The relations $W = U \oplus V$ and $\mathcal{F}(\{Y(t), m\}, \{F_j(t)\}_{j \in J}, \mathcal{T}_1)$ will be defined as in Chap. 4, The model will have no particles. The relation $\mathcal{G}(\alpha, \beta, t, F)$ is therefore never fulfilled.

Theorem 26.1. *The above described model fulfils all the axioms* I.1, ..., VI.3 *with the exception of Axiom* VI.1. *Because any model fulfilling all axioms has at least one particle, Axiom* VI.1 *is physically independent of all the other axioms.*

We leave the proof of this theorem (as well as those of the subsequent ones) to the reader.

27. INDEPENDENCE OF AXIOM VI.2

Let us have an E_3 with the metric ϱ and a cartesian reference system $\mathcal{S}$ in it. The interval $\mathcal{T}$ will be the set of all real numbers. If U, V, W are vectors in E_3, we shall define that $W = U \oplus V$ if and only if (4.1) holds. Our model will have particles

$$\ldots, \alpha_{-2}, \alpha_{-1}, \alpha_0, \alpha_1, \alpha_2, \alpha_3, \ldots$$

the mass of each of them will be 1. The position of the particle α_i will be

$$\mathsf{P}_{\alpha_i} = [i, 0, 0], \quad i = \ldots -2, -1, 0, 1, 2, \ldots .$$

We will define that $\mathscr{G}(\alpha_i, \alpha_j, t, \boldsymbol{F})$ if and only if

$$\boldsymbol{F} = (\mathsf{P}_{\alpha_j}(t) - \mathsf{P}_{\alpha_i}(t))/(\varrho(\mathsf{P}_{\alpha_i}(t), \mathsf{P}_{\alpha_j}(t)))^3 .$$

If $\mathscr{T}_1 \subset \mathscr{T}$ is an open interval, we shall say that $\mathscr{F}(\{Y(t), m\}, \{\boldsymbol{F}_j(t)\}_{j\in J}, \mathscr{T}_1)$ if and only if the following conditions are fulfilled:

(27.1a) the set J is countable[6]);

(27.1b) if J is infinite then $\sum_{j\in J} \boldsymbol{F}_j(t)$ converges absolutely for all $t \in \mathscr{T}_1$;

(27.1c) the function $t \mapsto \sum_{j\in J} \boldsymbol{F}_j(t)$ is continuous on $\mathscr{T}_1$;

(27.1d) the function $t \mapsto Y(t)$ has a continuous derivative of the second order;

(27.1e) $$m\ddot{Y} = \sum_{j\in J} \boldsymbol{F}_j \quad \text{on} \quad \mathscr{T}_1 .$$

Theorem 27.1. *The above described model fulfils all the axioms* I.1, ..., VI.3 *with the exception of Axiom* VI.2. *As any model fulfilling all axioms has a finite number of particles, Axiom* VI.2 *is physically independent of all the other axioms.*

28. INDEPENDENCE OF AXIOM VI.3

Let us have an E_3 and a cartesian reference system $\mathscr{S}$ in it. The set $\mathscr{T}$ will be the set of all real numbers. The relations $W = U \oplus V$ and $\mathscr{F}(\{Y(t), m\}, \{\boldsymbol{F}_j(t)\}_{j\in J}, \mathscr{T}_1)$ will be defined as in Chap. 4. Our model will have 2 particles α, β, the mass of each of them will be 1 and their positions will be

$$\mathsf{P}_\alpha(t) = \mathsf{P}_\beta(t) = [0, 0, 0] \quad \text{for all} \quad t \in \mathscr{T} .$$

We will define that $\mathscr{G}(\alpha, \beta, t, \boldsymbol{F})$ or $\mathscr{G}(\beta, \alpha, t, \boldsymbol{F})$ if and only if $\boldsymbol{F} = \boldsymbol{0}$.

Theorem 28.1. *The above described model fulfils all the axioms* I.1, ..., VI.3 *with the exception of Axiom* VI.3. *Because* $\mathsf{P}_\alpha(t) \neq \mathsf{P}_\beta(t)$ *for* $\alpha \neq \beta$ *and for all* $t \in \mathscr{T}$ *in any model fulfilling all axioms, Axiom* VI.3 *is physically independent of all the other axioms.*

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[6]) Finite sets and the empty set are also considered countable.

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