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SMOOTHNESS OF A TYPICAL CONVEX FUNCTION

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Introduction. Let $G \subset R^m$ be a nonempty open bounded convex set. Denote by $\mathscr{F} = \mathscr{F}(G)$ the system of all bounded convex functions on G and put $\mathscr{F}^j = \mathscr{F} \cap \cap C^j(G)$. The set $\mathscr{F}$ equipped with the usual supremum metric is obviously a complete metric space.

P. M. Gruber [2] 1977 proved that a typical function from $\mathscr{F}$ is smooth but not too smooth. More specifically, one of his results reads as follows:

Sets $\mathscr{F} \setminus \mathscr{F}^1$ and $\mathscr{F}^2$ are of the first category.

The aim of this note is to give a more detailed information concerning the "gap" between $\mathscr{F}^1$ and $\mathscr{F}^2$. A special case of our result shows that a typical function of $\mathscr{F}$ is of the class $C^{1+\alpha}$ on no (nonempty) open subset of G.

Notation. Let Ω stand for the set of all continuous increasing functions $\omega : : [0, \infty[\to [0, \infty[$ such that $\omega(0) = 0$ and $\omega > 0$ on $]0, \infty[$. For $\omega_1, \omega_2 \in \Omega$, we write $\omega_1 \prec \omega_2$, if $\omega_1(t) = O(\omega_2(t))$, $t \to 0+$. A set $\Omega' \subset \Omega$ is said to be *majorized*, if there is $\omega_0 \in \Omega$ such that $\omega \prec \omega_0$, whenever $\omega \in \Omega'$.

If $M \subset R^m$, $\omega \in \Omega$, then $\mathscr{D}_\omega(M)$ is the set of functions g defined on M and satisfying

$$|g(x) - g(y)| \leqq \omega(|x - y|)\,, \quad x, y \in M\,.$$

In what follows, J denotes the set $\{1, 2, \ldots, m\}$ and $\partial_j f$ means the j-th partial derivative of f.

Theorem. *Let $\Omega^* \subset \Omega$ be majorized and let $\mathscr{F}^*$ be the set of all $f \in \mathscr{F}$ possessing the following property: There exist $j \in J$, $\omega \in \Omega^*$ and a nonempty open set $G^* \subset G$ such that $\partial_j f \in \mathscr{D}_\omega(G^*)$. Then the set $\mathscr{F}^*$ is of the first category.*

Remark. The proof of Theorem is postponed to the end of this note.

Denote by Ω_H the set of all functions ω for which there are $K > 0$ and $\alpha > 0$ such that $\omega(t) = Kt^\alpha$, $t \in [0, \infty[$. It is easily seen that Ω_H is majorized. (For instance, if $\omega_0(t) = -1/\log t$ for $t \in]0, 1/e[$, $\omega_0(0) = 0$, $\omega_0 = 1$ on $[1/e, \infty[$, then $\omega_0 \in \Omega$ and $\omega \prec \omega_0$ for every $\omega \in \Omega_H$.)

It follows from Theorem that, in an obvious sense, *a typical function from $\mathscr{F}$ has nowhere Hölder continuous partial derivatives.* In particular, $\mathscr{F}^2$ *is of the first category.*

Similar assertions could be stated for other scales of moduli of continuity. In this connection, the following notion seems to be appropriate.

A set $\Omega_1 \subset \Omega$ is said to have *countable character,* if there is a countable set $\Omega_2 \subset \Omega_1$ such that for every $\omega_1 \in \Omega_1$ there is $\omega_2 \in \Omega_2$ with $\omega_1 \prec \omega_2$. (Clearly, Ω_H has countable character.) The proof of the following assertion is left to the reader:

Every subset of Ω having countable character is majorized.

The proof of Theorem is based on two lemmas. In Lemma 2, $\mathscr{F}^1$ is considered as the subspace of the metric space $\mathscr{F}$.

Lemma 1. *Let $j \in J$, $\omega \in \Omega$ and let B be an open ball with center at $z_0 \in R^m$. Suppose that $\varepsilon > 0$, $d > 0$. Then there is a convex function φ with the following properties:*

$$\varphi \in C^1(R^m), \quad \sup \{|\varphi(z)|;\ |z - z_0| < d\} < \varepsilon$$

and there are distinct points $x_0, y_0 \in B$ such that

$$\partial_j\varphi(x_0) - \partial_j\varphi(y_0) \geqq 3\omega(|x_0 - y_0|).$$

Proof. Without loss of generality we can suppose that $B = \{z \in R^m;\ |z| < 2r\}$ and $d > r > 0$. Define $\omega(t) = 0$ for $t \leqq 0$ and

$$h(s) = \int_0^s \omega^{1/2}(t - r)\, dt, \quad s \geqq 0.$$

Notice that $\alpha = h(d) > 0$ and that h is a continuously differentiable convex function on $[0, \infty[$ vanishing near the origin. Consequently, the function $\varphi : x \to (\varepsilon/\alpha)(h|x|)$ is convex in R^m, $\varphi \in C^1(R^m)$ and $|\varphi(z)| < \varepsilon$ provided $|z| < d$.

Put $e_j = (0, \ldots, 1, \ldots, 0)$ (1 is on the j-th place) and find $t_0 \in (0, r)$ such that $\omega(t_0) < (\varepsilon/3\alpha)^2$. If $x_0 = (r + t_0)\, e_j$, $y_0 = re_j$, then $x_0, y_0 \in B$, $|x_0 - y_0| = t_0$ and

$$\partial_j\varphi(x_0) - \partial_j\varphi(y_0) = (\varepsilon/\alpha)\,(h'(r + t_0) - h'(r)) = (\varepsilon/\alpha)\,\omega^{1/2}(t_0).$$

Since $(3\alpha/\varepsilon)\,\omega^{1/2}(t_0) \leqq 1$, we conclude that

$$\partial_j\varphi(x_0) - \partial_j\varphi(y_0) \geqq 3\omega(t_0) = 3\omega(|x_0 - y_0|).$$

Lemma 2. *Suppose that $\omega \in \Omega$, $j \in J$ and $B \subset G$ is an open ball. Denote by $\mathscr{A}(j, \omega, B)$ the set of all $f \in \mathscr{F}^1$ such that $\partial_j f \in \mathscr{D}_\omega(B)$. Then $\mathscr{A}(j, \omega, B)$ is nowhere dense in $\mathscr{F}^1$.*

Proof. Let N be the set of positive integers. Write $\mathscr{A} = \mathscr{A}(j, \omega, B)$ and prove first that $\mathscr{A}$ is a closed subset of $\mathscr{F}^1$.

To this end assume that the sequence $\{f_n\}$ of functions belonging to $\mathscr{A}$ converges uniformly on G to a function $f \in \mathscr{F}^1$. We are going to show that $f \in \mathscr{A}$.

Fix $x, y \in B$ and choose $\delta > 0$ such that $x + \delta e_j$, $y + \delta e_j \in B$. For $t \in [0, \delta[$ and $n \in N$ define

$$\alpha_n(t) = f_n(x + te_j) - f_n(x) - (f_n(y + te_j) - f_n(y)) .$$

Notice that $\alpha_n(0) = 0$ and for $s \in]0, \delta[$,

$$\alpha_n'(s) = \partial_j f_n(x + se_j) - \partial_j f_n(y + se_j) .$$

Since $f_n \in \mathscr{A}$, the inequality $|\alpha_n'(s)| \leqq \omega(|x - y|)$ holds whenever $s \in]0, \delta[$. Thus we have

$$|\alpha_n(t)| \leqq t \,.\, \omega(|x - y|) \quad \text{for every} \quad t \in [0, \delta[\,.$$

Given $t \in]0, \delta[$,

$$\lim_{n \to \infty} \alpha_n(t) = f(x + te_j) - f(x) - (f(y + te_j) - f(y))$$

so that

$$\left| \frac{f(x + te_j) - f(x)}{t} - \frac{f(y + te_j) - f(y)}{t} \right| \leqq \omega(|x - y|) \,.$$

Letting $t \to 0+$, we conclude that $|\partial_j f(x) - \partial_j f(y)| \leqq \omega(|x - y|)$. Consequently, $f \in \mathscr{A}$ and $\mathscr{A}$ is closed (in $\mathscr{F}^1$).

Fix now $\varepsilon > 0$ and $f \in \mathscr{A}$. To finish the proof of the lemma, it is sufficient to find a function $g \in \mathscr{F}^1$ such that $g \notin \mathscr{A}$ and the distance $\varrho(f, g)$ of f and g is less than ε.

Let d be the diameter of G and φ, x_0, y_0 have the same meaning as in Lemma 1. Define $g(z) = f(z) + \varphi(z)$, $z \in G$. Then $g \in \mathscr{F}^1$ and $\varrho(f, g) < \varepsilon$. We have

$$\partial_j g(x_0) - \partial_j g(y_0) = \partial_j f(x_0) - \partial_j f(y_0) + \partial_j \varphi(x_0) - \partial_j \varphi(y_0) \geqq$$
$$\geqq -\omega(|x_0 - y_0|) + 3\omega(|x_0 - y_0|) = 2\omega(|x_0 - y_0|) \,.$$

Consequently, $g \notin \mathscr{A}$.

The proof of the lemma is complete.

Proof of Theorem. Choose a countable system $\{B_i;\, i \in I\}$ of open balls $B_i \subset G$ such that $G = \bigcup B_i$ and for every nonempty open set $G' \subset G$ there is $B_i \subset G'$. Since Ω^* is majorized, there is $\omega^* \in \Omega$ such that $\omega \prec \omega^*$ whenever $\omega \in \Omega^*$. It is easily seen that for every $\omega \in \Omega^*$ there is $k \in N$ such that $\omega \leqq k\omega^*$ on $[0, d]$. It follows that

$$\mathscr{F}^* \subset (\mathscr{F} \setminus \mathscr{F}^1) \cup \big(\bigcup_{i \in I} \bigcup_{j \in J} \bigcup_{k \in N} \mathscr{A}(j, k\omega^*, B_i)\big) \,.$$

Gruber's result states that $\mathscr{F} \setminus \mathscr{F}^1$ is of the first category. By Lemma 2, $\mathscr{A}(j, k\omega^*, B_i)$ is nowhere dense in $\mathscr{F}^1$ and, *a fortiori*, in $\mathscr{F}$. We conclude that $\mathscr{F}^*$ is of the first category.

Remark. Various questions related to differential properties of convex functions are studied e.g. in [1]–[4] where further references can be found.

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