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THE LATTICE OF EQUATIONAL THEORIES PART II: THE LATTICE OF FULL SETS OF TERMS

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0. INTRODUCTION

In order to obtain some results on definability in the lattice $\mathscr{L}_\Delta$ of equational theories (in a further part of this treatment), it is advantageous first to investigate definability in the lattice $\mathscr{F}_\Delta$ of full sets of Δ-terms. The present Part II is of auxiliary character and it is devoted to this investigation. We give a long list of first-order formulas in the language of lattice theory; some of them describe (if interpreted in $\mathscr{F}_\Delta$) the structure of terms, other describe codes of finite sequences of terms or the consequence relation between equations. We find all automorphisms of the lattice $\mathscr{F}_\Delta$ and prove that every finitely generated member of $\mathscr{F}_\Delta$ is first-order definable in $\mathscr{F}_\Delta$ up to automorphisms.

We preserve the terminology and notation introduced in Section 1 of [1]. Moreover, the following notation will be used.

Let Δ be a type. For every symbol $F \in \Delta$ we denote by n_F the arity of F; put $\Delta_k = \{F \in \Delta;\ n_F = k\}$ for any $k \geqq 0$. We denote by $\Delta^{(1)}$ the set of ordered pairs (F, i) such that $F \in \Delta$ and $i \in \{1, \ldots, n_F\}$. Notice that if $(F, i) \in \Delta^{(1)}$ then $n_F \geqq 1$. The set $\{(F, i) \in \Delta^{(1)};\ n_F \geqq 2\}$ will be denoted by $\Delta^{(2)}$. We denote by $\Delta^{(-)}$ the set of finite (not necessarily non-empty) sequences of unary symbols from Δ. A type Δ is said to be *unary* if $n_F = 1$ for all $F \in \Delta$; it is said to be *strictly large* if it contains a symbol of arity $\geqq 2$.

For every term t we define a non-negative integer $\lambda_0(t)$ as follows: if $t \in V$ then $\lambda_0(t) = 0$; if $t = F(t_1, \ldots, t_{n_F})$ then $\lambda_0(t) = 1 + \lambda_0(t_1) + \ldots + \lambda_0(t_{n_F})$. Thus $\lambda_0(t)$ is the number of occurrences of symbols from Δ in t; we have $\lambda_0(t) \leqq \lambda(t)$.

Whenever a lemma is not followed by its proof, it is either regarded to be evident or follows easily from the preceding lemmas.

1. DEFINABILITY IN GENERAL LATTICES

By a formula we shall always mean a first-order formula in the language of lattice theory. Thus formulas are inscriptions composed of the symbols $\neg$, &, VEL, $\rightarrow$, $\leftrightarrow$, $\forall$, $\exists$, (,), =, $\leqq$ and the variable symbols $X, Y, Z, A, B, C, X', X_1, \ldots$ (These "variable symbols" are different from the variables $x_1, x_2, x_3, \ldots$ introduced in [1].)

We shall work with very long formulas and so it is necessary to introduce abbreviations. Instead of saying that A is an abbreviation for a formula f, we shall write $A \equiv f$. For example:

Definition. (i) $X \neq Y \equiv \neg X = Y$.

(ii) $X < Y \equiv X \leqq Y \,\&\, X \neq Y$.

(iii) $X_1 \leqq X_2 \leqq \ldots \leqq X_n \equiv X_1 \leqq X_2 \,\&\, \ldots \,\&\, X_{n-1} \leqq X_n$.

(iv) $X_1 < X_2 < \ldots < X_n \equiv X_1 < X_2 \,\&\, \ldots \,\&\, X_{n-1} < X_n$.

(v) $X = Y_1 \vee \ldots \vee Y_n \equiv \forall Z(X \leqq Z \leftrightarrow (Y_1 \leqq Z \,\&\, \ldots \,\&\, Y_n \leqq Z))$.

(vi) $X = Y_1 \wedge \ldots \wedge Y_n \equiv \forall Z(Z \leqq X \leftrightarrow (Z \leqq Y_1 \,\&\, \ldots \,\&\, Z \leqq Y_n))$.

(vii) $\omega_0(X) \equiv \forall Y X \leqq Y$.

(viii) $\omega_1(X) \equiv \forall Y Y \leqq X$.

Usually, every definition introducing an abbreviation for a formula will be followed by a lemma explaining how to interpret this formula in a given lattice. If we wanted to be precise, the lemma corresponding to $\omega_0(X)$ would have to look as follows: Given a lattice L and an element $a \in L$, the formula $\omega_0(X)$ is satisfied in L under the interpretation $X \mapsto a$ iff a is the least element of L. However, in order to be brief, we shall express this less accurately as follows: Given a lattice L, $\omega_0(X)$ in L iff X is the least element of L. Similarly, $\omega_1(X)$ in L iff X is the greatest element of L.

For every formula $f(X, \ldots)$ we introduce the following abbreviations:

$\exists!\, X f(X, \ldots) \equiv \forall X\, \forall Y((f(X, \ldots) \,\&\, f(Y, \ldots)) \rightarrow X = Y)$.

$\exists!!\, X f(X, \ldots) \equiv \exists X f(X, \ldots) \,\&\, \exists!\, X f(X, \ldots)$.

$\forall X_1, \ldots, X_n f \equiv \forall X_1\, \forall X_2 \ldots \forall X_n f$.

$\exists X_1, \ldots, X_n f \equiv \exists X_1\, \exists X_2 \ldots \exists X_n f$.

$\exists(X_1, \ldots, X_n)^{\neq} f \equiv \exists X_1, \ldots, X_n(f \,\&\, X_1 \neq X_2 \,\&\, X_1 \neq X_3 \,\&\, \ldots \,\&\, X_1 \neq X_n \,\&$
$\&\, X_2 \neq X_3 \,\&\, \ldots \,\&\, X_2 \neq X_n \,\&\, \ldots \,\&\, X_{n-1} \neq X_n)$.

A subset A of a lattice L is said to be *definable* if there exists a formula $f(X)$ (with a single free variable symbol X) such that an element of L satisfies $f(X)$ in L iff it belongs to A. Evidently, every definable subset of L is closed under the automorphisms of L. An element $a \in L$ is called *definable* if the set $\{a\}$ is definable. An element $a \in L$ is called *definable up to automorphisms* if the set $\{p(a);\ p \in \mathrm{Aut}\,(L)\}$ is definable.

2. THE LATTICE $\mathscr{F}_\Delta$

Throughout this paper let Δ be a fixed type.

Recall that by a full subset of W_Δ we mean any set U of Δ-terms such that if $a \in U$, $b \in W_\Delta$ and $a \leqq b$ then $b \in U$. Evidently, the union and the intersection of any system of full subsets of W_Δ is a full subset of W_Δ. The set of full subsets of W_Δ is thus a complete distributive lattice; the empty set and the set W_Δ are its extreme elements. The lattice of full subsets of W_Δ will be denoted by $\mathscr{F}_\Delta$.

For every set $U \subseteq W_\Delta$ we denote by U^* the full subset generated by U, i.e. $U^* = \{a \in W_\Delta;\ b \leqq a \text{ for some } b \in U\}$. For every term t put $t^* = \{t\}^* = \{a \in W_\Delta;\ t \leqq a\}$. If t, u are two terms, then $t^* \subseteq u^*$ iff $u \leqq t$; consequently, $t^* = u^*$ iff $t \sim u$.

Two subsets U_1, U_2 of W_Δ are said to be similar if every term from U_1 is similar to a term from U_2 and every term from U_2 is similar to a term from U_1. For every $U \subseteq W_\Delta$ put $U^\sim = \{a \in W_\Delta;\ a \sim b \text{ for some } b \in U\}$; evidently, $U^\sim$ is just the greatest subset of W_Δ which is similar to U. For every term t put $t^\sim = \{t\}^\sim$. By a representative subset of a set $U \subseteq W_\Delta$ we mean any minimal subset of U which is similar to U; thus R is a representative subset of U iff $R \subseteq U$ and every term from U is similar to exactly one term from R. By an irreducible subset of W_Δ we mean a subset U such that there is no pair a, b of elements of U with $a < b$.

For every $U \in \mathscr{F}_\Delta$ denote by $I(U)$ the set of all the terms $a \in U$ such that there is no term $b \in U$ with $b < a$. Evidently, $I(U)$ is an irreducible generating subset of U and every two irreducible generating subsets of U are similar. For every $U \in \mathscr{F}_\Delta$ fix a representative subset of $I(U)$ and denote it by $\bar{I}(U)$.

Evidently, if U_1, U_2 are two irreducible subsets of W_Δ then $U_1^* = U_2^*$ iff U_1, U_2 are similar. We have $I(t^*) = t^\sim$ for any term t.

Definition. (i) $\tau(X) \equiv \neg\omega_0(X) \,\&\, \forall Y, Z\, (X = Y \vee Z \rightarrow (X = Y \,\mathrm{VEL}\, X = Z))$.

(ii) $X \ll Y \equiv \tau(X) \,\&\, \tau(Y) \,\&\, Y \leqq X$.

(iii) $X_1 \ll X_2 \ll \ldots \ll X_n \equiv X_1 \ll X_2 \,\&\, \ldots \,\&\, X_{n-1} \ll X_n$.

(iv) $\varphi_1(X, Y) \equiv \tau(X) \,\&\, X \leqq Y \,\&\, \neg\exists Z\, (Z \ll X \,\&\, Z \leqq Y \,\&\, X \neq Z)$.

2.1. Lemma. (i) *$\tau(X)$ in $\mathscr{F}_\Delta$ iff $X = a^*$ for some term a.*

(ii) *$X \ll Y$ in $\mathscr{F}_\Delta$ iff $X = a^*$ and $Y = b^*$ for some terms a, b with $a \leqq b$.*

(iii) *$X_1 \ll X_2 \ll \ldots \ll X_n$ in $\mathscr{F}_\Delta$ iff $X_1 = a_1^*$, $X_2 = a_2^*, \ldots, X_n = a_n^*$ for some terms $a_1, a_2, \ldots, a_n$ with $a_1 \leqq a_2 \leqq \ldots \leqq a_n$.*

(iv) *$\varphi_1(X, Y)$ in $\mathscr{F}_\Delta$ iff $X = a^*$ for some $a \in I(Y)$.*

(v) *$\omega_0(X)$ in $\mathscr{F}_\Delta$ iff $X = \emptyset$.*

(vi) *$\omega_1(X)$ in $\mathscr{F}_\Delta$ iff $X = x^*$ for some (or any) $x \in V$.*

3. COVERS OF TERMS

Let a, b be two terms. We write $a \prec b$ (and say that b is a cover of a or that a is covered by b) if $a < b$ and there is no term c with $a < c < b$.

Let $(F, i) \in \Delta^{(1)}$, $t \in W_\Delta$ and $k \geqq 0$. We define a set $t\begin{bmatrix} k \\ F, i \end{bmatrix}$ of similar terms as follows: $t\begin{bmatrix} 0 \\ F, i \end{bmatrix} = \{t\}$; $a \in t\begin{bmatrix} k+1 \\ F, i \end{bmatrix}$ iff $a = F(y_1, \ldots, y_{i-1}, b, y_{i+1}, \ldots, y_{n_F})$ for some $b \in t\begin{bmatrix} k \\ F, i \end{bmatrix}$ and some pairwise different variables $y_1, \ldots, y_{n_F}$ not belonging to var (b). Moreover, $t\begin{bmatrix} k \\ F, i \end{bmatrix}^{\sim}$ denotes the set of terms similar to a term from $t\begin{bmatrix} k \\ F, i \end{bmatrix}$.

Let t be a term, $x \in V$ and $F \in \Delta$. The term $\sigma^x_{F(y_1, \ldots, y_{n_F})}(t)$ where $y_1, \ldots, y_{n_F}$ are pairwise different variables not contained in var (t) will be denoted by $\sigma^x_F(t)$. (It is determined by t, x, F only up to similarity; for every triple t, x, F we fix one such term $\sigma^x_F(t)$.)

3.1. Lemma. *Let* $(F, i) \in \Delta^{(1)}$, $k \geqq 0$, $x \in V$, $t \in x\begin{bmatrix} k \\ F, i \end{bmatrix}$, $a \in W_\Delta$. *Then* $a \leqq t$ *iff* $a \in x\begin{bmatrix} l \\ F, i \end{bmatrix}^{\sim}$ *for some* $l \in \{0, \ldots, k\}$.

3.2. Lemma. *Let* $(F, i) \in \Delta^{(1)}$, $t \in W_\Delta$, $k \geqq 0$, $u \in t\begin{bmatrix} k \\ F, i \end{bmatrix}$, $a \in W_\Delta$, $t \leqq a \leqq u$. *Then* $a \in t\begin{bmatrix} l \\ F, i \end{bmatrix}^{\sim}$ *for some* $l \in \{0, \ldots, k\}$.

Proof. By induction on t. If $t \in V$, we can use 3.1. Let $t = G(t_1, \ldots, t_{n_G})$. Suppose that there is a term a for which the assertion is not true and let us take a minimal such term a. There are substitutions f, g such that $f(t)$ is a subterm of a and $g(a)$ is a subterm of u. Evidently, there is a $j \in \{0, \ldots, k\}$ with $g(a) \in t\begin{bmatrix} j \\ F, i \end{bmatrix}$. If $j = 0$ then $a \sim t$, a contradiction. Hence $j > 0$; since $a \notin V$, we get $a = F(y_1, \ldots, y_{i-1}, b, y_{i+1}, \ldots, y_{n_F})$ for some term b and pairwise different variables $y_1, \ldots, y_{n_F}$ not contained in var (b). If $f(t)$ is a subterm of b then $t \leqq b < a \leqq u$ and $b \in t\begin{bmatrix} l \\ F, i \end{bmatrix}$ for some l by the minimality of a; but then $a \in t\begin{bmatrix} l+1 \\ F, i \end{bmatrix}^{\sim}$, a contradiction. Thus $f(t)$ is not a subterm of b and so $f(t) = a$. This implies that $G = F$ and $t_1, \ldots, t_{i-1}, t_{i+1}, \ldots, t_{n_F}$ are pairwise different variables not contained in var (t_i). Hence $u \in t_i\begin{bmatrix} k+1 \\ F, i \end{bmatrix}$ and

$t_i < a < u$. By the induction hypothesis, $a \in t_i \begin{bmatrix} l \\ F, i \end{bmatrix}^{\sim}$ for some $l \geqq 1$; hence $a \in t \begin{bmatrix} l-1 \\ F, i \end{bmatrix}^{\sim}$, a contradiction.

3.3. Lemma. *Let $t \in W_\Delta$, $x \in V$, $F \in \Delta$, $u = F(y_1, \ldots, y_{n_F})$ where $y_1, \ldots, y_{n_F}$ are pairwise different variables not belonging to* var (t). *Let b be a subterm of t such that $h(t) = \sigma_u^x(b)$ for some substitution h. Then either $b = t$ or $t \in x \begin{bmatrix} k \\ F, i \end{bmatrix}$ for some $k \geqq 0$ and some $i \in \{1, \ldots, n_F\}$.*

Proof. By induction on t. If $t \in V$, then evidently $b = t$. Let $t = G(t_1, \ldots, t_{n_G})$. Suppose $b \neq t$. There exists a $j \in \{1, \ldots, n_G\}$ such that b is a subterm of t_j.

Assume first $b \in V$. Then $b = x$, $h(t) = \sigma_u^x(b) = u$ and evidently $t \in x \begin{bmatrix} 1 \\ F, i \end{bmatrix}$ (for some i).

Now assume that $b \notin V$, so that $b = G(b_1, \ldots, b_{n_G})$ for some $b_1, \ldots, b_{n_G}$. We have $h(t_j) = \sigma_u^x(b_j)$ and b_j is a proper subterm of t_j. By the induction hypothesis, $t_j \in x \begin{bmatrix} k \\ F, i \end{bmatrix}$ for some $k \geqq 0$ and some $i \in \{1, \ldots, n_F\}$. We have $k \geqq 1$. Since b is a subterm of t_j, we get $G = F$, $i = j$ and $b \in x \begin{bmatrix} l \\ F, i \end{bmatrix}$ for some $l \in \{1, \ldots, k\}$. Hence $h(t) = \sigma_u^x(b) \in x \begin{bmatrix} l+1 \\ F, i \end{bmatrix}$. This implies that $t \in x \begin{bmatrix} p \\ F, i \end{bmatrix}$ for some p.

3.4. Proposition. *Let t, w be two terms. Then $t \prec w$ iff at least one of the following four cases takes place:*

(1) $w \in t \begin{bmatrix} 1 \\ F, i \end{bmatrix}^{\sim}$ *for some* $(F, i) \in \Delta^{(1)}$;

(2) $w \sim \sigma_F^x(t)$ *for some* $x \in$ var (t) *and some* $F \in \Delta$ *with* $n_F \geqq 1$;

(3) $w \sim \sigma_F^x(t)$ *for some* $x \in$ var (t) *and some* $F \in \Delta$ *with* $n_F = 0$;

(4) $w \sim \sigma_y^x(t)$ *for some* $x, y \in$ var (t) *with* $x \neq y$.

Proof. If (1) takes place, then $t \prec w$ by 3.2. Let (2) take place and put $u = F(y_1, \ldots, y_{n_F})$ where $y_1, \ldots, y_{n_F}$ are pairwise different variables not belonging to var (t). If $t \in x \begin{bmatrix} k \\ F, i \end{bmatrix}$ for some k, i, then $t \prec w$ follows from 3.1. Let $t \notin x \begin{bmatrix} k \\ F, i \end{bmatrix}$ for any k, i. Evidently $t < w$. Let $t \leqq a \leqq w$. There are substitutions f, g such that $f(t)$ is a subterm of a and $g(a)$ is a subterm of $\sigma_u^x(t)$. Hence $g\,f(t)$ is a subterm of $\sigma_u^x(t)$, so that either $g\,f(t)$ is a subterm of u or $g\,f(t) = \sigma_u^x(b)$ for some subterm b of t. It follows from 3.3 that $g\,f(t) = \sigma_u^x(t)$. Hence $f(t) = a$ and $g(a) = \sigma_u^x(t)$. This easily yields that either $a \sim t$ or $a \sim w$, so that $t \prec w$. In the cases (3) and (4) it is easy to prove $t \prec w$, as well.

Conversely, let t, w be two terms such that $t \prec w$. There is a substitution f such that $f(t)$ is a subterm of w. Since $t \leqq f(t) \leqq w$, we can assume that either t is a subterm of w or $f(t) = w$. If t is a subterm of w, then evidently (1) takes place. Let $f(t) = w$.

Assume first that there is a variable $x \in \operatorname{var}(t)$ with $f(x) \notin V$. Then $f(x) = F(t_1, \ldots, t_{n_F})$ for some $F \in \Delta$ and $t_1, \ldots, t_{n_F} \in W_\Delta$. We have $t < \sigma_F^x(t) \leqq f(t)$ and so $w \sim \sigma_F^x(t)$.

Finally, let $f(x) \in V$ for all $x \in \operatorname{var}(t)$. Since $t < w$, we have $f(x) = f(y)$ for some $x, y \in \operatorname{var}(t)$ with $x \neq y$. Then $t < \sigma_y^x(t) \leqq f(t)$ and so $w \sim \sigma_y^x(t)$.

We say that w is a cover of t of the first (second, third, fourth) kind if $t \prec w$ and in 3.4 the case (1) (the case (2), (3), (4), resp.) takes place.

3.5. Lemma. *Let* $(F, i) \in \Delta^{(1)}$, $t \in W_\Delta$, $x \in V$, $k \geqq 2$, $u \in x\begin{bmatrix} k \\ F, i \end{bmatrix}$, $\operatorname{var}(u) \cap \operatorname{var}(t) \subseteq \{x\}$. *Let b be a subterm of t such that $h(t) = \sigma_u^x(b)$ for some substitution h. Then either $b = t$ or $t \in x\begin{bmatrix} l \\ F, i \end{bmatrix}$ for some $l \geqq 0$.*

Proof. By induction on t. If $t \in V$, then evidently $b = t$. Let $t = G(t_1, \ldots, t_{n_G})$. Suppose $b \neq t$. There exists a $j \in \{1, \ldots, n_G\}$ such that b is a subterm of t_j.

Assume first that $b \in V$. Then $b = x$, $h(t) = \sigma_u^x(b) = u$ and we can use 3.1.

Now assume that $b \notin V$, so that $b = G(b_1, \ldots, b_{n_G})$ for some $b_1, \ldots, b_{n_G}$. We have $h(t_j) = \sigma_u^x(b_j)$ and b_j is a proper subterm of t_j. By the induction hypothesis, $t_j \in x\begin{bmatrix} l \\ F, i \end{bmatrix}$ for some $l \geqq 0$; we have $l \geqq 1$. Since b is a subterm of t_j, we get $G = F$, $j = i$ and $b \in x\begin{bmatrix} m \\ F, i \end{bmatrix}$ for some $m \geqq 0$. It is easy to see that $\sigma_u^x(b) \in x\begin{bmatrix} k+m \\ F, i \end{bmatrix}$. Hence $h(t) \in x\begin{bmatrix} k+m \\ F, i \end{bmatrix}$; by 3.1 we get $t \in x\begin{bmatrix} p \\ F, i \end{bmatrix}$ for some $p \geqq 0$.

3.6. Lemma. *Let* $(F, i) \in \Delta^{(1)}$, $t \in W_\Delta$, $x \in \operatorname{var}(t)$, $k \geqq 2$, $u \in x\begin{bmatrix} k \\ F, i \end{bmatrix}$, $\operatorname{var}(u) \cap \operatorname{var}(t) = \{x\}$. *Let there exist no $p \geqq 0$ with $t \in x\begin{bmatrix} p \\ F, i \end{bmatrix}^{\sim}$. Let $a \in W_\Delta$, $t \leqq a \leqq \sigma_u^x(t)$. Then there exist an $l \in \{0, \ldots, k\}$ and a $v \in x\begin{bmatrix} l \\ F, i \end{bmatrix}$ with* $\operatorname{var}(v) \cap \operatorname{var}(t) = \{x\}$ *such that $a \sim \sigma_v^x(t)$.*

Proof. There are substitutions f, g such that $f(t)$ is a subterm of a and $g(a)$ is a subterm of $\sigma_u^x(t)$. The term $g\,f(t)$ is a subterm of $\sigma_u^x(t)$ and so either $g\,f(t)$ is a subterm of u or $g\,f(t) = \sigma_u^x(b)$ for some subterm b of t. By 3.1 and 3.5 we get $g\,f(t) = \sigma_u^x(t)$. Hence $f(t) = a$ and $g(a) = \sigma_u^x(t)$. This implies the result.

Definition. (i) $X \prec Y \equiv X \leqslant Y \,\&\, X \neq Y \,\&\, \forall Z(X \leqslant Z \leqslant Y \rightarrow (Z = X \text{ VEL } Z = Y))$.
(ii) $X \lessdot Y \equiv X \prec Y \,\&\, \exists Z(Y \prec Z \,\&\, \forall U((X \leqslant U \leqslant Z \,\&\, X \neq U \,\&\, U \neq Z) \rightarrow \rightarrow U = Y))$.

3.7. Lemma. (i) $X \prec Y$ *in* $\mathscr{F}_\Delta$ *iff* $X = t^*$ *and* $Y = w^*$ *for some terms* t, w *with* $t \prec w$.

(ii) $X \lessdot Y$ *in* $\mathscr{F}_\Delta$ *iff* $X = t^*$ *and* $Y = w^*$ *for some terms* t, w *such that* w *is a cover of* t *of either the first or the second kind.*

Proof. (i) is evident. Let us prove (ii). If w is a cover of t of the first kind then $t^* \lessdot w^*$ follows from 3.2. Let w be a cover of t of the second kind. If there exists an $i \in \{1, \ldots, n_F\}$ such that $t \notin x\begin{bmatrix} p \\ F, i \end{bmatrix}^\sim$ for any $p \geqq 0$ then $t^* \lessdot w^*$ follows from 3.6. If there is no such i then evidently either $t \in V$ or $t \sim F(x_1, \ldots, x_{n_F})$ or $n_F = 1$ and $t = F^p x$ for some $p \geqq 0$. However, in all these singular cases we easily get $t^* \lessdot w^*$.

Now let $X \lessdot Y$. There are terms t, w, a such that $X = t^*$, $Y = w^*$, $t \prec w \prec a$ and whenever $t < b < a$ then $b \sim w$. Suppose $w = \sigma_y^x(t)$ for some $x, y \in \operatorname{var}(t)$ with $x \neq y$. If $a \sim F(y_1, \ldots, y_{i-1}, w, y_{i+1}, \ldots, y_{n_F})$ then $t < F(y_1, \ldots, y_{i-1}, t, y_{i+1}, \ldots, y_{n_F}) < a$ implies $w \sim F(y_1, \ldots, y_{i-1}, t, y_{i+1}, \ldots, y_{n_F})$, a contradiction. If $a \sim \sigma_F^z(w)$ then $t < \sigma_F^z(t) < a$ implies $w \sim \sigma_F^z(t)$, a contradiction. If $a \sim \sigma_{z_1}^z(w)$ then $t < \sigma_{z_1}^z(t) < a$ implies $w \sim \sigma_{z_1}^z(t)$, a contradiction again. We have proved that the case when w is a cover of the fourth kind is impossible. Similarly, w cannot be a cover of the third kind.

4. SOME FORMULAS DESCRIBING THE STRUCTURE OF TERMS

For every symbol $F \in \Delta$ put $F^* = t^*$ where $t = F(x_1, \ldots, x_{n_F})$. Moreover, for every pair $(F, i) \in \Delta^{(1)}$ put $(F, i)^* = t^*$ where $t \in x\begin{bmatrix} 2 \\ F, i \end{bmatrix}$ and $x \in V$.

Definition. (i) $\alpha(X) \equiv \exists Y(\omega_1(Y) \,\&\, Y \prec X)$.
(ii) $\varphi_2(X, Y) \equiv \alpha(X) \,\&\, \tau(Y) \,\&\, \forall Z((\alpha(Z) \,\&\, Z \leqslant Y) \rightarrow Z = X)$.
(iii) $\varphi_3(X, Y) \equiv \varphi_2(X, Y) \,\&\, X \lessdot Y$.
(iv) $\varphi_4(X) \equiv \exists Y \varphi_3(Y, X)$.
(v) For every $n \geqq 1$ put

$$\bar{\alpha}_n(X) \equiv \alpha(X) \,\&\, \exists(X_1, \ldots, X_n)^{\neq} \, (\varphi_3(X, X_1) \,\&\, \ldots \,\&\, \varphi_3(X, X_n)) .$$

Moreover, put $\bar{\alpha}_0(X) \equiv \alpha(X)$.
(vi) For every $n \geqq 0$ put $\alpha_n(X) \equiv \bar{\alpha}_n(X) \,\&\, \neg\bar{\alpha}_{n+1}(X)$.

4.1. Lemma. (i) *$\alpha(X)$ in $\mathscr{F}_\Delta$ iff $X = F^*$ for some $F \in \Delta$.*

(ii) *$\varphi_2(X, Y)$ in $\mathscr{F}_\Delta$ iff $X = F^*$ for some $F \in \Delta$ and $Y = t^*$ for some term t containing no symbol from Δ other than F.*

(iii) *$\varphi_3(X, Y)$ in $\mathscr{F}_\Delta$ iff there is a pair $(F, i) \in \Delta^{(1)}$ such that $X = F^*$ and $Y = (F, i)^*$.*

(iv) *$\varphi_4(X)$ in $\mathscr{F}_\Delta$ iff $X = (F, i)^*$ for some $(F, i) \in \Delta^{(1)}$.*

(v) *Let $n \geqq 0$. Then $\bar{\alpha}_n(X)$ in $\mathscr{F}_\Delta$ iff $X = F^*$ for some $F \in \Delta$ of arity $\geqq n$.*

(vi) *Let $n \geqq 0$. Then $\alpha_n(X)$ in $\mathscr{F}_\Delta$ iff $X = F^*$ for some $F \in \Delta_n$.*

Definition. (i) $\delta_1 \equiv \forall X(\alpha(X) \to \alpha_0(X))$.

(ii) $\delta_2 \equiv \forall X(\alpha(X) \to \alpha_1(X))$.

(iii) $\delta_3 \equiv \exists X, Y(\alpha_1(X) \& \alpha_1(Y) \& X \neq Y)$.

(iv) $\delta_4 \equiv \exists X\, \bar{\alpha}_2(X)$.

(v) $\delta_5 \equiv \delta_3$ VEL δ_4.

4.2. Lemma. (i) *δ_1 in $\mathscr{F}_\Delta$ iff Δ contains only nullary symbols.*

(ii) *δ_2 in $\mathscr{F}_\Delta$ iff Δ is a unary type.*

(iii) *δ_3 in $\mathscr{F}_\Delta$ iff Δ contains at least two different unary symbols.*

(iv) *δ_4 in $\mathscr{F}_\Delta$ iff Δ is strictly large.*

(v) *δ_5 in $\mathscr{F}_\Delta$ iff Δ is large.*

A term t is said to be *balanced* if it contains no nullary symbol from Δ and every variable has at most one occurrence in t.

Definition. (i) $\varphi_5(X) \equiv \tau(X) \& \forall Y(\alpha_0(Y) \to \neg Y \ll X)$.

(ii) $\varphi_6(X) \equiv \varphi_5(X) \& \forall Y((X \prec Y \& \varphi_5(Y)) \to X \lessdot Y)$.

(iii) $\varphi_7(X) \equiv \varphi_5(X) \& \forall Y, Z((Y \prec Z \& Z \ll X) \to Y \lessdot Z)$.

4.3. Lemma. (i) *$\varphi_5(X)$ in $\mathscr{F}_\Delta$ iff $X = t^*$ for some term t containing no nullary symbol.*

(ii) *$\varphi_6(X)$ in $\mathscr{F}_\Delta$ iff $X = t^*$ for some term t containing no nullary symbol and containing a single variable.*

(iii) *$\varphi_7(X)$ in $\mathscr{F}_\Delta$ iff $X = t^*$ for some balanced term t.*

For any term t we define a set $Q(t)$ of terms as follows: if t is either a variable or a nullary symbol from Δ then $Q(t) = V$; if $t = F(t_1, \ldots, t_{n_F})$ where $n_F \geqq 1$, then we take terms $u_1 \in Q(t_1), \ldots, u_{n_F} \in Q(t_{n_F})$ such that the sets var $(u_1), \ldots,$ var (u_{n_F}) are pairwise disjoint and put $Q(t) = \{F(u_1, \ldots, u_{n_F})\}^\sim$. Evidently, $Q(t)$ is a non-empty set of similar balanced terms; the terms $u \in Q(t)$ are just the greatest balanced terms u with the property $u \leqq t$.

For any term t and any variable x define a term $K_x(t)$ as follows: if t is either a variable or a nullary symbol from Δ then $K_x(t) = x$; if $t = F(t_1, \ldots, t_{n_F})$ where $n_F \geqq 1$ then $K_x(t) = F(K_x(t_1), \ldots, K_x(t_{n_F}))$. Moreover, put $K(t) = \{K_x(t);\ x \in V\}$.

Definition. (i) $\varphi_8(X, Y) \equiv \tau(X) \& \varphi_7(Y) \& \forall Z(\varphi_7(Z) \rightarrow (Z \lll X \leftrightarrow Z \lll Y))$.
(ii) $\varphi_9(X, Y) \equiv \varphi_6(Y) \& \exists Z(\varphi_8(X, Z) \& \varphi_8(Y, Z))$.

4.4. Lemma. (i) *$\varphi_8(X, Y)$ in $\mathscr{F}_\Delta$ iff $X = t^*$ for some term t and $Y = u^*$ for some $u \in Q(t)$.*
(ii) *$\varphi_9(X, Y)$ in $\mathscr{F}_\Delta$ iff $X = t^*$ for some term t and $Y = u^*$ for some $u \in K(t)$.*

Definition. $\varphi_{10}(X, Y) \equiv \varphi_7(X) \& X \lessdot Y \& \exists Z_1, Z_2(\varphi_9(X, Z_1) \& \varphi_9(Y, Z_2) \& \& Z_1 \lll Z_2)$.

4.5. Lemma. *$\varphi_{10}(X, Y)$ in $\mathscr{F}_\Delta$ iff there exist two balanced terms t, u and a pair $(F, i) \in \Delta^{(1)}$ such that $X = t^*$, $Y = u^*$ and either $u \in t\begin{bmatrix} 1 \\ F, i \end{bmatrix}$ or $t = sx$, $u = sF(y_1, \ldots, y_{n_F})$ for some $s \in \Delta^{(-)}$ and $x, y_1, \ldots, y_{n_F} \in V$.*

Proof. The converse implication is evident. Let $\varphi_{10}(X, Y)$, $X = t^*$, $Y = u^*$. Evidently t, u are balanced terms and $t \prec u$. Since $t^* \lessdot u^*$, it is enough to consider the case $u = \sigma_v^x(t)$ where $x \in \operatorname{var}(t)$, $v = F(y_1, \ldots, y_{n_F})$, $n_F \geqq 1$ and $y_1, \ldots, y_{n_F}$ are pairwise different variables not contained in $\operatorname{var}(t)$. Put $h = \sigma_v^x$; for every term a put $a' = K_x(a)$. Since $\varphi_{10}(X, Y)$ is satisfied, we have $t' \leqq u'$. There exists a substitution f such that $f(t')$ is a subterm of u'. Evidently, either $f(x) = F(x, \ldots, x)$ and t' contains a single occurrence of x or $f(x) = x$. In the first case $t = sx$ for some $s \in \Delta^{(-)}$ and we are through. Consider the second case; t' is a subterm of u'.

Let us prove by induction on a that if a is a balanced term not containing $y_1, \ldots, y_{n_F}$ and such that a' is a subterm of $(h(a))'$ and $x \in \operatorname{var}(a)$ then $a \in x\begin{bmatrix} p \\ F, i \end{bmatrix}$ for some $p \geqq 0$ and some $i \in \{1, \ldots, n_F\}$. If $a \in V$, this is evident. Let $a = G(a_1, \ldots, a_{n_G})$. There is a unique $j \in \{1, \ldots, n_G\}$ with $x \in \operatorname{var}(a_j)$. We have $G(a'_1, \ldots, a'_{n_G}) = a'$ and a' is a subterm of $(h(a))' = G(a'_1, \ldots, a'_{j-1}, (h(a_j))', a'_{j+1}, \ldots, a'_{n_G})$ and so $G(a'_1, \ldots, a'_{n_G})$ is a subterm of $(h(a_j))'$. Hence a'_j is a subterm of $(h(a_j))'$. By the induction hypothesis, $a_j \in x\begin{bmatrix} p \\ F, i \end{bmatrix}$ for some $p \geqq 0$ and $i \in \{1, \ldots, n_F\}$. Since $G(a'_1, \ldots, a'_{n_G})$ is a subterm of $(h(a_j))'$, we get $G = F$ and $a'_1 = \ldots = a'_{i-1} = a'_{i+1} = \ldots = a'_{n_F} = x$. If $\{a_1, \ldots, a_{n_F}\} \subseteq V$, we get $a \in x\begin{bmatrix} 1 \\ F, j \end{bmatrix}$; in the remaining case we get $i = j$ and $a \in x\begin{bmatrix} p+1 \\ F, i \end{bmatrix}$.

Particularly, $t \in x\begin{bmatrix} p \\ F, i \end{bmatrix}$ for some $p \geqq 0$ and $i \in \{1, \ldots, n_F\}$. But then $u \in x\begin{bmatrix} p+1 \\ F, i \end{bmatrix}^{\sim}$ and so $u \in t\begin{bmatrix} 1 \\ F, i \end{bmatrix}^{\sim}$.

Definition. (i) $\varphi_{11}(X, Y) \equiv \varphi_{10}(X, Y) \,\&\, (\forall U((\alpha(U) \,\&\, U \preccurlyeq X) \to \alpha_1(U)) \,\&$
$\& \,\exists U'(\bar{\alpha}_2(U') \,\&\, U' \preccurlyeq Y)) \to \exists Z, Z', Y'(\varphi_{10}(Y, Y') \,\&\, \bar{\alpha}_2(Z) \,\&\, \varphi_3(Z, Z') \,\&\, Z' \preccurlyeq Y')$.

(ii) $\varphi_{12}(X, Y) \equiv \varphi_{11}(X, Y) \,\&\, ((\forall Z((\alpha(Z) \,\&\, Z \preccurlyeq Y) \to \alpha_1(Z)) \,\&\, \exists Z', X', Y'(\bar{\alpha}_2(Z') \,\&$
$\& \,\varphi_{11}(X, X') \,\&\, \varphi_{11}(Y, Y') \,\&\, Z' \preccurlyeq X' \preccurlyeq Y')) \to \forall U_1, U_2((\alpha(U_1) \,\&\, \alpha(U_2) \,\&\, U_1 \preccurlyeq Y \,\&$
$\& \,U_2 \preccurlyeq Y) \to U_1 = U_2))$.

4.6. Lemma. (i) *$\varphi_{11}(X, Y)$ in $\mathscr{F}_\Delta$ iff there exist two balanced terms t, u and a pair $(F, i) \in \Delta^{(1)}$ such that $X = t^*$, $Y = u^*$ and either $u \in t\begin{bmatrix} 1 \\ F, i \end{bmatrix}$ or $n_F = 1$ and $t = sx$, $u = sFx$ for some $s \in \Delta^{(-)}$ and $x \in V$.*

(ii) *$\varphi_{12}(X, Y)$ in $\mathscr{F}_\Delta$ iff there exist two balanced terms t, u and a pair $(F, i) \in \Delta^{(1)}$ such that $X = t^*$, $Y = u^*$ and either $u \in t\begin{bmatrix} 1 \\ F, i \end{bmatrix}$ or $n_F = 1$, Δ contains no symbol of arity $\geqq 2$ and $t = sx$, $u = sFx$ for some $s \in \Delta^{(-)}$ and $x \in V$.*

Definition. (i) $\varphi_{13}(X, Y) \equiv \bar{\alpha}_1(X) \,\&\, \varphi_7(Y) \,\&\, X \preccurlyeq Y \,\&\, \forall Z_1, Z_2((Z_1 \preccurlyeq Y \,\&$
$\& \,Z_2 \preccurlyeq Y) \to (Z_1 \preccurlyeq Z_2 \text{ VEL } Z_2 \preccurlyeq Z_1))$.

(ii) $\varphi_{14}(X, Y, Z) \equiv \varphi_{13}(X, Y) \,\&\, Y \preccurlyeq Z \,\&\, \forall U_1, U_2((Y \preccurlyeq U_1 \,\&\, U_1 \prec U_2 \,\&\, U_2 \preccurlyeq$
$\preccurlyeq Z) \to \neg U_1 \lessdot U_2)$.

(iii) $\varphi_{15}(X, Y, Z) \equiv \varphi_{14}(X, Y, Z) \,\&\, X \neq Y \,\&\, \exists Y', Z_1, Z_2(Z \lessdot Z_1 \,\&\, Z \lessdot Z_2 \,\&$
$\& \,Z_1 \neq Z_2 \,\&\, \varphi_{13}(X, Y') \,\&\, Y \prec Y' \,\&\, \varphi_{14}(X, Y', Z_1) \,\&\, \varphi_{14}(X, Y', Z_2))$.

(iv) $\varphi_{16}(X, Y) \equiv X \lessdot Y \,\&\, \exists X_1, Y_1(\varphi_{15}(X_1, Y_1, Y) \,\&\, \forall Z((X \lessdot Z \,\&\, \forall Z'((\bar{\alpha}_1(Z') \,\&$
$\& \,Z' \preccurlyeq Z) \to Z' = X_1)) \to \exists Z''(Y \lessdot Z'' \,\&\, Z \lessdot Z'')))$.

4.7. Lemma. (i) *$\varphi_{13}(X, Y)$ in $\mathscr{F}_\Delta$ iff there are $(F, i) \in \Delta^{(1)}$, $x \in V$, $k \geqq 1$ and $t \in x\begin{bmatrix} k \\ F, i \end{bmatrix}$ such that $X = F^*$ and $Y = t^*$.*

(ii) *$\varphi_{14}(X, Y, Z)$ in $\mathscr{F}_\Delta$ iff there are $(F, i) \in \Delta^{(1)}$, $x \in V$, $k \geqq 1$, $t \in x\begin{bmatrix} k \\ F, i \end{bmatrix}$ and a substitution f mapping V into $V \cup \Delta_0$ such that $X = F^*$, $Y = t^*$ and $Z = (f(t))^*$.*

(iii) *$\varphi_{15}(X, Y, Z)$ in $\mathscr{F}_\Delta$ iff there are $(F, i) \in \Delta^{(1)}$, $x \in V$, $k \geqq 2$, $t \in x\begin{bmatrix} k \\ F, i \end{bmatrix}$ and a substitution f such that $f(x) = x$, f maps $V \setminus \{x\}$ into $(V \setminus \{x\}) \cup \Delta_0$, $f(t)$ is not a balanced term and $X = F^*$, $Y = t^*$, $Z = (f(t))^*$.*

(iv) *$\varphi_{16}(X, Y)$ in $\mathscr{F}_\Delta$ iff there are $(F, i) \in \Delta^{(1)}$, $x \in V$, $k \geqq 1$, $t \in x\begin{bmatrix} k \\ F, i \end{bmatrix}$ and a substitution f such that $f(x) = x$, f maps $V \setminus \{x\}$ into $(V \setminus \{x\}) \cup \Delta_0$, $f(t)$ is not a balanced term and $X = (f(t))^*$, $Y = (\sigma^x_{F(y_1, \ldots, y_{n_F})} f(t))^*$ where $y_1, \ldots, y_{n_F}$ are pairwise different variables not contained in* var $(f(t))$.

Proof. The assertions (i), (ii) and (iii) and the converse implication in (iv) are easy. Let $\varphi_{16}(X, Y)$. There exist $(F, i) \in \Delta^{(1)}$, $x \in V$, $l \geqq 2$, $u \in x\begin{bmatrix} l \\ F, i \end{bmatrix}$ and a substitution g such that $g(x) = x$, $g(V \setminus \{x\}) \subseteq (V \setminus \{x\}) \cup \Delta_0$, $g(u)$ is not balanced and $Y = (g(u))^*$. Put $k = l - 1$, so that $k \geqq 1$. We evidently have $X = (f(t))^*$ for some term $t \in x\begin{bmatrix} k \\ F, i \end{bmatrix}$ and some substitution f with $f(x) = x$ and $f(V \setminus \{x\}) \subseteq$ $\subseteq (V \setminus \{x\}) \cup \Delta_0$. Since $g(u)$ is not balanced, $f(t)$ is not balanced and we have $n_F \geqq 2$. Let us fix a $j \in \{1, \ldots, n_F\}$ with $j \neq i$. Evidently, we have either $g(u) \sim \sigma^x_{F(y_1, \ldots, y_{n_F})} f(t)$ for some pairwise different variables $y_1, \ldots, y_{n_F}$ not contained in var $(f(t))$ or $g(u) \in f(t)\begin{bmatrix} 1 \\ F, i \end{bmatrix}^\sim$. It is enough to exclude the second possibility. Suppose $g(u) \in$ $\in f(t)\begin{bmatrix} 1 \\ F, i \end{bmatrix}^\sim$. Let us take a term $a \in f(t)\begin{bmatrix} 1 \\ F, j \end{bmatrix}$. Since $\varphi_{16}(X, Y)$, there exists a term b such that $(g(u))^* \lessdot b^*$ and $a^* \lessdot b^*$.

Suppose first that $b \in g(u)\begin{bmatrix} 1 \\ G, s \end{bmatrix}^\sim$ for some G, s. There exists a substitution h such that $h(a)$ is a subterm of b; evidently $h(f(t)) = g(u) \in f(t)\begin{bmatrix} 1 \\ F, i \end{bmatrix}^\sim$. It is easy to prove by induction on w that if w is a term such that some substitution maps w onto a term from $w\begin{bmatrix} 1 \\ F, i \end{bmatrix}$ then $w \in x\begin{bmatrix} p \\ F\ i, \end{bmatrix}^\sim$ for some $p \geqq 0$. Hence $f(t)$ is a balanced term, a contradiction, since $(f(t))^* \lessdot (g(u))^*$ and $g(u)$ is not balanced.

Now suppose that $b \in \begin{bmatrix} 1 \\ G, s \end{bmatrix}^\sim$ for some G, s. There exists a substitution h such that $h(g(u))$ is a subterm of b. Evidently $h(f(t)) = a \in f(t)\begin{bmatrix} 1 \\ F, j \end{bmatrix}$; hence it follows similarly as above that $f(t) \in x\begin{bmatrix} p \\ F, j \end{bmatrix}$ for some $p \geqq 0$, a contradiction.

Finally, suppose that $b \sim \sigma^y_v\, g(u)$ and $b \sim \sigma^z_w(a)$ for some y, z and $v =$ $= G(z_1, \ldots, z_{n_G})$, $w = H(z'_1, \ldots, z'_{n_H})$. We have $g(u) \sim F(y_1, \ldots, y_{i-1}, f(t),$ $y_{i+1}, \ldots, y_{n_F})$ for some $y_1, \ldots, y_{n_F}$, $a = F(y'_1, \ldots, y'_{j-1}, f(t), y'_{j+1}, \ldots, y'_{n_F})$ for some $y'_1, \ldots, y'_{n_F}$, $\sigma^y_v(y_j) \sim \sigma^z_w f(t)$, $\sigma^y_v(y_j) \notin V$, $y = y_j$, $G(z_1, \ldots, z_{n_G}) \sim \sigma^z_w f(t)$, evidently a contradiction.

Definition. $\varphi_{17}(X, Y) \equiv X \lessdot Y \,\&\, \neg\varphi_{16}(X, Y) \,\&\, \exists X', Y'(\varphi_8(X, X') \,\&\, \varphi_8(Y, Y') \,\&$ $\&\, \varphi_{12}(X', Y'))$.

4.8. Lemma. *$\varphi_{17}(X, Y)$ in $\mathscr{F}_\Delta$ iff there exist two terms t, u and a pair $(F, i) \in \Delta^{(1)}$ such that $X = t^*$, $Y = u^*$ and either $u \in t\begin{bmatrix} 1 \\ F, i \end{bmatrix}$ or $n_F = 1$, Δ contains no symbol of arity $\geqq 2$ and $t = sx$, $u = sFx$ for some $s \in \Delta^{(-)}$ and $x \in V$.*

Proof. The converse implication is easy (it follows from 4.7 that we cannot have $\varphi_{16}(X, Y)$). Let $\varphi_{17}(X, Y)$ and suppose that the assertion is false. We have $X = t^*$, $Y = u^*$ and $u = \sigma_F^x(t)$ for some non-balanced terms t, u, some $x \in V$ and some non-nullary symbol $F \in \Delta$. There exist terms $t' \in Q(t)$ and $u' \in Q(u)$ with $u' \in t'\begin{bmatrix} 1 \\ G, i \end{bmatrix}$ for some G, i. Evidently $G = F$ and x has exactly one occurrence in t. It is easy to prove by induction on a that if a is a term containing a single occurrence of x and such that $Q(\sigma_F^x(a)) = Q(F(y_1, \ldots, y_{i-1}, a, y_{i+1}, \ldots, y_{n_F}))$ (where $y_1, \ldots, y_{n_F}$ are pairwise different variables not contained in var (a)), then $a = f(b)$ for some $b \in x\begin{bmatrix} p \\ F, i \end{bmatrix}$ with $p \geqq 0$ and some substitution f such that $f(x) = x$ and $f(V \setminus \{x\}) \subseteq \subseteq (V \setminus \{x\}) \cup \Delta_0$. In particular, $t = f(b)$ for some b and f with these properties. We get $\varphi_{16}(X, Y)$, a contradiction.

Definition. $\varphi_{18}(X, Y) \equiv \varphi_{17}(X, Y) \,\&\, ((\neg\, \exists Z\bar{\alpha}_2(Z)) \rightarrow \forall U, X', Y'((\alpha_0(U) \,\&\, \&\, \varphi_8(X', X) \,\&\, \varphi_8(Y', Y) \,\&\, U \ll X' \,\&\, U \ll Y') \rightarrow X' \prec Y'))$.

4.9. Lemma. *$\varphi_{18}(X, Y)$ in $\mathscr{F}_\Delta$ iff there exist two terms t, u and a pair $(F, i) \in \Delta^{(1)}$ such that $X = t^*$, $Y = u^*$ and either $u \in t\begin{bmatrix} 1 \\ F, i \end{bmatrix}$ or Δ contains only unary symbols and $t = sx$, $u = sFx$ for some $s \in \Delta^{(-)}$ and $x \in V$.*

Definition. $\varphi_{19}(X, Y) \equiv X \ll Y \,\&\, \forall Z_1, Z_2((X \ll Z_1 \ll Y \,\&\, X \ll Z_2 \ll Y) \rightarrow \rightarrow (Z_1 \ll Z_2 \text{ VEL } Z_2 \ll Z_1))$.

4.10. Lemma. *$\varphi_{19}(X, Y)$ in $\mathscr{F}_\Delta$ iff $X = a^*$ and $Y = b^*$ for some terms a, b such that $a \leqq b$ and whenever $a \leqq c \leqq b$ and $a \leqq d \leqq b$ then either $c \leqq d$ or $d \leqq c$.*

4.11. Lemma. *Let a, b be two terms such that $a \parallel b$ (i.e. neither $a \leqq b$ nor $b \leqq a$). Let $n > \lambda_0(a)$, $(F, i) \in \Delta^{(1)}$, $t \in b\begin{bmatrix} n \\ F, i \end{bmatrix}$. Then we do not have $\varphi_{19}(a^*, t^*)$ in $\mathscr{F}_\Delta$.*

Proof. Suppose $\varphi_{19}(a^*, t^*)$. Denote by m the least non-negative integer such that $a \leqq t_1$ for some $t_1 \in b\begin{bmatrix} m \\ F, i \end{bmatrix}$. We have $1 \leqq m \leqq n$ and $t_1 = F(y_1, \ldots, y_{i-1}, t_2, y_{i+1}, \ldots, y_{n_F})$ for some $t_2 \in b\begin{bmatrix} m-1 \\ F, i \end{bmatrix}$ and pairwise different variables $y_1, \ldots, y_{n_F}$ not contained in var (t_2). Let $a_0 \in a\begin{bmatrix} 1 \\ F, i \end{bmatrix}$, $a_0 = F(z_1, \ldots, z_{i-1}, a, z_{i+1}, \ldots, z_{n_F})$. If it were $a_0 \leqq t_1$ then evidently $a \leqq t_2$, a contradiction with the minimality of m.

If it were $t_1 \leqq a_0$ then $a < t_1 \leqq a_0$ and so $t_1 \sim a_0$, since $a \prec a_0$; hence $t_2 \sim a$, a contradiction. We have proved $a_0 \parallel t_1$. Since $a \leqq t_1 \leqq t$ and $a \leqq a_0$, this implies $a_0 \nleqq t$. Hence $m = n$. Since $n > \lambda_0(a)$, it follows that $a \in x\begin{bmatrix} k \\ F, i \end{bmatrix}$ for some $x \in V$ and $k = \lambda_0(a)$. Hence $a \leqq t_3$ for some $t_3 \in b\begin{bmatrix} k \\ F, i \end{bmatrix}$. Consequently $m \leqq k$, a contradiction with $m = n$ and $k = \lambda_0(a)$.

Definition. (i) $\varphi_{20}(X, Y) \equiv \forall Z(\varphi_1(Z, X) \to \exists!!\, Z'(\varphi_1(Z', Y) \mathbin{\&} Z \ll Z')) \mathbin{\&} \mathbin{\&} \forall U(\varphi_1(U, Y) \to \exists!!\, U'(\varphi_1(U', X) \mathbin{\&} U' \ll U))$.

(ii) $\varphi_{21}(X, Y) \equiv \forall Z(\varphi_1(Z, X) \to \exists!!\, Z'(\varphi_1(Z', Y) \mathbin{\&} \varphi_{19}(Z, Z'))) \mathbin{\&} \forall U(\varphi_1(U, Y) \to \to \exists!!\, U'(\varphi_1(U', X) \mathbin{\&} \varphi_{19}(U', U)))$.

(iii) $\varphi_{22}(X, Y) \equiv \exists X_1, X_2, X_3, X_4, Y_1, Y_2, Y_3, Y_4, Z(\varphi_{21}(X, X_1) \mathbin{\&} \varphi_{21}(X_1, X_2) \mathbin{\&} \mathbin{\&} \varphi_{21}(X_2, X_3) \mathbin{\&} \varphi_{21}(X_3, X_4) \mathbin{\&} \varphi_{21}(Y, Y_1) \mathbin{\&} \varphi_{21}(Y_1, Y_2) \mathbin{\&} \varphi_{21}(Y_2, Y_3) \mathbin{\&} \varphi_{21}(Y_3, Y_4) \mathbin{\&} \mathbin{\&} \varphi_{20}(Z, X_4) \mathbin{\&} \varphi_{20}(Z, Y_4))$.

4.12. Lemma. *Let Δ be a large type. Let $X, Y \in \mathscr{F}_\Delta$ be such that the sets $\bar{I}(X), \bar{I}(Y)$ are finite. Then $\varphi_{22}(X, Y)$ in $\mathscr{F}_\Delta$ iff* Card $(\bar{I}(X)) =$ Card $(\bar{I}(Y))$.

Proof. The direct implication is obvious. Let Card $(\bar{I}(X)) =$ Card $(\bar{I}(Y)) = k$, $\bar{I}(X) = \{a_1, \ldots, a_k\}$, $\bar{I}(Y) = \{b_1, \ldots, b_k\}$. Let $n > \operatorname{Max}(\lambda_0(a_1), \ldots, \lambda_0(a_k), \lambda_0(b_1), \ldots \ldots, \lambda_0(b_k))$. Since Δ is large, there exist two different pairs (F, i), (G, j) in $\Delta^{(1)}$. Put

$$X_1 = \{c_1, \ldots, c_k\}^* \quad \text{where} \quad c_m \in a_m\begin{bmatrix} n \\ F, i \end{bmatrix} \quad \text{for all } m\,,$$

$$X_2 = \{d_1, \ldots, d_k\}^* \quad \text{where} \quad d_m \in c_m\begin{bmatrix} 1 \\ G, j \end{bmatrix},$$

$$X_3 = \{e_1, \ldots, e_k\}^* \quad \text{where} \quad e_m \in d_m\begin{bmatrix} 2n + m \\ F, i \end{bmatrix},$$

$$X_4 = \{f_1, \ldots, f_k\}^* \quad \text{where} \quad f_m \in e_m\begin{bmatrix} 1 \\ G, j \end{bmatrix},$$

$$Y_1 = \{\bar{c}_1, \ldots, \bar{c}_k\}^* \quad \text{where} \quad \bar{c}_m \in b_m\begin{bmatrix} n \\ F, i \end{bmatrix},$$

$$Y_2 = \{\bar{d}_1, \ldots, \bar{d}_k\}^* \quad \text{where} \quad \bar{d}_m \in \bar{c}_m\begin{bmatrix} 1 \\ G, j \end{bmatrix},$$

$$Y_3 = \{\bar{e}_1, \ldots, \bar{e}_k\}^* \quad \text{where} \quad \bar{e}_m \in \bar{d}_m\begin{bmatrix} 2n + m \\ F, i \end{bmatrix},$$

$$Y_4 = \{\bar{f}_1, \ldots, \bar{f}_k\}^* \quad \text{where} \quad \bar{f}_m \in \bar{e}_m\begin{bmatrix} 1 \\ G, j \end{bmatrix},$$

$$Z = \{g_1, \ldots, g_k\}^* \quad \text{where} \quad g_m \in F(x_1, \ldots, x_{n_F})\begin{bmatrix} 1 \\ G, j \end{bmatrix}\begin{bmatrix} 2n + m \\ F, i \end{bmatrix}\begin{bmatrix} 1 \\ G, j \end{bmatrix}.$$

$\left(\text{If } T \text{ is a set of terms then } T\begin{bmatrix} p \\ F, i \end{bmatrix} \text{ denotes the set } \left\{t\begin{bmatrix} p \\ F, i \end{bmatrix};\ t \in T\right\}.\right)$ Using 3.2 and 4.11, it is easy to verify $\varphi_{22}(X, Y)$.

Definition. $\varphi_{23}(X, Y, Z) \equiv \varphi_{19}(X, Y) \mathbin{\&} \forall Z_1(\varphi_1(Z_1, Z) \rightarrow X \ll Z_1) \mathbin{\&} \forall X_1(X \ll \ll X_1 \ll Y \rightarrow \exists!!\, Z_1(\varphi_1(Z_1, Z) \mathbin{\&} \forall X_2(X \ll X_2 \ll Y \rightarrow (X_2 \ll Z_1 \leftrightarrow X_2 \ll X_1))))$.

4.13. Lemma. *Let Δ be a large type. Let $X, Y \in \mathscr{F}_\Delta$ be such that $\varphi_{19}(X, Y)$, $\neg\alpha(X)$, $X \neq W_\Delta$. Then there exists a $Z \in \mathscr{F}_\Delta$ such that $\varphi_{23}(X, Y, Z)$ is satisfied in $\mathscr{F}_\Delta$.*

Proof. We have $X = a^*$ and $Y = b^*$ for some terms a, b such that if $x \in V$ then neither $a = x$ nor $x \prec a$. There are terms $a = a_0 \prec a_1 \prec \ldots \prec a_k = b$ such that any term u with $a \leqq u \leqq b$ is similar to some term from $\{a_0, \ldots, a_k\}$. For every $j \in \{0, \ldots, k\}$ put $d_j = \lambda_0(a_j)$; we have $d_0 \leqq d_1 \leqq \ldots \leqq d_k$. Put $b_k = b$. Moreover, for every $j \in \{0, \ldots, k-1\}$ we shall define a term b_j as follows.

Consider first the case $a_{j+1} \in a_j\begin{bmatrix} 1 \\ F, i \end{bmatrix}^\sim$ for some $(F, i) \in \Delta^{(1)}$. Since Δ is large, there exists a pair $(G, i_0) \in \Delta^{(1)}$ different from (F, i); let b_j be any term from $a_j\begin{bmatrix} d_k - d_j + k - j \\ G, i_0 \end{bmatrix}$.

Now consider the case $a_{j+1} \sim \sigma_F^x(a_j)$ for some $x \in \operatorname{var}(a_j)$ and $F \in \Delta$. Evidently, there exists a pair $(G, i) \in \Delta^{(1)}$ such that $a_j \notin x\begin{bmatrix} p \\ G, i \end{bmatrix}^\sim$ for any $p \geqq 0$; let b_j be any term from $a_j\begin{bmatrix} d_k - d_j + k - j \\ G, i \end{bmatrix}$.

Finally, consider the case $a_{j+1} \sim \sigma_y^x(a_j)$ for some $x, y \in \operatorname{var}(a_j)$ with $x \neq y$. Take any pair $(F, i) \in \Delta^{(1)}$ and let b_j be any term from $a_j\begin{bmatrix} d_k - d_j + k - j \\ F, i \end{bmatrix}$.

Evidently, $a_j \leqq b_j$. Let us prove $a_{j+1} \nleqq b_j$. In the first and the last cases it is evident. Consider the case $a_{j+1} = \sigma_F^x(a_j)$. If $F \neq G$, it is evident that $a_{j+1} \nleqq b_j$. Let $F = G$. It is easy to prove by induction on t that if t is a term and there exists a substitution h such that $h(t) \in t\begin{bmatrix} p \\ F, i \end{bmatrix}$ for some $p \geqq 1$ then $t \in x\begin{bmatrix} q \\ F, i \end{bmatrix}^\sim$ for some $q \geqq 0$. If $a_{j+1} \leqq b_j$ then $f(a_{j+1})$ is a subterm of b_j for some substitution f; evidently $f(a_{j+1}) = f\,\sigma_F^x(a_j) \in a_j\begin{bmatrix} p \\ F, i \end{bmatrix}$ for some p, so that $a_j \in x\begin{bmatrix} q \\ F, i \end{bmatrix}^\sim$ for some q, a contradiction.

Let us prove that if $j_1, j_2 \in \{0, \ldots, k\}$ and $j_1 < j_2$ then $b_{j_1} \parallel b_{j_2}$. By the above proved, $b_{j_2} \nleqq b_{j_1}$. Since $\lambda_0(b_{j_1}) = d_k + k - j_1 > d_k + k - j_2 = \lambda_0(b_{j_2})$, we cannot have $b_{j_1} \leqq b_{j_2}$.

Put $Z = \{b_0, \ldots, b_k\}^*$. Now it is evident that $\varphi_{23}(X, Y, Z)$ is satisfied in $\mathscr{F}_\Delta$.

Definition. $\varphi_{24}(X_1, Y_1, X_2, Y_2) \equiv \varphi_{19}(X_1, Y_1) \,\&\, \varphi_{19}(X_2, Y_2) \,\&\, ((X_1 = Y_1 \,\&\, X_2 = Y_2)$ VEL $(X_1 \prec Y_1 \,\&\, X_2 \prec Y_2)$ VEL $\exists A_1, A_2, B_1, B_2, C_1, C_2(X_1 \prec A_1 \,\&\, A_1 \prec B_1 \,\&\, B_1 \leqslant Y_1 \,\&\, X_2 \prec A_2 \,\&\, A_2 \prec B_2 \,\&\, B_2 \leqslant Y_2 \,\&\, \varphi_{23}(B_1, Y_1, Z_1) \,\&\, \varphi_{23}(B_2, Y_2, Z_2) \,\&\, \varphi_{22}(Z_1, Z_2)))$.

4.14. Lemma. *Let Δ be a large type. Then $\varphi_{24}(X_1, Y_1, X_2, Y_2)$ in $\mathscr{F}_\Delta$ iff there are $n \geqq 0$ and terms $a_0 \prec a_1 \prec \ldots \prec a_n$, $b_0 \prec b_1 \prec \ldots \prec b_n$ such that $X_1 = a_0^*$, $Y_1 = a_n^*$, $X_2 = b_0^*$, $Y_2 = b_n^*$, every term u with $a_0 \leqq u \leqq a_n$ is similar to some a_j and every term v with $b_0 \leqq v \leqq b_n$ is similar to some b_j.*

Definition. (i) $\varphi_{25}(X, X', U, Y, Z) \equiv \bar{\alpha}_2(X) \,\&\, \alpha(U) \,\&\, X \neq U \,\&\, \varphi_3(X, X') \,\&\, \varphi_{13}(X, Y) \,\&\, X' \leqslant Y \,\&\, Y \prec Z \,\&\, U \leqslant Z \,\&\, \neg\varphi_{18}(Y, Z)$.

(ii) $\varphi_{26}(X, X', U, Y, Z) \equiv \varphi_{25}(X, X', U, Y, Z) \,\&\, \forall Y', Z'((\varphi_{25}(X, X', U, Y', Z') \,\&\, Z \lessdot Z') \to \varphi_{18}(Z, Z'))$.

(iii) $\varphi_{27}(X, X', U, Y, Z) \equiv \bar{\alpha}_2(X) \,\&\, \varphi_3(X, X') \,\&\, \varphi_{13}(X, Y) \,\&\, X' \leqslant Y \,\&\, Y \prec Z \,\&\, (X = U \to \varphi_{13}(X, Z)) \,\&\, (X \neq U \to \varphi_{26}(X, X', U, Y, Z))$.

(iv) $\varphi_{28}(X, X', Y, Z) \equiv \varphi_3(X, X') \,\&\, \varphi_{18}(Y, Z) \,\&\, (\omega_1(Y) \to X = Z) \,\&\, \forall A((\neg\omega_1(Y) \,\&\, \varphi_{13}(X, A) \,\&\, X' \leqslant A) \to \exists B, C, D, E(\omega_1(E) \,\&\, \varphi_{24}(E, A, Y, B) \,\&\, A \leqslant B \,\&\, Z \leqslant B \,\&\, (\bar{\alpha}_2(X) \to (D \leqslant B \,\&\, \varphi_{27}(X, X', C, A, D))) \,\&\, \forall Z_1, Z_2((Y \leqslant Z_1 \,\&\, Z_1 \prec Z_2 \,\&\, Z_2 \leqslant B) \to \varphi_{18}(Z_1, Z_2))))$.

4.15. Lemma. (i) *$\varphi_{25}(X, X', U, Y, Z)$ in $\mathscr{F}_\Delta$ iff there are $(F, i) \in \Delta^{(1)}$, $G \in \Delta$, $x \in V$, $k \geqq 2$, $t \in x\begin{bmatrix} k \\ F, i \end{bmatrix}$ such that $n_F \geqq 2$, $F \neq G$, $X = F^*$, $X' = (F, i)^*$, $U = G^*$, $Y = t^*$ and $Z = (\sigma_G^y(t))^*$ for some $y \in \mathrm{var}\,(t)$.*

(ii) *$\varphi_{26}(X, X', U, Y, Z)$ in $\mathscr{F}_\Delta$ iff there are $(F, i) \in \Delta^{(1)}$, $G \in \Delta$, $x \in V$, $k \geqq 2$, $t \in x\begin{bmatrix} k \\ F, i \end{bmatrix}$ such that $n_F \geqq 2$, $F \neq G$, $X = F^*$, $X' = (F, i)^*$, $U = G^*$, $Y = t^*$ and $Z = (\sigma_G^x(t))^*$.*

(iii) *$\varphi_{27}(X, X', U, Y, Z)$ in $\mathscr{F}_\Delta$ iff there are $(F, i) \in \Delta^{(1)}$, $G \in \Delta$, $x \in V$, $k \geqq 2$, $t \in x\begin{bmatrix} k \\ F, i \end{bmatrix}$ such that $n_F \geqq 2$, $X = F^*$, $X' = (F, i)^*$, $U = G^*$, $Y = t^*$ and $Z = (\sigma_G^x(t))^*$.*

(iv) *Let Δ be a large type. Then $\varphi_{28}(X, X', Y, Z)$ in $\mathscr{F}_\Delta$ iff there are $(F, i) \in \Delta^{(1)}$ and $t, u \in W_\Delta$ such that $X = F^*$, $X' = (F, i)^*$, $Y = t^*$, $Z = u^*$ and either $u \in t\begin{bmatrix} 1 \\ F, i \end{bmatrix}$ or Δ is a unary type and $t = sx$, $u = sFx$ for some $s \in \Delta^{(-)}$ and $x \in V$.*

Proof. (i) is evident.

(ii) Let $\varphi_{26}(X, X', U, Y, Z)$, $X = F^*$, $X' = (F, i)^*$, $U = G^*$, $Y = t^*$, $t \in x\begin{bmatrix} k \\ F, i \end{bmatrix}$, $Z = (\sigma_G^y(t))^*$. We must prove $y = x$. If $y \neq x$, put $Y' = (\sigma_F^x(t))^*$ and $Z' = (\sigma_F^x \sigma_G^y(t))^*$; we evidently have $\varphi_{25}(X, X', U, Y', Z')$ and $Z \lessdot Z'$, but not $\varphi_{18}(Z, Z')$, a contradiction. The converse is easy.

(iii) is evident.

(iv) The converse implication is easy $\left(\text{if } A = a^* \text{ where } a \in x\begin{bmatrix} k \\ F, i \end{bmatrix}, \text{ put } B = b^* \text{ where } b \in t\begin{bmatrix} k \\ F, i \end{bmatrix}\right)$. Now let $\varphi_{28}(X, X', Y, Z)$, $X = F^*$, $X' = (F, i)^*$, $Y = t^*$, $Z = u^*$. Everything is evident if $t \in V$. Let $t \notin V$. Take a $k > \lambda_0(u)$, a variable x and a term $a \in x\begin{bmatrix} k \\ F, i \end{bmatrix}$. Since $\varphi_{28}(X, X', Y, Z)$ is satisfied, there exist a finite sequence $b_0, \ldots, b_k$ of terms and a finite sequence $(G_1, i_1), \ldots, (G_k, i_k)$ of pairs from $\Delta^{(1)}$ such that $b_0 \prec b_1 \prec \ldots \prec b_k$, any term v with $b_0 \leqq v \leqq b_k$ is similar to some term from $\{b_0, \ldots, b_k\}$, $b_0 = t$, $b_1 = u$, $a \leqq b_k$ and if $j \in \{1, \ldots, k\}$ then either $b_j \in b_{j-1}\begin{bmatrix} 1 \\ G_j, i_j \end{bmatrix}$ or Δ is unary, $x \in \operatorname{var}(b_{j-1})$ and $b_j = \sigma_{G_j(x)}^x(b_{j-1})$; moreover, if $n_F \geqq 2$ then there exists a symbol $H \in \Delta$ such that $\sigma_H^x(a) \leqq b_k$.

Consider first the case when Δ is unary. We have $t = sx$ for some $s \in \Delta^{(-)}$ and $x \in V$; there is a $K \in \Delta$ such that either $u = Ksx$ or $u = sKx$; it is enough to prove $K = F$ in the first case, since in the case $u = sKx$ we could prove $K = F$ analogously. If s contains no symbols other than K, then $K = F$ is evident. Let s contain other symbols than K. Then $b_1 = Ksx$, $b_2 = K_2Ksx$ for some $K_2 \in \Delta, \ldots, b_k = K_k \ldots \ldots K_2Ksx$ for some $K_2, \ldots, K_k \in \Delta$. Since $F^k x \leqq K_k \ldots K_2Ksx$ and k is greater than the length of s, we get $K = F$.

Now consider the case when Δ is not unary. Then $b_j \in b_{j-1}\begin{bmatrix} 1 \\ G_j, i_j \end{bmatrix}$ for all j. If $n_F = 1$, it is easy to prove $G_1 = F$, so that $u \in t\begin{bmatrix} 1 \\ F, 1 \end{bmatrix}$. Let $n_F \geqq 2$. Since $\sigma_H^x(a) \leqq \leqq b_k$, there exists a substitution f such that $f(a)$ is a subterm of b_k and $f(x) \notin V$. Now it is evident that $f(x)$ is a subterm of t and there exists a $p \in \{0, \ldots, k-1\}$ such that $t = f(a')$ where a' is the subterm of a belonging to $x\begin{bmatrix} p \\ F, i \end{bmatrix}$; we have $b_1 = f(a'')$ where a'' is the subterm of a belonging to $x\begin{bmatrix} p+1 \\ F, i \end{bmatrix}$. Hence $(G_1, i_1) = (F, i)$ and $u \in t\begin{bmatrix} 1 \\ F, i \end{bmatrix}$.

Definition. $\varphi_{29}(X, Y, Z) \equiv (\delta_5 \,\&\, \exists X'\, \varphi_{28}(X', X, Y, Z))$ VEL $(\neg\delta_5 \,\&\, \varphi_4(X) \,\&\, \& \, Y \lessdot Z)$.

4.16. Lemma. *$\varphi_{29}(X, Y, Z)$ in $\mathscr{F}_\Delta$ iff there are $(F, i) \in \Delta^{(1)}$ and $t, u \in W_\Delta$ such that $X = (F, i)^*$, $Y = t^*$, $Z = u^*$ and either $u \in t\begin{bmatrix} 1 \\ F, i \end{bmatrix}$ or Δ is a unary type and $t = sx$, $u = sFx$ for some $s \in \Delta^{(-)}$ and $x \in V$.*

Definition. $\varphi_{30}(X, X', Y, Z, U) \equiv \varphi_3(X, X') \,\&\, \varphi_{13}(X, Y) \,\&\, X' \preccurlyeq Y \,\&\, \exists A(\omega_1(A) \,\&$ $\&\, \varphi_{24}(A, Y, Z, U)) \,\&\, \forall B, C((Z \preccurlyeq B \,\&\, B \prec C \,\&\, C \preccurlyeq U) \to \varphi_{28}(X, X', B, C))$.

4.17. Lemma. *Let Δ be a large type. Then $\varphi_{30}(X, X', Y, Z, U)$ in $\mathscr{F}_\Delta$ iff there are $(F, i) \in \Delta^{(1)}$, $k \geqq 2$, $x \in V$, $a \in x\begin{bmatrix} k \\ F, i \end{bmatrix}$ and $t, u \in W_\Delta$ such that $X = F^*$, $X' = (F, i)^*$, $Y = a^*$, $Z = t^*$, $U = u^*$ and either $u \in t\begin{bmatrix} k \\ F, i \end{bmatrix}$ or Δ is unary and $t = sx$, $u = sF^k x$ for some $s \in \Delta^{(-)}$.*

Definition. $\varphi_{31}(X, Y) \equiv X \preccurlyeq Y \,\&\, ((\neg\delta_5 \,\&\, \exists A(\alpha_0(A) \,\&\, A \preccurlyeq X)) \to X = Y) \,\&$ $\&\, (\delta_5 \to \forall X_1, X_1', X_2, B, C, D, C', D'((\varphi_{30}(X_1, X_1', B, X, C) \,\&\, \varphi_{30}(X_1, X_1', B, Y, D) \,\&$ $\&\, \varphi_{29}(X_2, C, C') \,\&\, \varphi_{29}(X_2, D, D')) \to C' \preccurlyeq D'))$.

4.18. Lemma. *Let Δ be not a large unary type. Then $\varphi_{31}(X, Y)$ in $\mathscr{F}_\Delta$ iff $X = t^*$ and $Y = (f(t))^*$ for some term t and substitution f.*

Proof. The converse implication is evident. Let $\varphi_{31}(X, Y)$, $X = t^*$, $Y = u^*$; we have $t \leqq u$. If Δ is not large, it is evident that $u = f(t)$ for some substitution f. Let Δ be large (and not unary). There exist two different pairs (F, i), (G, j) in $\Delta^{(1)}$. Let us take an integer k such that $k \geqq 2$ and $k > \lambda_0(u)$; let $c \in t\begin{bmatrix} k \\ F, i \end{bmatrix}$, $d \in u\begin{bmatrix} k \\ F, i \end{bmatrix}$, $c' \in c\begin{bmatrix} 1 \\ G, j \end{bmatrix}$, $d' \in d\begin{bmatrix} 1 \\ G, j \end{bmatrix}$. Since $\varphi_{31}(X, Y)$, we have $c' \leqq d'$. There exists a substitution f such that $f(c')$ is a subterm of d'. Since $k > \lambda_0(u)$, $f(c')$ is not a subterm of u. Since $(F, i) \neq (G, j)$, $f(c')$ is not a subterm of d. Hence $f(c') = d'$ and so $f(t) = u$.

Definition. $\varphi_{32}(X, Y, Z) \equiv (\neg\delta_4 \,\&\, \varphi_{29}(X, Y, Z))$ VEL $(\delta_4 \,\&\, \varphi_4(X) \,\&\, \tau(Y) \,\&\, \tau(Z) \,\&$ $\&\, \forall A(\tau(A) \to (\varphi_{31}(A, Y) \leftrightarrow \exists B(\varphi_{29}(X, A, B) \,\&\, \varphi_{31}(B, Z)))))$.

4.19. Lemma. *$\varphi_{32}(X, Y, Z)$ in $\mathscr{F}_\Delta$ iff there are $(F, i) \in \Delta^{(1)}$ and terms $a_1, \ldots, a_{n_F}$ such that $X = (F, i)^*$, $Y = a_i^*$ and either $Z = (F(a_1, \ldots, a_{n_F}))^*$ or Δ is unary and $Z = (\sigma^x_{F(x)}(a_1))^*$ where x is the variable contained in a_1.*

5. FINITE SEQUENCES OF TERMS AND THE CONSEQUENCE RELATION; THE CASE OF A LARGE BUT NOT STRICTLY LARGE TYPE

Denote by $\Delta^{(4)}$ the set of quadruples (F, G, w, x) such that $F, G \in \Delta_1$, $F \neq G$, $w \in \Delta^{(-)}$, $x \in V$ and either $w = GF$ or Δ is unary and $w = FG$.

Definition. $\varphi_{33}(X_1, X_2, Y) \equiv \alpha_1(X_1) \,\&\, \alpha_1(X_2) \,\&\, X_1 \neq X_2 \,\&\, \exists X_2' \, \varphi_{28}(X_2, X_2', X_1, Y)$.

5.1. Lemma. *$\varphi_{33}(X_1, X_2, Y)$ in $\mathscr{F}_\Delta$ iff there is a quadruple $(F, G, w, x) \in \Delta^{(4)}$ such that $X_1 = F^*$, $X_2 = G^*$, $Y = (wx)^*$.*

Definition. (i) $\varphi_{34}(X_1, X_2, Y, X_1', X_2', Z) \equiv \varphi_{33}(X_1, X_2, Y) \,\&\, \varphi_{13}(X_1, X_1') \,\&$
$\& \, \varphi_{13}(X_2, X_2') \,\&\, X_1 \neq X_1' \,\&\, X_2 \neq X_2' \,\&\, Y \leqslant Z \,\&\, \exists A \, \varphi_{30}(X_2, A, X_2', X_1', Z)$.

(ii) $\varphi_{35}(X_1, X_2, Y, Z) \equiv \exists X_1', X_2' \, \varphi_{34}(X_1, X_2, Y, X_1', X_2', Z)$.

(iii) $\varphi_{36}(X_1, X_2, Y, A, B) \equiv \varphi_{33}(X_1, X_2, Y) \,\&\, A \leqslant B \,\&\, \varphi_7(B) \,\&$
$\& \, \forall X_1', X_2', Z \, \exists X_1'', X_2'', A', A'', B', B''(\varphi_{34}(X_1, X_2, Y, X_1', X_2', Z) \rightarrow$
$\rightarrow (\varphi_{30}(X_1, X_1'', X_1', A, A') \,\&\, \varphi_{30}(X_2, X_2'', X_2', A', A'') \,\&\, \varphi_{30}(X_1, X_1'', X_1', B, B') \,\&$
$\& \, \varphi_{30}(X_2, X_2'', X_2', B', B'') \,\&\, Z \leqslant A'' \,\&\, Z \leqslant B'' \,\&\, A'' \leqslant B'' \,\&\, (\varphi_{35}(X_1, X_2, Y, A'') \rightarrow$
$\rightarrow \varphi_2(X_1, A)) \,\&\, (\varphi_{35}(X_1, X_2, Y, B'') \rightarrow \varphi_2(X_1, B))))$.

(iv) $\varphi_{37}(X_1, X_2, Y, A, B) \equiv \varphi_{33}(X_1, X_2, Y) \,\&\, \exists Z(\varphi_{33}(X_1, X_2, Z) \,\&\, Y \neq Z \,\&$
$\& \, \varphi_{36}(X_1, X_2, Z, A, B))$.

(v) $\varphi_{38}(X_1 X_2, Y, X, A, B) \equiv \varphi_{33}(X_1, X_2, Y) \,\&\, \exists X'(X \prec X' \,\&\, \varphi_{29}(X', A, B)) \,\&$
$\& \, (\delta_2 \rightarrow (\varphi_{36}(X_1, X_2, Y, X, B) \,\&\, \varphi_{37}(X_1, X_2, Y, A, B)))$.

(vi) $\varphi_{39}(X_1, X_2, Y, X, X', A, B) \equiv \varphi_{33}(X_1, X_2, Y) \,\&\, \exists X'' \, \varphi_{30}(X, X'', X', A, B) \,\&$
$\& \, (\delta_2 \rightarrow (\varphi_{36}(X_1, X_2, Y, X', B) \,\&\, \varphi_{37}(X_1, X_2, Y, A, B)))$.

5.2. Lemma. *Let Δ be a large but not strictly large type. Then:*

(i) $\varphi_{34}(X_1, X_2, Y, X_1', X_2', Z)$ in $\mathscr{F}_\Delta$ iff there are $(F, G, w, x) \in \Delta^{(4)}$ and $n, m \geqq 2$ such that $X_1 = F^*$, $X_2 = G^*$, $Y = (wx)^*$, $X_1' = (F^n x)^*$, $X_2' = (G^m x)^*$ and either $w = GF$, $Z = (G^m F^n x)^*$ or $w = FG$, $Z = (F^n G^m x)^*$.

(ii) *$\varphi_{35}(X_1, X_2, Y, Z)$ in $\mathscr{F}_\Delta$ iff there are $(F, G, w, x) \in \Delta^{(4)}$ and $n, m \geqq 2$ such that $X_1 = F^*$, $X_2 = G^*$, $Y = (wx)^*$ and either $w = GF$, $Z = (G^m F^n x)^*$ or $w = FG$, $Z = (F^n G^m x)^*$.*

(iii) *$\varphi_{36}(X_1, X_2, Y, A, B)$ in $\mathscr{F}_\Delta$ iff there are $(F, G, w, x) \in \Delta^{(4)}$ and $s_1, s_2 \in \Delta^{(-)}$ such that $X_1 = F^*$, $X_2 = G^*$, $Y = (wx)^*$, $A = (s_1 x)^*$, $B = (s_2 x)^*$ and either $w = GF$, s_1 is a beginning of s_2 or $w = FG$, s_1 is an end of s_2.*

(iv) *$\varphi_{37}(X_1, X_2, Y, A, B)$ in $\mathscr{F}_\Delta$ iff Δ is unary and there are $(F, G, w, x) \in \Delta^{(4)}$ and $s_1, s_2 \in \Delta^{(-)}$ such that $X_1 = F^*$, $X_2 = G^*$, $Y = (wx)^*$, $A = (s_1 x)^*$, $B = (s_2 x)^*$ and either $w = GF$, s_1 is an end of s_2 or $w = FG$, s_1 is a beginning of s_2.*

(v) *$\varphi_{38}(X_1, X_2, Y, X, A, B)$ in $\mathscr{F}_\Delta$ iff there are $(F, G, w, x) \in \Delta^{(4)}$, $H \in \Delta_1$, $s \in \Delta^{(-)}$ and $y \in V \cup \Delta_0$ such that $X_1 = F^*$, $X_2 = G^*$, $Y = (wx)^*$, $X = H^*$, $A = (sy)^*$ and either $w = GF$, $B = (Hsy)^*$ or $w = FG$, $B = (sHy)^*$.*

(vi) $\varphi_{39}(X_1, X_2, Y, X, X', A, B)$ *in* $\mathscr{F}_\Delta$ *iff there are* $(F, G, w, x) \in \Delta^{(4)}$, $H \in \Delta_1$, $n \geqq 2$, $s \in \Delta^{(-)}$ *and* $y \in V \cup \Delta_0$ *such that* $X_1 = F^*$, $X_2 = G^*$, $Y = (wx)^*$, $X = H^*$, $X' = (H^n x)^*$, $A = (sy)^*$ *and either* $w = GF$, $B = (H^n sy)^*$ *or* $w = FG$, $B = (sH^n y)^*$.

Proof. We shall prove only the direct implication in (iii); everything else is evident. Let $\varphi_{36}(X_1, X_2, Y, A, B)$; let $X_1 = F^*$, $X_2 = G^*$, $Y = (wx)^*$, $A = (s_1 x)^*$, $B = (s_2 x)^*$. If Δ is not unary then $w = GF$ and it is evident that s_1 is a beginning of s_2. Let Δ be unary. It is enough to consider the case $w = GF$ (the case $w = FG$ would be similar). Take an $n \geqq 2$ such that $n > \lambda_0(s_2 x)$. Put $X_1' = (F^n x)^*$, $X_2' = (G^n x)^*$, $Z = (G^n F^n x)^*$. Since $\varphi_{36}(X_1, X_2, Y, A, B)$ is satisfied, there are sequences $a', a'', b', b'' \in \Delta^{(-)}$ such that $a' \in \{F^n s_1, s_1 F^n\}$, $a'' \in \{G^n a', a' G^n\}$, $b' \in \{F^n s_2, s_2 F^n\}$, $b'' \in \{G^n b', b' G^n\}$, $G^n F^n x \leqq a'' x$, $G^n F^n x \leqq b'' x$, $a'' x \leqq b'' x$ and such that if $a'' = G^k F^l$ for some $k, l \geqq 2$ then s_1 contains only F and if $b'' = G^k F^l$ for some $k, l \geqq 2$ then s_2 contains only F. Since $n > \lambda_0(s_2 x)$, it is evident that $a'' = G^n F^n s_1$ and $b'' = G^n F^n s_2$; since $a'' \leqq b''$, it follows that s_1 is a beginning of s_2.

Let Δ be not strictly large; let $(F, G, w, x) \in \Delta^{(4)}$ and let $t_1, \ldots, t_n$ be a non-empty finite sequence of terms. A pair (A, D) is said to be an (F, G, w, x)-code of $t_1, \ldots, t_n$ (in $\mathscr{F}_\Delta$) if $A \in \mathscr{F}_\Delta$, $D = (F^n x)^*$ and there are positive integers $k_1, \ldots, k_n$ such that either $w = GF$ and $I(A) = \{F^{k_1} GFGt_1, \ldots, F^{k_n} GF^n Gt_n\}^\sim$ or $w = FG$ and $I(A) = \{s_1 GFGF^{k_1} y_1, \ldots, s_n GF^n GF^{k_n} y_n\}^\sim$ where $t_i = s_i y_i$, $y_i \in V$.

Let $F \in \Delta_1$ and let $t = sy$ be a term (where $s \in \Delta^{(-)}$ and $y \in V \cup \Delta_0$). Define a term z as follows: if $y \in \Delta_0$ then $z = y$; if $y = x_i$ for some $i \geqq 1$ then $z = F^i y$. The pair (t^*, z^*) is called the fine F-code of t.

Let $(F, G, w, x) \in \Delta^{(4)}$ and let $t_1, \ldots, t_n$ be a non-empty finite sequence of terms. A triple (A, B, D) is called a fine (F, G, w, x)-code of $t_1, \ldots, t_n$ (in $\mathscr{F}_\Delta$) if (A, D) is an (F, G, w, x)-code of $t_1, \ldots, t_n$ and (B, D) is an (F, G, w, x)-code of a sequence $z_1, \ldots, z_n$ such that for every $i \in \{1, \ldots, n\}$ the pair (t_i^*, z_i^*) is the fine F-code of t_i.

5.3. Lemma. *Let* Δ *be not strictly large and let* $(F, G, w, x) \in \Delta^{(4)}$. *Then every non-empty finite sequence of terms has at least one* (F, G, w, x)*-code and at least one fine* (F, G, w, x)*-code. If* (A, D) *is an* (F, G, w, x)*-code of two sequences* $t_1, \ldots, t_n$ *and* $u_1, \ldots, u_m$, *then* $n = m$ *and* $t_1 \sim u_1, \ldots, t_n \sim u_n$. *If* (A, B, D) *is a fine* (F, G, w, x)*-code of two sequences* $t_1, \ldots, t_n$ *and* $u_1, \ldots, u_m$, *then* $n = m$ *and* $t_1 = u_1, \ldots, t_n = u_n$.

Proof. Let $t_1, \ldots, t_n$ be a non-empty finite sequence of terms. Take an integer $k > \operatorname{Max}(\lambda(t_1), \ldots, \lambda(t_n))$ and put $D = (F^n x)^*$. If $w = GF$, put $A = \{F^k GFGt_1, \ldots \ldots, F^k GF^n Gt_n\}^*$; if $w = FG$, put $A = \{s_1 GFGF^k y_1, \ldots, s_n GF^n GF^k y_n\}^*$ where $t_i = s_i y_i$. Since $k > \lambda(t_i)$ for all i, it is evident that the terms $F^k GF^i Gt_i$ (the terms $s_i GF^i GF^k y_i$, resp.) are pairwise uncomparable and so (A, D) is an (F, G, w, x)-code of $t_1, \ldots, t_n$. The existence of a fine code follows easily. The rest is obvious.

Definition. (i) $\varphi_{40}(X_1, X_2, Y, A, Z, B, C) \equiv \varphi_{33}(X_1, X_2, Y) \,\&\, \varphi_{13}(X_1, Z) \,\&\, \varphi_1(B, A) \,\&$

$\& \exists Z', C_1, C_2, C_3(\varphi_{13}(X_1, Z') \& \varphi_{38}(X_1, X_2, Y, X_2, C, C_1) \& \varphi_{39}(X_1, X_2, Y, X_1, Z, C_1, C_2) \& \varphi_{38}(X_1, X_2, Y, X_2, C_2, C_3) \& \varphi_{39}(X_1, X_2, Y, X_1, Z', C_3, B))$.

(ii) $\varphi_{41}(X_1, X_2, Y, A, D) \equiv \varphi_{33}(X_1, X_2, Y) \& \varphi_{13}(X_1, D) \& \forall B \exists Z, C(\varphi_1(B, A) \to \to (Z \preccurlyeq D \& \varphi_{40}(X_1, X_2, Y, A, Z, B, C))) \& \forall Z(X_1 \preccurlyeq Z \preccurlyeq D \to \exists!! B \exists C\ \varphi_{40}(X_1, X_2, Y, A, Z, B, C))$.

(iii) $\varphi_{42}(X_1, X_2, Y, A, D) \equiv \varphi_{41}(X_1, X_2, Y, A, D) \& \forall Z_1, Z_2, B_1, B_2, C((\varphi_{40}(X_1, X_2, Y, A, Z_1, B_1, C) \& \varphi_{40}(X_1, X_2, Y, A, Z_2, B_2, C)) \to Z_1 = Z_2)$.

(iv) $\varphi_{43}(U, A, B) \equiv \alpha_1(U) \& ((\varphi_7(A) \& \varphi_{13}(U, B))\ \mathrm{VEL}\ (\alpha_0(B) \& B \preccurlyeq A))$.

(v) $\varphi_{44}(X_1, X_2, Y, A, B, D) \equiv \varphi_{41}(X_1, X_2, Y, A, D) \& \varphi_{41}(X_1, X_2, Y, B, D) \& \& \forall Z \exists A_1, A_2, B_1, B_2(X_1 \preccurlyeq Z \preccurlyeq D \to (\varphi_{40}(X_1, X_2, Y, A, Z, A_1, A_2) \& \varphi_{40}(X_1, X_2, Y, B, Z, B_1, B_2) \& \varphi_{43}(X_1, A_2, B_2)))$.

5.4. Lemma. *Let Δ be a large but not strictly large type. Then*:

(i) $\varphi_{40}(X_1, X_2, Y, A, Z, B, C)$ *in* $\mathscr{F}_\Delta$ *iff there are* $(F, G, w, x) \in \Delta^{(4)}$, $n, m \geqq 1$, $s_1, s_2 \in \Delta^{(-)}$ *and* $y \in V \cup \Delta_0$ *such that* $X_1 = F^*$, $X_2 = G^*$, $Y = (wx)^*$, $Z = (F^n x)^*$, $B = (s_1 y)^*$, $C = (s_2 y)^*$, $s_1 y \in I(A)$ *and either* $w = GF$, $s_1 = F^m G F^n G s_2$ *or* $w = FG$, $s_1 = s_2 G F^n G F^m$.

(ii) $\varphi_{41}(X_1, X_2, Y, A, D)$ *in* $\mathscr{F}_\Delta$ *iff there are* $(F, G, w, x) \in \Delta^{(4)}$ *and a non-empty finite sequence* $t_1, \ldots, t_n$ *of terms such that* $X_1 = F^*$, $X_2 = G^*$, $Y = (wx)^*$ *and* (A, D) *is an* (F, G, w, x)*-code of* $t_1, \ldots, t_n$.

(iii) $\varphi_{42}(X_1, X_2, Y, A, D)$ *in* $\mathscr{F}_\Delta$ *iff there are* $(F, G, w, x) \in \Delta^{(4)}$ *and a non-empty finite sequence* $t_1, \ldots, t_n$ *of pairwise non-similar terms such that* $X_1 = F^*$, $X_2 = G^*$, $Y = (wx)^*$ *and* (A, D) *is an* (F, G, w, x)*-code of* $t_1, \ldots, t_n$.

(iv) $\varphi_{43}(U, A, B)$ *in* $\mathscr{F}_\Delta$ *iff there are* $F \in \Delta_1$ *and a term* t *such that* $U = F^*$ *and* (A, B) *is the fine* F*-code of* t.

(v) $\varphi_{44}(X_1, X_2, Y, A, B, D)$ *in* $\mathscr{F}_\Delta$ *iff there are* $(F, G, w, x) \in \Delta^{(4)}$ *and a non-empty finite sequence* $t_1, \ldots, t_n$ *of terms such that* $X_1 = F^*$, $X_2 = G^*$, $Y = (wx)^*$ *and* (A, B, D) *is a fine* (F, G, w, x)*-code of* $t_1, \ldots, t_n$.

Definition. (i) $\varphi_{45}(A, B) \equiv \exists X_1, X_2, Y, A', B', A_1, A_2, D(\varphi_8(A, A') \& \varphi_8(B, B') \& \& \varphi_{42}(X_1, X_2, Y, A_1, D) \& \varphi_{42}(X_1, X_2, Y, A_2, D) \& \forall C(\varphi_{36}(X_1, X_2, Y, C, A') \leftrightarrow \leftrightarrow \exists Z, U \varphi_{40}(X_1, X_2, Y, A_1, Z, U, C)) \& \forall C(\varphi_{36}(X_1, X_2, Y, C, B') \leftrightarrow \exists Z, U \varphi_{40}(X_1, X_2, Y, A_2, Z, U, C))$.

(ii) $\varphi_{46}(X_1, X_2, Y, A, B) \equiv \exists A', D, U_1, U_2(\varphi_{41}(X_1, X_2, Y, A', D) \& \varphi_{40}(X_1, X_2, Y, A', X_1, U_1, A) \& \varphi_{40}(X_1, X_2, Y, A', D, U_2, B) \& \forall Z_1, Z_2 \exists X, B_1, B_2, C_1, C_2((X_1 \preccurlyeq \preccurlyeq Z_1 \& Z_1 \prec Z_2 \& Z_2 \preccurlyeq D) \to (\varphi_{40}(X_1, X_2, Y, A', Z_1, B_1, C_1) \& \varphi_{40}(X_1, X_2, Y, A', Z_2, B_2, C_2) \& \varphi_{38}(X_1, X_2, Y, X, C_1, C_2))))$.

(iii) $\varphi_{47}(X_1, X_2, Y, A, B, C) \equiv \varphi_{36}(X_1, X_2, Y, A, C) \& \varphi_{46}(X_1, X_2, Y, B, C) \& \& (\omega_1(A) \to B = C) \& (\alpha_1(A) \to \varphi_{38}(X_1, X_2, Y, A, B, C)) \& ((\neg\omega_1(A) \& \neg\alpha_1(A)) \to \to \exists A_1, C_1(\varphi_{39}(X_1, X_2, Y, X_1, A_1, B, C_1) \& \varphi_{45}(A, A_1) \& \varphi_{45}(C, C_1)))$.

5.5. Lemma. *Let Δ be a large but not strictly large type. Then*:

(i) $\varphi_{45}(A, B)$ *in* $\mathscr{F}_\Delta$ *iff* $A = t^*$ *and* $B = u^*$ *for some terms* t, u *with* $\lambda(t) = \lambda(u)$.

(ii) $\varphi_{46}(X_1, X_2, Y, A, B)$ *in* $\mathscr{F}_\Delta$ *iff there are* $(F, G, w, x) \in \Delta^{(4)}$, $s_1, s_2 \in \Delta^{(-)}$ *and* $y \in V \cup \Delta_0$ *such that* $X_1 = F^*$, $X_2 = G^*$, $Y = (wx)^*$, $A = (s_1y)^*$, $B = (s_2y)^*$ *and either* $w = GF$, s_1 *is an end of* s_2 *or* $w = FG$, s_1 *is a beginning of* s_2.

(iii) $\varphi_{47}(X_1, X_2, Y, A, B, C)$ *in* $\mathscr{F}_\Delta$ *iff there are* $(F, G, w, x) \in \Delta^{(4)}$ *and* $s_1, s_2 \in \Delta^{(-)}$ *such that* $X_1 = F^*$, $X_2 = G^*$, $Y = (wx)^*$, $A = (s_1x)^*$, $B = (s_2x)^*$ *and either* $w = GF$, $C = (s_1s_2x)^*$ *or* $w = FG$, $C = (s_2s_1x)^*$.

Definition. (i) $\varphi_{48}(X_1, X_2, Y, A, U_1, B, U_2, C, U_3, D, U_4) \equiv A \ll C \,\&\, B \ll D \,\&\, \&\, \varphi_{43}(X_1, A, U_1) \,\&\, \varphi_{43}(X_1, B, U_2) \,\&\, \varphi_{43}(X_1, C, U_3) \,\&\, \varphi_{43}(X_1, D, U_4) \,\&\, \exists A', B', C', D', C_1, C_2, D_1, D_2, E(\varphi_8(A, A') \,\&\, \varphi_8(B, B') \,\&\, \varphi_8(C, C') \,\&\, \varphi_8(D, D') \,\&\, \varphi_{47}(X_1, X_2, Y, A', C_1, C_2) \,\&\, \varphi_{47}(X_1, X_2, Y, E, C_2, C') \,\&\, \varphi_{47}(X_1, X_2, Y, B', D_1, D_2) \,\&\, \varphi_{47}(X_1, X_2, Y, E, D_2, D') \,\&\, (\alpha_0(U_1) \to \omega_1(C_1)) \,\&\, (\alpha_0(U_2) \to \omega_1(D_1)) \,\&\, ((\omega_1(U_1) \,\&\, U_1 = U_2) \to \to (C_1 = D_1 \,\&\, U_3 = U_4)))$.

In the following two definitions let $s(A, B, D)$ be an abbreviation for $\varphi_{44}(X_1, X_2, Y, A, B, D)$ and let $A(Z) = M$ be an abbreviation for $\exists H\, \varphi_{40}(X_1, X_2, Y, A, Z, H, M)$.

(ii) $\varphi_{49}(X_1, X_2, Y, A_1, B_1, A_2, B_2, D, C_1, D_1, C_2, D_2) \equiv s(A_1, B_1, D) \,\&\, \&\, s(A_2, B_2, D) \,\&\, \exists P, Q, E(s(P, Q, E) \,\&\, P(X_1) = C_1 \,\&\, Q(X_1) = D_1 \,\&\, P(E) = C_2 \,\&\, \&\, Q(E) = D_2 \,\&\, \forall Z_1, Z_2\, \exists Z, M_1, N_1, M_2, N_2, P_1, Q_1, P_2, Q_2((X_1 \ll Z_1 \,\&\, Z_1 \prec Z_2 \,\&\, \&\, Z_2 \ll E) \to (A_1(Z) = M_1 \,\&\, B_1(Z) = N_1 \,\&\, A_2(Z) = M_2 \,\&\, B_2(Z) = N_2 \,\&\, P(Z_1) = = P_1 \,\&\, Q(Z_1) = Q_1 \,\&\, P(Z_2) = P_2 \,\&\, Q(Z_2) = Q_2 \,\&\, (\varphi_{48}(X_1, X_2, Y, M_1, N_1, M_2, N_2, P_1, Q_1, P_2, Q_2) \text{ VEL } \varphi_{48}(X_1, X_2, Y, M_1, N_1, M_2, N_2, P_2, Q_2, P_1, Q_1)))))$.

(iii) $\varphi_{50}(X_1, X_2, Y, A, U_1, B, U_2, C, U_3, D, U_4) \equiv \exists A_1, B_1, A_2, B_2(\varphi_{49}(X_1, X_2, Y, A_1, B_1, A_2, B_2, X_1, C, U_3, D, U_4) \,\&\, A_1(X_1) = A \,\&\, B_1(X_1) = U_1 \,\&\, A_2(X_1) = B \,\&\, \&\, B_2(X_1) = U_2)$.

5.6. Lemma. *Let Δ be a large but not strictly large type. Let* $F, G \in \Delta_1$, $F \neq G$, $x \in V$, $X_1 = F^*$, $X_2 = G^*$, $Y = (GFx)^*$. *Then*:

(i) $\varphi_{48}(X_1, X_2, Y, A, U_1, B, U_2, C, U_3, D, U_4)$ *in* $\mathscr{F}_\Delta$ *iff there are terms* a, b, c, d *such that* (A, U_1), (B, U_2), (C, U_3), (D, U_4) *are the fine F-codes of* a, b, c, d, *respectively, and* (c, d) *is an immediate consequence of* (a, b).

(ii) $\varphi_{49}(X_1, X_2, Y, A_1, B_1, A_2, B_2, D, C_1, D_1, C_2, D_2)$ *in* $\mathscr{F}_\Delta$ *iff there are* $n \geqq 1$ *and terms* $t, u, t_1, \ldots, t_n, u_1, \ldots, u_n$ *such that* (A_1, B_1, D) *is a fine* (F, G, GF, x)*-code of* $t_1, \ldots, t_n$, (A_2, B_2, D) *is a fine* (F, G, GF, x)*-code of* $u_1, \ldots, u_n$, (C_1, D_1) *is the fine F-code of* t, (C_2, D_2) *is the fine F-code of* u *and* (t, u) *is a consequence of* $\{(t_1, u_1), \ldots, (t_n, u_n)\}$.

(iii) $\varphi_{50}(X_1, X_2, Y, A, U_1, B, U_2, C, U_3, D, U_4)$ *in* $\mathscr{F}_\Delta$ *iff there are terms* a, b, c, d *such that* (A, U_1), (B, U_2), (C, U_3), (D, U_4) *are the fine F-codes of* a, b, c, d, *respectively, and* (c, d) *is a consequence of* (a, b).

6. FINITE SEQUENCES OF TERMS AND THE CONSEQUENCE RELATION; THE CASE OF A STRICTLY LARGE TYPE

Let $(a_1, \ldots, a_n)$ and $(b_1, \ldots, b_m)$ be two finite sequences of terms. We write $(a_1, \ldots, a_n) \sim (b_1, \ldots, b_m)$ if $n = m$ and there is an automorphism f of W_Δ with $f(a_1) = b_1, \ldots, f(a_n) = b_n$.

Let $(F, i) \in \Delta^{(2)}$ and $x \in V$. Then for every finite sequence $a_1, \ldots, a_n$ of terms we define a term $H_{F,i,x}(a_1, \ldots, a_n)$ as follows: if $n = 0$, this term equals x; if $n \geqq 1$ then $H_{F,i,x}(a_1, \ldots, a_n) = F(u_1, \ldots, u_{i-1}, t, u_{i+1}, \ldots, u_{n_F})$ where $u_1 = \ldots = u_{n_F} = a_n$ and $t = H_{F,i,x}(a_1, \ldots, a_{n-1})$.

Let $(F, i) \in \Delta^{(2)}$ and let $a_1, \ldots, a_n$ be a finite sequence of terms. Then we put $H_{F,i}(a_1, \ldots, a_n) = t^*$ where $t = H_{F,i,x}(a_1, \ldots, a_n)$ and x is a variable not belonging to $\operatorname{var}(a_1) \cup \ldots \cup \operatorname{var}(a_n)$. Evidently, $H_{F,i}(a_1, \ldots, a_n) = H_{F,i}(b_1, \ldots, b_m)$ iff $(a_1, \ldots, a_n) \sim (b_1, \ldots, b_m)$.

Definition. (i) $\varphi_{51}(X, Y, Z, X', X'', Z') \equiv \varphi_4(X) \,\&\, \varphi_4(X') \,\&\, \varphi_4(X'') \,\&\, X \neq X' \,\&$ $\& \, X \neq X'' \,\&\, X' \neq X'' \,\&\, \exists U(\alpha(U) \,\&\, U \leqslant X \,\&\, U \leqslant X' \,\&\, U \leqslant X'') \,\&\, \varphi_{29}(X, Y, Z) \,\&$ $\& \, Z \prec Z' \,\&\, \neg Z \lessdot Z' \,\&\, \exists A(\varphi_{32}(X', A, Z') \,\&\, \varphi_{32}(X'', A, Z'))$.

(ii) $\varphi_{52}(X, Y, U) \equiv \varphi_4(X) \,\&\, \tau(Y) \,\&\, \tau(U) \,\&\, \forall A(\varphi_{31}(U, A) \leftrightarrow \exists Z(\varphi_{29}(X, Y, Z) \,\&$ $\& \, \varphi_{31}(Z, A) \,\&\, \forall X', X'', Z'(\varphi_{51}(X, Y, Z, X', X'', Z') \rightarrow \varphi_{31}(Z', A))))$.

(iii) $\varphi_{53}(X, Y) \equiv \varphi_4(X) \,\&\, \exists X', Y'(\varphi_{13}(X', Y') \,\&\, X' \leqslant X \leqslant Y' \,\&\, Y \leqslant Y' \,\&\, \bar{\alpha}_2(X'))$.

(iv) $\varphi_{54}(X, Y, Z) \equiv \varphi_{53}(X, Z) \,\&\, \varphi_8(Y, Z) \,\&\, \forall A((A \leqslant Y \,\&\, \neg\varphi_8(A, Z)) \rightarrow$ $\rightarrow \exists B(\varphi_{52}(X, A, B) \,\&\, B \leqslant Y))$.

(v) $\varphi_{55}(X, Y) \equiv \exists Z(\varphi_{54}(X, Y, Z) \,\&\, \forall Y'(\varphi_{54}(X, Y', Z) \rightarrow Y \leqslant Y'))$.

6.1. Lemma. *Let Δ be a strictly large type. Then:*

(i) *$\varphi_{51}(X, Y, Z, X', X'', Z')$ in $\mathscr{F}_\Delta$ iff there are $F \in \Delta$, three pairwise different numbers $i, j, k \in \{1, \ldots, n_F\}$, a term t and pairwise different variables $y_1, \ldots, y_{n_F}$ not belonging to* $\operatorname{var}(t)$ *such that $X = (F, i)^*$, $X' = (F, j)^*$, $X'' = (F, k)^*$, $Y = t^*$, $Z = u^*$ where $u = F(y_1, \ldots, y_{i-1}, t, y_{i+1}, \ldots, y_{n_F})$ and $Z' = (\sigma_z^y(u))^*$ where $y = y_j$ and $z = y_k$.*

(ii) *$\varphi_{52}(X, Y, U)$ in $\mathscr{F}_\Delta$ iff there are $(F, i) \in \Delta^{(1)}$, $t \in W_\Delta$ and $x \in V \setminus \operatorname{var}(t)$ such that $X = (F, i)^*$, $Y = t^*$ and $U = (F(y_1, \ldots, y_{i-1}, t, y_{i+1}, \ldots, y_{n_F}))^*$ where $y_1 = \ldots$ $\ldots = y_{n_F} = x$.*

(iii) *$\varphi_{53}(X, Y)$ in $\mathscr{F}_\Delta$ iff there are $(F, i) \in \Delta^{(2)}$, $k \geqq 0$, $x \in V$ and $a \in x\begin{bmatrix} k \\ F, i \end{bmatrix}$ such that $X = (F, i)^*$ and $Y = a^*$.*

(iv) *$\varphi_{55}(X, Y)$ in $\mathscr{F}_\Delta$ iff there are $(F, i) \in \Delta^{(2)}$ and a finite sequence $y_1, \ldots, y_n$ of pairwise different variables such that $X = (F, i)^*$ and $Y = H_{F,i}(y_1, \ldots, y_n)$.*

Proof. Only (iv) is not quite obvious. Let $\varphi_{55}(X, Y)$. There are $(F, i) \in \Delta^{(2)}$,

$t \in W_\Delta$, $x \in V$, $n \geqq 0$ and $a \in x\begin{bmatrix} n \\ F, i \end{bmatrix}$ such that $X = (F, i)^*$, $Y = t^*$ and $\varphi_{54}(X, Y, a^*)$ is satisfied. Let $x, y_1, \ldots, y_n$ be pairwise different variables. It is easy to prove by induction on $k \in \{0, \ldots, n\}$ that $H_{F,i,x}(y_1, \ldots, y_k) \leqq t$. Hence $H_{F,i,x}(y_1, \ldots, y_n) \leqq t$. On the other hand, we evidently have $\varphi_{54}(X, H_{F,i}(y_1, \ldots, y_n), a^*)$ and so $t \leqq$ $\leqq H_{F,i,x}(y_1, \ldots, y_n)$. This proves $t \sim H_{F,i,x}(y_1, \ldots, y_n)$, i.e. $Y = H_{F,i}(y_1, \ldots, y_n)$.

Conversely, let $X = (F, i)^*$ and $Y = H_{F,i}(y_1, \ldots, y_n)$; put $Z = a^*$ where $a \in$ $\in x\begin{bmatrix} n \\ F, i \end{bmatrix}$. Evidently, $\varphi_{54}(X, Y, Z)$ is satisfied; if $Y' = u^*$ and $\varphi_{54}(X, Y', Z)$, then $H_{F,i,x}(y_1, \ldots, y_k) \leqq u$ can be proved by induction on $k \in \{0, \ldots, n\}$, so that $Y \lessdot Y'$.

Definition. (i) $\varphi_{56}(X, Y, U) \equiv \exists Z, U', Y_1, Y', Y_1'(\varphi_{55}(X, Z) \,\&\, \varphi_{53}(X, U) \,\&$ $\&\, \varphi_{53}(X, U') \,\&\, U \prec U' \,\&\, \varphi_8(Z, U) \,\&\, Y \lessdot Y_1 \,\&\, \varphi_8(Y, Y') \,\&\, \varphi_8(Y_1, Y_1') \,\&\, Y' \lessdot Y_1' \,\&$ $\&\, \varphi_{31}(Z, Y) \,\&\, \varphi_{31}(U', Y_1) \,\&\, \neg\varphi_{31}(U', Y) \,\&(\varphi_{53}(X, Y) \text{ VEL } \neg\varphi_{29}(X, Y, Y_1)) \,\&$ $\&\, (\omega_1(U) \rightarrow \omega_1(Y)))$.

(ii) $\varphi_{57}(X, Y) \equiv \exists U\, \varphi_{56}(X, Y, U)$.

6.2. Lemma. *Let Δ be a strictly large type. Then*:

(i) $\varphi_{56}(X, Y, U)$ *in* $\mathscr{F}_\Delta$ *iff there are* $(F, i) \in \Delta^{(2)}$, $x \in V$ *and a finite sequence* $a_1, \ldots, a_n$ *of terms such that* $X = (F, i)^*$, $Y = H_{F,i}(a_1, \ldots, a_n)$ *and* $U = a^*$ *where* $a \in x\begin{bmatrix} n \\ F, i \end{bmatrix}$.

(ii) $\varphi_{57}(X, Y)$ *in* $\mathscr{F}_\Delta$ *iff there are* $(F, i) \in \Delta^{(2)}$ *and a finite sequence* $a_1, \ldots, a_n$ *of terms such that* $X = (F, i)^*$ *and* $Y = H_{F,i}(a_1, \ldots, a_n)$.

Proof. Let $\varphi_{56}(X, Y, U)$, so that there are Z, U', Y_1, Y', Y_1' as above. We have $X = (F, i)^*$ for some $(F, i) \in \Delta^{(2)}$, $Z = t^*$ where $t = H_{F,i,x}(y_1, \ldots, y_n)$ for some $n \geqq 0$ and pairwise different variables $x, y_1, \ldots, y_n$, $U = a^*$ where $a \in x\begin{bmatrix} n \\ F, i \end{bmatrix}$, $U' = b^*$ where $b \in x\begin{bmatrix} n+1 \\ F, i \end{bmatrix}$ and $Y = (f(t))^*$ for some substitution f. If $n = 0$, then $t \in V$ and everything is evident. Let $n \geqq 1$. Then evidently $Y_1 = (\sigma_G^z f(t))^*$ for some $z \in \operatorname{var}(f(t))$ and $G \in \Delta \setminus \Delta_0$. Since there are terms $c \in Q(f(t))$ and $d \in$ $\in Q(\sigma_G^z f(t))$ with $c \prec d$, z has a single occurrence in $f(t)$. Since $\varphi_{31}(U', Y_1) \,\&$ $\&\, \neg\varphi_{31}(U', Y)$, we have $z = f(x)$. Hence $Y = H_{F,i}(f(y_1), \ldots, f(y_n))$.

Conversely, if $X = (F, i)^*$, $Y = H_{F,i}(a_1, \ldots, a_n)$ and $U = a^*$ where $a \in x\begin{bmatrix} n \\ F, i \end{bmatrix}$, we can put $Z = (H_{F,i,x}(y_1, \ldots, y_n))^*$ where $x, y_1, \ldots, y_n$ are pairwise different variables, $U' = b^*$ where $b \in x\begin{bmatrix} n+1 \\ F, i \end{bmatrix}$ and $Y_1 = (\sigma_F^x(H_{F,i,x}(y_1, \ldots, y_n)))^*$ and define Y', Y_1' by $\varphi_8(Y, Y')$ and $\varphi_8(Y_1, Y_1')$. (ii) is evident.

Definition. (i) $\varphi_{58}(X, Y, Z, U) \equiv \varphi_{53}(X, Y) \,\&\, \tau(Z) \,\&\, \tau(U) \,\&\, (\omega_1(Y) \to Z = U) \,\&$ $\&\, (Y \prec X \to \varphi_{32}(X, Z, U)) \,\&\, (X \preccurlyeq Y \to \forall A(\varphi_{31}(A, Z) \leftrightarrow \exists B, C(\varphi_{30}(B, X, Y, A, C) \,\&$ $\&\, \varphi_{31}(C, U)))).$

(ii) $\varphi_{59}(X, Y_1, Y_2) \equiv \exists U_1, U_2, U_3(\varphi_{56}(X, Y_1, U_1) \,\&\, \varphi_{56}(X, Y_2, U_2) \,\&\, \varphi_{58}(X, U_3, Y_1, Y_2) \,\&\, \varphi_{58}(X, U_3, U_1, U_2)).$

(iii) $\varphi_{60}(X, Y, Z, U) \equiv \exists A, B, C(\varphi_{56}(X, Y, A) \,\&\, \varphi_{56}(X, B, Z) \,\&\, \varphi_{59}(X, B, Y) \,\&$ $\&\, \varphi_4(C) \,\&\, \varphi_{32}(C, U, B)).$

(iv) $\varphi_{61}(X, Y_1, Y_2) \equiv \exists Z_1, Z_2(\varphi_{56}(X, Y_1, Z_1) \,\&\, \varphi_{56}(X, Y_2, Z_2) \,\&\, Z_2 \preccurlyeq Z_1 \,\&$ $\&\, \varphi_{31}(Y_2, Y_1) \,\&\, \forall Y_3((\varphi_{56}(X, Y_3, Z_2) \,\&\, \varphi_{31}(Y_3, Y_1)) \to \varphi_{31}(Y_3, Y_2))).$

6.3. Lemma. *Let Δ be a strictly large type. Then:*

(i) $\varphi_{58}(X, Y, Z, U)$ *in* $\mathscr{F}_\Delta$ *iff there are* $(F, i) \in \Delta^{(2)}$, $k \geqq 0$, $x \in V$, $a \in x\begin{bmatrix} k \\ F, i \end{bmatrix}$ *and terms* $t_0, \ldots, t_k$ *such that* $X = (F, i)^*$, $Y = a^*$, $Z = t_0^*$, $U = t_k^*$ *and such that whenever* $j \in \{1, \ldots, k\}$ *then* $t_j = F(p_1, \ldots, p_{i-1}, t_{j-1}, p_{i+1}, \ldots, p_{n_F})$ *for some terms* $p_1, \ldots, p_{n_F}$.

(ii) $\varphi_{59}(X, Y_1, Y_2)$ *in* $\mathscr{F}_\Delta$ *iff there are* $(F, i) \in \Delta^{(2)}$, *a finite sequence* $a_1, \ldots, a_n$ *of terms and a number* $k \in \{0, \ldots, n\}$ *such that* $X = (F, i)^*$, $Y_1 = H_{F,i}(a_1, \ldots, a_k)$ *and* $Y_2 = H_{F,i}(a_1, \ldots, a_n)$.

(iii) $\varphi_{60}(X, Y, Z, U)$ *in* $\mathscr{F}_\Delta$ *iff there are* $(F, i) \in \Delta^{(2)}$, *a finite sequence* $a_1, \ldots, a_n$ *of terms and a number* $k \in \{1, \ldots, n\}$ *such that* $X = (F, i)^*$, $Y = H_{F,i}(a_1, \ldots, a_n)$, $Z = a^*$ *where* $a \in x\begin{bmatrix} k \\ F, i \end{bmatrix}$ *and* $x \in V$, *and* $U = a_k^*$.

(iv) $\varphi_{61}(X, Y_1, Y_2)$ *in* $\mathscr{F}_\Delta$ *iff there are* $(F, i) \in \Delta^{(2)}$, *a finite sequence* $a_1, \ldots, a_n$ *of terms and a number* $k \in \{1, \ldots, n + 1\}$ *such that* $X = (F, i)^*$, $Y_1 = H_{F,i}(a_1, \ldots, a_n)$ *and* $Y_2 = H_{F,i}(a_k, \ldots, a_n)$.

Definition. (i) $\varphi_{62}(X, Y) \equiv \exists Z(\varphi_{56}(X, Y, Z) \,\&\, \forall Y'((\varphi_{56}(X, Y', Z) \,\&$ $\&\, \forall Z', A, B((\varphi_{60}(X, Y, Z', A) \,\&\, \varphi_{60}(X, Y', Z', B)) \to A = B)) \to \varphi_{31}(Y, Y'))).$

(ii) $\varphi_{63}(X, Y_1, Y_2, Y_3) \equiv \varphi_{62}(X, Y_3) \,\&\, \exists Z_1, Z_2, Z_3, X'(\varphi_{56}(X, Y_1, Z_1) \,\&$ $\&\, \varphi_{56}(X, Y_2, Z_2) \,\&\, \varphi_{56}(X, Y_3, Z_3) \,\&\, \varphi_{30}(X', X, Z_1, Z_2, Z_3) \,\&\, \varphi_{59}(X, Y_1, Y_3) \,\&$ $\&\, \varphi_{61}(X, Y_3, Y_2)).$

(iii) $\varphi_{64}(X, A, Y) \equiv \exists Z(\alpha(Z) \,\&\, \varphi_{56}(X, Y, Z) \,\&\, \varphi_{60}(X, Y, Z, A)).$

(iv) $\varphi_{65}(X, Y, A, Y') \equiv \exists Y_1(\varphi_{64}(X, A, Y_1) \,\&\, \varphi_{63}(X, Y, Y_1, Y')).$

(v) $\varphi_{66}(X, Y, A, Y') \equiv \exists Y_1(\varphi_{64}(X, A, Y_1) \,\&\, \varphi_{63}(X, Y_1, Y, Y')).$

6.4. Lemma. *Let Δ be a strictly large type. Then:*

(i) $\varphi_{62}(X, Y)$ *in* $\mathscr{F}_\Delta$ *iff there are* $(F, i) \in \Delta^{(2)}$ *and a finite sequence* $a_1, \ldots, a_n$ *of terms with pairwise disjoint sets of variables such that* $X = (F, i)^*$ *and* $Y = H_{F,i}(a_1, \ldots, a_n)$.

(ii) $\varphi_{63}(X, Y_1, Y_2, Y_3)$ *in* $\mathscr{F}_\Delta$ *iff there are* $(F, i) \in \Delta^{(2)}$, *a finite sequence* $a_1, \ldots, a_n$ *of terms with pairwise disjoint sets of variables and a number* $k \in \{0, \ldots, n\}$ *such* *that* $X = (F, i)^*$, $Y_1 = H_{F,i}(a_1, \ldots, a_k)$, $Y_2 = H_{F,i}(a_{k+1}, \ldots, a_n)$ *and* $Y_3 = H_{F,i}(a_1, \ldots, a_n)$.

(iii) $\varphi_{64}(X, A, Y)$ *in* $\mathscr{F}_\Delta$ *iff there are* $(F, i) \in \Delta^{(2)}$ *and a term a such that* $X = (F, i)^*$, $A = a^*$, $Y = H_{F,i}(a)$.

(iv) $\varphi_{65}(X, Y, A, Y')$ *in* $\mathscr{F}_\Delta$ *iff there are* $(F, i) \in \Delta^{(2)}$, *a term a and a finite sequence* $a_1, \ldots, a_n$ *of terms such that the terms* $a_1, \ldots, a_n$, *a have pairwise disjoint sets of variables,* $X = (F, i)^*$, $Y = H_{F,i}(a_1, \ldots, a_n)$, $A = a^*$ *and* $Y' = H_{F,i}(a_1, \ldots, a_n, a)$.

(v) $\varphi_{66}(X, Y, A, Y')$ *in* $\mathscr{F}_\Delta$ *iff there are* $(F, i) \in \Delta^{(2)}$, *a term a and a finite sequence* $a_1, \ldots, a_n$ *of terms such that the terms* $a, a_1, \ldots, a_n$ *have pairwise disjoint sets of variables,* $X = (F, i)^*$, $Y = H_{F,i}(a_1, \ldots, a_n)$, $A = a^*$ *and* $Y' = H_{F,i}(a, a_1, \ldots, a_n)$.

For every term t and every finite sequence e of elements of $\Delta^{(1)}$ we define (by induction on the length of e) an element $t\langle e\rangle$ of $W_\Delta \cup \{\emptyset\}$ as follows: if e is empty, put $t\langle e\rangle = t$; if $e = ((G_1, j_1), \ldots, (G_k, j_k))$ is non-empty and $t\langle (G_1, j_1), \ldots, (G_{k-1}, j_{k-1})\rangle = G_k(a_1, \ldots, a_n)$ (where $n = n_{G_k}$) for some terms $a_1, \ldots, a_n$, put $t\langle e\rangle = a_{j_k}$; in all other cases put $t\langle e\rangle = \emptyset$. We denote by $E(t)$ the set of all the sequences e such that $t\langle e\rangle$ is a term. Evidently, $E(t)$ is finite and $\{t\langle e\rangle; e \in E(t)\}$ is just the set of subterms of t.

If $(F, i) \in \Delta^{(2)}$ and $e = ((G_1, j_1), \ldots, (G_k, j_k))$ is a finite sequence of elements of $\Delta^{(1)}$, put $H^{(1)}_{F,i}(e) = H_{F,i}(a_1, \ldots, a_k)$ where $a_1, \ldots, a_k$ are terms with pairwise disjoint sets of variables such that $a_1^* = (G_1, j_1)^*, \ldots, a_k^* = (G_k, j_k)^*$.

Definition. (i) $\varphi_{67}(X, A, Y, B) \equiv \varphi_{62}(X, Y) \mathbin{\&} \exists Y', Z, Z', Z_1(\varphi_{56}(X, Y, Z) \mathbin{\&} \varphi_{56}(X, Y', Z') \mathbin{\&} Z \prec Z' \mathbin{\&} \alpha(Z_1) \mathbin{\&} Z_1 \leqslant X \mathbin{\&} \varphi_{60}(X, Y', Z_1, A) \mathbin{\&} \varphi_{60}(X, Y', Z', B) \mathbin{\&} \forall Z_2, Z_3 \exists U_1, U_2, U_3((Z_1 \leqslant Z_2 \mathbin{\&} Z_2 \prec Z_3 \mathbin{\&} Z_3 \leqslant Z') \to (\varphi_{60}(X, Y, Z_2, U_1) \mathbin{\&} \varphi_{60}(X, Y', Z_2, U_2) \mathbin{\&} \varphi_{60}(X, Y', Z_3, U_3) \mathbin{\&} \varphi_{32}(U_1, U_3, U_2))))$.

(ii) $\varphi_{68}(X, A, Y_1, Y_2, B) \equiv \varphi_{67}(X, A, Y_1, B) \mathbin{\&} \varphi_{67}(X, A, Y_2, B) \mathbin{\&} \forall A', B'((\varphi_{31}(A, A') \mathbin{\&} \varphi_{62}(X, A', Y_1, B')) \to \varphi_{67}(X, A', Y_2, B'))$.

6.5. Lemma. *Let* Δ *be a strictly large type. Then*:

(i) $\varphi_{67}(X, A, Y, B)$ *in* $\mathscr{F}_\Delta$ *iff there are* $(F, i) \in \Delta^{(2)}$, *a term t and a sequence* $e \in E(t)$ *such that* $X = (F, i)^*$, $A = t^*$, $Y = H^{(1)}_{F,i}(e)$ *and* $B = (t\langle e\rangle)^*$.

(ii) $\varphi_{68}(X, A, Y_1, Y_2, B)$ *in* $\mathscr{F}_\Delta$ *iff there are* $(F, i) \in \Delta^{(2)}$, *a term t and two sequences* $e, f \in E(t)$ *such that* $X = (F, i)^*$, $A = t^*$, $Y_1 = H^{(1)}_{F,i}(e)$, $Y_2 = H^{(1)}_{F,i}(f)$, $B = (t\langle e\rangle)^* = (t\langle f\rangle)^*$ *and* $t\langle e\rangle = t\langle f\rangle$.

Definition. (i) $\varphi_{69}(X, Y_1, Y_2) \equiv \exists Y_3, Z, Z', I, J, X', B(\varphi_{56}(X, Y_3, Z) \mathbin{\&} \varphi_{56}(X, Y_1, Z') \mathbin{\&} X \leqslant Z' \mathbin{\&} Z' \prec Z \mathbin{\&} \varphi_{59}(X, Y_1, Y_3) \mathbin{\&} \varphi_{56}(X, Y_2, X) \mathbin{\&} \varphi_{61}(X, Y_3, Y_2) \mathbin{\&} \varphi_{56}(X, I, Z) \mathbin{\&} \forall Z_1((\neg\omega_1(Z_1) \mathbin{\&} Z_1 \leqslant Z') \to \varphi_{60}(X, I, Z_1, X)) \mathbin{\&} \varphi_4(X') \mathbin{\&} X \neq X' \mathbin{\&} \exists X_0(\alpha(X_0) \mathbin{\&} X_0 \leqslant X \mathbin{\&} X_0 \leqslant X') \mathbin{\&} \varphi_{60}(X, I, Z, X') \mathbin{\&} \varphi_{64}(X, X', J) \mathbin{\&} \varphi_{68}(X, Y_3, I, J, B))$.

(ii) $\varphi_{70}(X, Y) \equiv \exists Z_0, Z_1, Z_2, Z_3, Z_4, X', J_1, J_2, J_3, J_1', J_2', I, I_1, I_2, I_1', B_1,$

$B_2(\varphi_{56}(X, Y, Z_4) \& \omega_1(Z_0) \& Z_0 \prec Z_1 \& Z_1 \prec Z_2 \& Z_2 \prec Z_3 \& Z_3 \prec Z_4 \& \varphi_4(X') \&$
$\& Z_1 \preccurlyeq X' \& X \neq X' \& \varphi_{64}(X, X, J_1) \& \varphi_{65}(X, J_1, X, J_2) \& \varphi_{65}(X, J_2, X, J_3) \&$
$\& \varphi_{65}(X, J_3, X', J_1') \& \varphi_{65}(X, J_2, X', J_2') \& \varphi_{66}(X, I, X', I_2) \& \varphi_{66}(X, I, X', I_1') \&$
$\& \varphi_{66}(X, I_1', X, I_1) \& \varphi_{68}(X, Y, J_1', I_1, B_1) \& \varphi_{68}(X, Y, J_2', I_2, B_2) \&$
$\& \forall K, K_1, K_2, K_1'((\varphi_{66}(X, K, X', K_2) \& \varphi_{66}(X, K, X', K_1') \& \varphi_{66}(X, K_1', X, K_1) \&$
$\& \neg\varphi_{59}(X, K, I) \& \neg\varphi_{59}(X, I, K) \& (\exists C_1\, \varphi_{67}(X, Y, K_1, C_1)\ \mathrm{VEL}\ \exists C_2\, \varphi_{67}(X, Y,$
$K_2, C_2))) \to \exists C\, \varphi_{68}(X, Y, K_1, K_2, C)))$.

(iii) $\varphi_{71}(X, Y_1, Y_2) \equiv \varphi_{56}(X, Y_1, X) \& \varphi_{56}(X, Y_2, X) \& \exists Y_3, Y_4(\varphi_{56}(X, Y_3, X) \&$
$\& \varphi_{31}(Y_1, Y_3) \& \varphi_{70}(X, Y_4) \& \varphi_{59}(X, Y_3, Y_4) \& \varphi_{61}(X, Y_4, Y_2))$.

(iv) $\varphi_{72}(X, Y) \equiv \varphi_{56}(X, Y, X) \& \exists X_0, X', I, J, B(\varphi_4(X') \& X_0 \prec X \& X_0 \prec X' \&$
$\& X \neq X' \& \varphi_{64}(X, X', J) \& \varphi_{66}(X, J, X, I) \& \varphi_{68}(X, Y, I, J, B))$.

6.6. Lemma. *Let Δ be a strictly large type. Then:*

(i) $\varphi_{69}(X, Y_1, Y_2)$ *in* $\mathscr{F}_\Delta$ *iff there are* $(F, i) \in \Delta^{(2)}$ *and a finite sequence* $a_1, \ldots, a_n$ *of terms such that* $n \geqq 2$, $X = (F, i)^*$, $Y_1 = H_{F,i}(a_1, \ldots, a_n)$ *and* $Y_2 = H_{F,i}(a_n, a_1)$.

(ii) $\varphi_{70}(X, Y)$ *in* $\mathscr{F}_\Delta$ *iff there are* $(F, i) \in \Delta^{(2)}$ *and terms* a, b, c, d *such that* $X = (F, i)^*$, $Y = H_{F,i}(a, b, c, d)$, *a is a subterm of c and d arises from c by replacing one occurrence of a by b.*

(iii) $\varphi_{71}(X, Y_1, Y_2)$ *in* $\mathscr{F}_\Delta$ *iff there are* $(F, i) \in \Delta^{(2)}$ *and terms* a, b, c, d *such that* $X = (F, i)^*$, $Y_1 = H_{F,i}(a, b)$, $Y_2 = H_{F,i}(c, d)$ *and* (c, d) *is an immediate consequence of* (a, b).

(iv) $\varphi_{72}(X, Y)$ *in* $\mathscr{F}_\Delta$ *iff there are* $(F, i) \in \Delta^{(2)}$ *and a term a such that* $X = (F, i)^*$ *and* $Y = H_{F,i}(a, a)$.

If $(F, i) \in \Delta^{(2)}$ and $(a_1, b_1), \ldots, (a_n, b_n)$ is a finite sequence of equations, put $H_{F,i}^{(2)}((a_1, b_1), \ldots, (a_n, b_n)) = H_{F,i}(u_1, \ldots, u_n)$ where $u_1, \ldots, u_n$ are terms with pairwise disjoint sets of variables such that $u_1^* = H_{F,i}(a_1, b_1), \ldots, u_n^* = H_{F,i}(a_n, b_n)$.

Definition. (i) $\varphi_{73}(X, Y) \equiv \varphi_{62}(X, Y) \& \forall Z, U(\varphi_{60}(X, Y, Z, U) \to \varphi_{56}(X, U, X))$.

(ii) $\varphi_{74}(X, Y_1, Y_2) \equiv \varphi_{73}(X, Y_1) \& \exists Y_3, Z_3(\varphi_{56}(X, Y_3, Z_3) \& \varphi_{69}(X, Y_3, Y_2) \&$
$\& \forall U_1, U_2\, \exists U_3, U_4((\varphi_{56}(X, U_1, X) \& \varphi_{59}(X, U_1, U_2) \& \varphi_{61}(X, Y_3, U_2)) \to$
$\to ((\varphi_{72}(X, U_3)\ \mathrm{VEL}\ \exists Z\, \varphi_{60}(X, Y_1, Z, U_3)) \& (U_3 = U_4\ \mathrm{VEL}\ \varphi_{69}(X, U_3, U_4)) \&$
$\& \varphi_{71}(X, U_4, U_1))))$.

(iii) $\varphi_{75}(X, Y_1, Y_2) \equiv \exists Y_3(\varphi_{64}(X, Y_1, Y_3) \& \varphi_{74}(X, Y_3, Y_2))$.

6.7. Lemma. *Let Δ be a strictly large type. Then:*

(i) $\varphi_{73}(X, Y)$ *in* $\mathscr{F}_\Delta$ *iff there are* $(F, i) \in \Delta^{(2)}$ *and a finite sequence* $(a_1, b_1), \ldots$ $\ldots, (a_n, b_n)$ *of equations such that* $X = (F, i)^*$ *and* $Y = H_{F,i}^{(2)}((a_1, b_1), \ldots, (a_n, b_n))$.

(ii) $\varphi_{74}(X, Y_1, Y_2)$ *in* $\mathscr{F}_\Delta$ *iff there are* $(F, i) \in \Delta^{(2)}$, *a finite sequence* $(a_1, b_1), \ldots$ $\ldots, (a_n, b_n)$ *of equations and an equation* (a, b) *such that* $X = (F, i)^*$, $Y_1 = H_{F,i}^{(2)}((a_1, b_1), \ldots, (a_n, b_n))$, $Y_2 = H_{F,i}(a, b)$ *and (a, b) is a consequence of* $\{(a_1, b_1), \ldots$ $\ldots, (a_n, b_n)\}$.

(iii) $\varphi_{75}(X, Y_1, Y_2)$ *in* $\mathscr{F}_\Delta$ *iff there are* $(F, i) \in \Delta^{(2)}$ *and terms* a, b, c, d *such that* $X = (F, i)^*$, $Y_1 = H_{F,i}(a, b)$, $Y_2 = H_{F,i}(c, d)$ *and* (c, d) *is a consequence of* (a, b).

7. DEFINABILITY UP TO AUTOMORPHISMS IN $\mathscr{F}_\Delta$

Denote by S_Δ the group of all permutations of Δ, by S_{Δ_0} the group of all permutations of Δ_0 and by $S_\Delta^{(1)}$ the group of permutations f of $\Delta^{(1)}$ with the following two properties: if $f(F, i) = (G, j)$ then $n_F = n_G$; if $f(F, i) = (G, j)$ and $f(F, k) = (H, l)$ then $G = H$. If $f \in S_\Delta^{(1)}$ and $F \in \Delta \setminus \Delta_0$, then the first member of $f(F, 1)$ will be denoted by $f(F)$.

For every type Δ define a group G_Δ as follows: if Δ is not a large unary type then $G_\Delta = S_{\Delta_0} \times S_\Delta^{(1)}$; if Δ is a large unary type then $G_\Delta = C_2 \times S_\Delta$ where C_2 is the two-element group $\{1, 2\}$ with unit 1.

For every pair $(c, f) \in G_\Delta$ define a permutation $P_{c,f}$ of W_Δ as follows:

(1) Let Δ be not a large unary type and let $t \in W_\Delta$. If $t \in V$, put $P_{c,f}(t) = t$. If $t \in \Delta_0$, put $P_{c,f}(t) = c(t)$. If $t = F(t_1, \ldots, t_n)$ where $F \in \Delta_n$, $n \geqq 1$ and $f(F, 1) = (G, i(1)), \ldots, f(F, n) = (G, i(n))$, put $P_{c,f}(t) = G(P_{c,f}(t_{i^{-1}(1)}), \ldots, P_{c,f}(t_{i^{-1}(n)}))$.

(2) Let Δ be a large unary type and $t = F_k \ldots F_1 x$ where $x \in V$ and $f(F_1) = G_1, \ldots, f(F_k) = G_k$. If $c = 1$, put $P_{c,f}(t) = G_k \ldots G_1 x$. If $c = 2$, put $P_{c,f}(t) = G_1 \ldots G_k x$.

7.1. Lemma. *We have* $P_{(c_2,f_2)(c_1,f_1)} = P_{c_2,f_2} P_{c_1,f_1}$. *If* h *is a substitution then* $P_{c,f}(h(t)) = k(P_{c,f}(t))$ *where* k *is the substitution with* $k(x) = P_{c,f}(h(x))$ *for all* $x \in V$. *We have* $t \leqq u$ *iff* $P_{c,f}(t) \leqq P_{c,f}(u)$.

For every pair $(c, f) \in G_\Delta$ define a mapping $\bar{P}_{c,f}$ of $\mathscr{F}_\Delta$ into $\mathscr{F}_\Delta$ as follows: $\bar{P}_{c,f}(U) = \{P_{c,f}(u);\ u \in U\}$.

7.2. Lemma. *For every* $(c, f) \in G_\Delta$ *the mapping* $\bar{P}_{c,f}$ *is an automorphism of* $\mathscr{F}_\Delta$. *Moreover, the mapping* $(c, f) \mapsto \bar{P}_{c,f}$ *is an isomorphism of* G_Δ *onto a subgroup of the automorphism group of* $\mathscr{F}_\Delta$.

Let $t_1, \ldots, t_n$ be a non-empty finite sequence of terms. By a supporting sequence for $t_1, \ldots, t_n$ we mean a finite sequence $((H_1, \ldots, H_m), ((F_1, p_1), \ldots, (F_k, p_k)), (e_{1,0}, \ldots, e_{1,s_1}), \ldots, (e_{n,0}, \ldots, e_{n,s_n}))$ such that $H_1, \ldots, H_m$ are all pairwise different nullary symbols occurring in a term from $\{t_1, \ldots, t_n\}$, $(F_1, p_1), \ldots, (F_k, p_k)$ are all pairwise different pairs from $\Delta^{(1)}$ whose first members are symbols occurring in a term from $\{t_1, \ldots, t_n\}$ and if $i \in \{1, \ldots, n\}$ then $e_{i,0}, \ldots, e_{i,s_i}$ are all pairwise different elements of $E(t_i)$ and $e_{i,0}$ is the empty sequence.

Let Δ be a strictly large type, $t_1, \ldots, t_n$ a non-empty finite sequence of terms from W_Δ and $r = ((H_1, \ldots, H_m), ((F_1, p_1), \ldots, (F_k, p_k)), (e_{1,0}, \ldots, e_{1,s_1}), \ldots, (e_{n,0}, \ldots, e_{n,s_n}))$ a supporting sequence for $t_1, \ldots, t_n$. We denote by

$$\mu^r_{t_1,\ldots,t_n}(X_1, \ldots, X_n, Y_1, \ldots, Y_m, Z_1, \ldots, Z_k)$$

the formula

$$\exists X, A_{1,0}, \ldots, A_{1,s_1}, \ldots, A_{n,0}, \ldots A_{n,s_n}, B_{1,0}, \ldots, B_{1,s_1}, \ldots, B_{n,0}, \ldots$$

$$\ldots, B_{n,s_n}(g_1 \,\&\, g_2 \,\&\, g_3 \,\&\, g_4 \,\&\, g_5 \,\&\, g_6 \,\&\, g_7 \,\&\, g_8 \,\&\, g_9 \,\&\, g_{10} \,\&\, g_{11})$$

where

g_1 is the conjunction of the formulas $\varphi_{67}(X, X_i, A_{i,j}, B_{i,j})$

$$(1 \leqq i \leqq n,\ 0 \leqq j \leqq s_i),$$

g_2 is the conjunction of the formulas $\omega_1(B_{i,j})$

$$(1 \leqq i \leqq n,\ 0 \leqq j \leqq s_i,\ t_i\langle e_{i,j}\rangle \in V),$$

g_3 is the conjunction of the formulas $B_{i,j} = Y_l \,\&\, \alpha_0(Y_l)$

$$(1 \leqq i \leqq n,\ 0 \leqq j \leqq s_i,\ 1 \leqq l \leqq m,\ t_i\langle e_{i,j}\rangle = H_l),$$

g_4 is the conjunction of the formulas $\varphi_{68}(X, X_i, A_{i,j}, A_{i,l}, B_{i,j})$

$$(1 \leqq i \leqq n,\ 0 \leqq j, l \leqq s_i,\ t_i\langle e_{i,j}\rangle = t_i\langle e_{i,l}\rangle),$$

g_5 is the conjunction of the formulas $\neg\varphi_{68}(X, X_i, A_{i,j}, A_{i,l}, B_{i,j})$

$$(1 \leqq i \leqq n,\ 0 \leqq j, l \leqq s_i,\ t_i\langle e_{i,j}\rangle \neq t_i\langle e_{i,l}\rangle),$$

g_6 is the conjunction of the formulas $\varphi_{65}(X, A_{i,j}, Z_h, A_{i,l})$

$$(1 \leqq i \leqq n,\ 0 \leqq j, l \leqq s_i,\ 1 \leqq h \leqq k,\ e_{i,l} = e_{i,j}(F_h, p_h)),$$

g_7 is the conjunction of the formulas $\omega_1(A_{i,0})$

$$(1 \leqq i \leqq n),$$

g_8 is the conjunction of the formulas $\exists U(\alpha_h(U) \,\&\, U \prec Z_i \,\&\, U \prec Z_j)$

$$(1 \leqq i, j \leqq k,\ F_i = F_j,\ h = n_{F_i}),$$

g_9 is the conjunction of the formulas $\neg\, \exists U(\alpha(U) \,\&\, U \prec Z_i \,\&\, U \prec Z_j)$

$$(1 \leqq i, j \leqq k,\ F_i \neq F_j),$$

g_{10} is the conjunction of the formulas $Z_i \neq Z_j$

$$(1 \leqq i, j \leqq k,\ i \neq j),$$

g_{11} is the conjunction of the formulas $Y_i \neq Y_j$

$$(1 \leqq i, j \leqq m,\ i \neq j).$$

7.3. Lemma. *Let Δ be a strictly large type, $t_1, \ldots, t_n$ a non-empty finite sequence of terms from W_Δ and $r = ((H_1, \ldots, H_m), ((F_1, p_1), \ldots, (F_k, p_k)), \ldots)$ a* *supporting sequence for* $t_1, \ldots, t_n$. *Then* $\mu^r_{t_1,\ldots,t_n}(X_1, \ldots, X_n, Y_1, \ldots, Y_m, Z_1, \ldots, Z_k)$ *in* $\mathcal{F}_\Delta$ *iff there is a pair* $(c, f) \in G_\Delta$ *such that if* $1 \leqq i \leqq n$ *then* $X_i = (P_{c,f}(t_i))^*$, *if* $1 \leqq i \leqq m$ *then* $Y_i = (c(H_i))^*$ *and if* $1 \leqq i \leqq k$ *then* $Z_i = (f(F_i, p_i))^*$.

7.4. Lemma. *Let* Δ *be a strictly large type and let* h *be an automorphism of* $\mathcal{F}_\Delta$. *Then* $h = \bar{P}_{c,f}$ *for some* $(c, f) \in G_\Delta$.

Proof. If $H \in \Delta_0$, then $\alpha_0(H^*)$ is satisfied in $\mathcal{F}_\Delta$, so that $\alpha_0(h(H^*))$ is satisfied and so $h(H^*) = (c(H))^*$ for some $c(H) \in \Delta_0$. If $(F, i) \in \Delta^{(1)}$, then $\varphi_4((F, i)^*)$ is satisfied in $\mathcal{F}_\Delta$, so that $\varphi_4(h((F, i)^*))$ is satisfied and $h((F, i)^*) = (f(F, i))^*$ for some $f(F, i) \in \Delta^{(1)}$. We get two mappings c, f and it is easy to see that $(c, f) \in G_\Delta$. Let $t \in W_\Delta$. There exists a supporting sequence $r = ((H_1, \ldots, H_m), ((F_1, p_1), \ldots, (F_k, p_k)), \ldots)$ for t. Evidently, $\mu^r_t(t^*, H^*_1, \ldots, H^*_m, (F_1, p_1)^*, \ldots, (F_k, p_k)^*)$ is satisfied in $\mathcal{F}_\Delta$. Hence $\mu^r_t(h(t^*), h(H^*_1), \ldots, h(H^*_m), h((F_1, p_1)^*), \ldots, h((F_k, p_k)^*))$ is satisfied in $\mathcal{F}_\Delta$, too. It follows from 7.3 that $h(t^*) = (P_{c,f}(t))^*$. Now let $A \in \mathcal{F}_\Delta$. For every term t we have $t \in A$ iff $t^* \subseteq A$ iff $h(t^*) \subseteq h(A)$ iff $(P_{c,f}(t))^* \subseteq h(A)$ iff $P_{c,f}(t) \in h(A)$, so that $h(A) = \bar{P}_{c,f}(A)$. We get $h = \bar{P}_{c,f}$.

Let Δ be a large but not strictly large type and let $t_1, \ldots, t_n$ be a non-empty finite sequence of terms from W_Δ. For every $i \in \{1, \ldots, n\}$, the term t_i can be uniquely expressed in the form $t_i = F_{i,k_i} \ldots F_{i,1} y_i$ where $y_i \in V \cup \Delta_0$ and $F_{i,1}, \ldots, F_{i,k_i} \in \Delta_1$. We denote by

$$\mu_{t_1,\ldots,t_n}(A_1, A_2, B, X_1, \ldots, X_n, Y_1, \ldots, Y_n, Z_{1,1}, \ldots, Z_{1,k_1}, \ldots, Z_{n,1}, \ldots, Z_{n,k_n})$$

the formula

$$\varphi_{33}(A_1, A_2, B) \mathbin{\&} \exists U_{1,0}, \ldots, U_{1,k_1}, \ldots, U_{n,0}, \ldots$$
$$\ldots, U_{n,k_n}(g_1 \mathbin{\&} g_2 \mathbin{\&} g_3 \mathbin{\&} g_4 \mathbin{\&} g_5 \mathbin{\&} g_6 \mathbin{\&} g_7 \mathbin{\&} g_8 \mathbin{\&} g_9)$$

where

g_1 is the conjunction of the formulas $\omega_1(Y_i)$

$$(1 \leqq i \leqq n,\ y_i \in V),$$

g_2 is the conjunction of the formulas $\alpha_0(Y_i)$

$$(1 \leqq i \leqq n,\ y_i \in \Delta_0),$$

g_3 is the conjunction of the formulas $\alpha_1(Z_{i,j})$

$$(1 \leqq i \leqq n,\ 1 \leqq j \leqq k_i),$$

g_4 is the conjunction of the formulas $\varphi_{38}(A_1, A_2, B, Z_{i,j}, U_{i,j-1}, U_{i,j})$

$$(1 \leqq i \leqq n,\ 1 \leqq j \leqq k_i),$$

g_5 is the conjunction of the formulas $Z_{i,j} = Z_{l,h}$

$$(1 \leqq i, l \leqq n,\ 1 \leqq j \leqq k_i,\ 1 \leqq h \leqq k_l,\ F_{i,j} = F_{l,h}),$$

g_6 is the conjunction of the formulas $Z_{i,j} \neq Z_{l,h}$

$$(1 \leqq i, l \leqq n,\ 1 \leqq j \leqq k_i,\ 1 \leqq h \leqq k_l,\ F_{i,j} \neq F_{l,h}),$$

g_7 is the conjunction of the formulas $Y_i = Y_j$

$$(1 \leqq i, j \leqq n,\ y_i = y_j \in \Delta_0),$$

g_8 is the conunction of the formulas $Y_i \neq Y_j$

$$(1 \leqq i, j \leqq n,\ y_i \neq y_j,\ y_i, y_j \in \Delta_0),$$

g_9 is the conjunction of the formulas $U_{i,0} = Y_i \,\&\, U_{i,k_i} = X_i$

$$(1 \leqq i \leqq n).$$

7.5. Lemma. *Let Δ be a large but not strictly large type and let $t_1, \ldots, t_n$ be a non-empty finite sequence of terms from W_Δ; let $t_i = F_{i,k_i} \ldots F_{i,1} y_i$ where $y_i \in V \cup \Delta_0$. Then $\mu_{t_1,\ldots,t_n}(A_1, A_2, B, X_1, \ldots, X_n, Y_1, \ldots, Y_n, Z_{1,1}, \ldots, Z_{1,k_1}, \ldots, Z_{n,1}, \ldots \ldots, Z_{n,k_n})$ in $\mathscr{F}_\Delta$ iff there are $(F, G, w, x) \in \Delta^{(4)}$ and $(c, f) \in G_\Delta$ such that $A_1 = F^*$, $A_2 = G^*$, $B = (wx)^*$, if $1 \leqq i \leqq n$ then $X_i = (P_{c,f}(t_i))^*$, $Y_i = (P_{c,f}(y_i))^*$. $Z_{i,1} = (f(F_{i,1}))^*, \ldots, Z_{i,k_i} = (f(F_{i,k_i}))^*$ and if Δ is unary then either $w = GF$, $c = 1$ or $w = FG$, $c = 2$.*

7.6. Lemma. *Let Δ be not a strictly large type and let h be an automorphism of $\mathscr{F}_\Delta$. Then $h = \bar{P}_{c,f}$ for some $(c, f) \in G_\Delta$.*

Proof. If Δ is small, all is evident. If Δ all large and not unary, the proof is analogous to that of 7.4. Let Δ be a large unary type. Similarly as in the proof of 7.4, there is a permutation f of Δ such that $h(F^*) = (f(F))^*$ for all $F \in \Delta$. Let F, G be two different symbols from Δ and let $x \in V$; put $F_1 = f(F)$, $G_1 = f(G)$. Evidently, $\varphi_{33}(F^*, G^*, (GFx)^*)$ is satisfied in $\mathscr{F}_\Delta$. Hence $\varphi_{33}(h(F^*), h(G^*), h((GFx)^*))$ is satisfied, too. By 5.1 we have either $h((GFx)^*) = (G_1F_1x)^*$ or $h((GFx)^*) = (F_1G_1x)^*$. In the former case put $c = 1$, while in the latter put $c = 2$. We have $(c, f) \in G_\Delta$. Using 5.2(v), it is easy to see that the definition of c does not depend on the choice of the pair F, G. Now we get easily from 7.5 that $h(t^*) = (P_{c,f}(t))^*$ for any term t; this implies $h = \bar{P}_{c,f}$ similarly as in the proof of 7.4.

Combining 7.2, 7.4 and 7.6, we get the following result.

7.7. Theorem. *Le Δ be any type. For every automorphism h of $\mathscr{F}_\Delta$ there exists a pair $(c, f) \in G_\Delta$ such that $h = \bar{P}_{c,f}$. The automorphism group of $\mathscr{F}_\Delta$ is isomorphic to G_Δ.*

For every type Δ and every non-empty finite sequence $t_1, \ldots, t_n$ of terms from W_Δ we define a formula $\vartheta_{\Delta,t_1,\ldots,t_n}(X)$ as follows. If Δ is strictly large, fix a supporting sequence $r = ((H_1, \ldots, H_m), ((F_1, p_1), \ldots, (F_k, p_k)), \ldots)$ for $t_1, \ldots, t_n$ and put

$$\vartheta_{\Delta,t_1,\ldots,t_n}(X) \equiv \exists X_1, \ldots, X_n, Y_1, \ldots, Y_m, Z_1, \ldots$$
$$\ldots, Z_k(\mu^r_{t_1,\ldots,t_n}(X_1, \ldots, X_n, Y_1, \ldots, Y_m, Z_1, \ldots, Z_k) \,\&\, X = X_1 \vee \ldots \vee X_n).$$

If Δ is large but not strictly large and $t_i = F_{i,k_i} \dots F_{i,1} y_i$ where $y_i \in V \cup \Delta_0$, put

$$\vartheta_{\Delta,t_1,\dots,t_n}(X) \equiv \exists A_1, A_2, B, X_1, \dots, X_n, Y_1, \dots, Y_n, Z_{1,1}, \dots$$
$$\dots, Z_{n,k_n}(\mu_{t_1,\dots,t_n}(A_1, A_2, B, X_1, \dots, X_n, Y_1, \dots, Y_n, Z_{1,1}, \dots, Z_{n,k_n}) \,\&$$
$$\& \, X = X_1 \vee \dots \vee X_n) \,.$$

Finally, let Δ be small. Then for every $i \in \{1, \dots, n\}$ we can express t_i in the form $t_i = F^{k_i} y_i$ where $y_i \in \Delta_0$, $k_i \geqq 0$ and if $k_i \neq 0$ then $F \in \Delta_1$. Put

$$\vartheta_{\Delta,t_1,\dots,t_n}(X) \equiv$$
$$\equiv \exists X_1, \dots, X_n, Y_1, \dots, Y_n(g_1 \,\&\, g_2 \,\&\, g_3 \,\&\, g_4 \,\&\, g_5 \,\&\, X = X_1 \vee \dots \vee X_n)$$

where

g_1 is the conjunction of the formulas $\omega_1(Y_i)$

$$(1 \leqq i \leqq n, \; y_i \in V) \,,$$

g_2 is the conjunction of the formulas $\alpha_0(Y_i)$

$$(1 \leqq i \leqq n, \; y_i \in \Delta_0) \,,$$

g_3 is the conjunction of the formulas $Y_i = Y_j$

$$(1 \leqq i, j \leqq n, \; y_i = y_j \in \Delta_0) \,,$$

g_4 is the conjunction of the formulas $Y_i \neq Y_j$

$$(1 \leqq i, j \leqq n, \; y_i \neq y_j, \; y_i, y_j \in \Delta_0) \,,$$

g_5 is the conjunction of the formulas $\exists Z_0, \dots, Z_{k_i}(Z_0 = Y_i \,\&\, Z_{k_i} = X_i \,\&\, Z_0 \lessdot \lessdot Z_1 \,\&\, \dots \,\&\, Z_{k_i - 1} \lessdot Z_{k_i})$

$$(1 \leqq i \leqq n) \,.$$

The following theorem is an easy combination of the above results.

7.8. Theorem. *Let Δ be any type and let $t_1, \dots, t_n$ be a non-empty finite sequence of terms from W_Δ. Then $\vartheta_{\Delta,t_1,\dots,t_n}(X)$ in $\mathscr{F}_\Delta$ iff $X = \bar{P}_{c,f}(\{t_1, \dots, t_n\}^*)$ for some $(c, f) \in G_\Delta$.*

7.9. Corollary. *Every finitely generated element of $\mathscr{F}_\Delta$ is definable up to automorphisms in $\mathscr{F}_\Delta$.*

Reference

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