Jiří Jarník; Jaroslav Kurzweil

Sets of solutions of differential relations

Czechoslovak Mathematical Journal, Vol. 31 (1981), No. 4, 554–568

Persistent URL: http://dml.cz/dmlcz/101773

Czechoslovak Mathematical Journal, 31 (106) 1981, Praha

SETS OF SOLUTIONS OF DIFFERENTIAL RELATIONS

JIŘÍ JARNÍK and JAROSLAV KURZWEIL, Praha

(Received December 12, 1979)

1

1.0. Introduction. Let F be a map from the $(n + 1)$-dimensional Euclidean space $\mathbb{R}^1 \times \mathbb{R}^n$ into the family $\mathscr{K}^n$ of all convex compact nonempty subsets of $\mathbb{R}^n$, i.e. $F : G \to \mathscr{K}^n$ where $G \subset \mathbb{R}^{n+1}$. We denote by Sol F the set of all solutions (in the usual sense) of the differential relation

$$\dot{x} \in F(t, x) . \tag{1.1}$$

Let U be a set of functions $u : I_u \to \mathbb{R}^n$ where the definition domain $I_u \subset \mathbb{R} = \mathbb{R}^1$ of u is an interval. It was shown in [1] that under certain conditions on U it is possible to find a map $Q : \mathbb{R}^{n+1} \to \mathscr{K}^n_0$ where $\mathscr{K}^n_0 = \mathscr{K}^n \cup \{\emptyset\}$, the map $Q(t, .)$ is upper semicontinuous for almost all t, $U \subset$ Sol Q and Q is the minimal map with this property, that is: if $H : \mathbb{R}^{n+1} \to \mathscr{K}^n_0$, $H(t, .)$ is upper semicontinuous for almost all t and $U \subset$ Sol H, then there is a set $N \subset \mathbb{R}$ of measure zero such that $Q(t, x) \subset \subset H(t, x)$ provided $t \notin N$.

The aim of the present paper is to find conditions under which $U =$ Sol Q or, more generally, to determine the structure of the set Sol Q. This is achieved in Part 1 by introducing a certain operation Θ which is described in the next section.

Roughly speaking, the operation Θ assigns a set U of continuous functions another set $\Theta(U)$ which contains uniform limits of sequences of functions that belong "piecewise" to U and whose jump functions approach zero uniformly.

The main results are Theorems 1.2 and 1.4. Theorem 1.2 states that $U \subset$ Sol F implies $\Theta(U) \subset$ Sol F while Theorem 1.4 asserts that, given a "reasonable" set U, there exists a differential relation $\dot{x} \in Q(t, x)$ such that Sol $Q = \Theta(U)$.

1.1. Notations and definitions. Given a set U of continuous functions $u : I_u \to \to \mathbb{R}^n$, $I_u \subset \mathbb{R}$ an interval, we denote by $\mathscr{W}(U)$ the set of all functions $w : I_w \to \mathbb{R}^n$ ($I_w \subset \mathbb{R}$ a compact interval) with the following property:
(1.2) For each $w \in \mathscr{W}(U)$ there exists a positive integer k and a decomposition of

the interval I_w,

$$\sigma_0 \leqq \tau_1 < \sigma_1 \leqq \tau_2 < \sigma_2 \leqq \ldots \leqq \tau_k < \sigma_k \leqq \tau_{k+1},$$

such that for each $i = 1, 2, \ldots, k$ there is a function $u_i \in U$, $[\tau_i, \sigma_i] \subset I_{u_i}$, satisfying

$$w(t) = \begin{cases} u_1(\tau_1) & \text{for } t \in [\sigma_0, \tau_1], \\ u_i(t) & \text{for } t \in (\tau_i, \sigma_i), \quad i = 1, \ldots, k, \\ u_i(\sigma_i) & \text{for } t \in [\sigma_i, \tau_{i+1}], \quad i = 1, \ldots, k. \end{cases}$$

The only possible points of discontinuity of the function w are the points τ_i; nevertheless, the one-sided limits at these points evidently exist. Hence each $w \in \mathscr{W}(U)$ can be assigned in a unique way a piecewise constant function $Jw : I_w \to \mathbb{R}^n$ so that $w - Jw$ is continuous, $(Jw)(t) = 0$ for $t \in [\sigma_0, \tau_2]$.

For $w \in \mathscr{W}(U)$ let us denote

$$\eta(w) = \sum_{i=0}^{k} (\tau_{i+1} - \sigma_i).$$

Further, we denote by $\Theta(U)$ the set of all functions $q : I_q \to \mathbb{R}^n$ ($I_q \subset \mathbb{R}$ an interval) which fulfil the following condition:

(1.3) If $[\alpha, \beta] \subset I_q$ then there exists a sequence of functions $w_j \in \mathscr{W}(U)$, $j = 1, 2, \ldots$, such that $I_{w_j} = [\alpha, \beta]$,

$$\lim_{j \to \infty} w_j = q, \quad \lim_{j \to \infty} Jw_j = 0$$

both uniformly on $[\alpha, \beta]$, and $\lim_{j \to \infty} \eta(w_j) = 0$.

1.2. Theorem. *Let U be a set of functions $u : I_u \to \mathbb{R}^n$ where I_u is an interval. Let $H : \mathbb{R} \times \mathbb{R}^n \to \mathscr{K}_0^n$ satisfy*

(1.4) *$H(t, .)$ is upper semicontinuous for almost every $t \in \mathbb{R}$;*

(1.5) *there exists a locally integrable function $\varrho : \mathbb{R} \to \mathbb{R}^+ = [0, +\infty)$ such that $H(t, x) \subset B(0, \varrho(t))$ for almost every $t \in \mathbb{R}$.*

($B(\xi, \delta)$ stands for a ball in $\mathbb{R}^n$ with centre ξ and radius δ.) Further, let

(1.6) $$U \subset \operatorname{Sol} H.$$

Then

(1.7) $$\Theta(U) \subset \operatorname{Sol} H.$$

1.3. Proof. Let us notice first of all that (1.6) and (1.5) imply

(1.8) u is absolutely continuous on I_u,

(1.9) $|\dot{u}(t)| \leqq \varrho(t)$ for almost all $t \in I_u$

provided $u \in U$.

Now let $q \in \Theta(U)$ have a definition interval I_q, let $[\alpha, \beta] \subset I_q$. Let w_j, $j = 1, 2, \ldots$ be the functions from (1.3) so that $\lim_{j\to\infty} \eta(w_j) = 0$ and

$$\lim_{j\to\infty} w_j = q\,, \quad \lim_{j\to\infty} Jwj = 0$$

both uniformly on $[\alpha, \beta]$. Denote $\omega_j = w_j - Jw_j$; then evidently

$$\lim_{j\to\infty} \omega_j = q$$

again uniformly on $[\alpha, \beta]$.

Denote by $T \subset [\alpha, \beta]$ a set with $m(T) = \beta - \alpha$ (m denotes the Lebesgue measure of a set in $\mathbb{R}$) such that the following conditions are satisfied for $t \in T$:

(i) $H(t, .)$ is upper semicontinuous,

(ii) t is not a point of discontinuity for any w_j, $j = 1, 2, \ldots$ (cf. (1.2)),

(iii) for all functions $u \in U$ appearing in the "decomposition" of w_j, $j = 1, 2, \ldots$ as described in (1.2), the derivatives $\dot{u}$ exist and $\dot{u}(t) \in H(t, u(t))$.

Existence of such a set (with full measure in $[\alpha, \beta]$) follows from (1.4), from the fact that the number of the points of discontinuity in (ii) as well as the number of the functions u in (iii) is at most countable, and from (1.6), (1.8).

Now we apply Lemma in [1, p. 313] or Lemma 2 in [2, p. 2] to the functions ω_j. We conclude that q is absolutely continuous and

$$\dot{q}(t) \in \bigcap_{j=1}^{\infty} \overline{\operatorname{conv}}\, \{\dot{\omega}_j(t), \dot{\omega}_{j+1}(t), \ldots\}\,, \tag{1.10}$$

where $\overline{\operatorname{conv}}$ stands for the closed convex hull, for $t \in T_1 \subset T$ with $m(T_1) = \beta - \alpha$. We want to prove that $\dot{q}(t) \in H(t, q(t))$ for almost all $t \in [\alpha, \beta]$. Let us assume, on the contrary, that there is a set $T_2 \subset T_1$, $m(T_2) = \nu > 0$, such that $\dot{q}(t) \notin H(t, q(t))$ for $t \in T_2$. Passing from the sequence w_j to a subsequence if necessary (but keeping the original notation) we find an index j so large that

$$\sum_{l=j}^{\infty} \eta(w_l) < \tfrac{1}{2}\nu$$

(cf. (1.3) for the notation). Consequently, if $D \subset [\alpha, \beta]$ is the union of such subintervals of $[\alpha, \beta]$ in which all w_l, $l \geqq j$, coincide with some functions $u \in U$, then $m(D) > \beta - \alpha - \frac{1}{2}\nu$. Since for $t \in T_0 = T_1 \cap D$ the values of $\dot{\omega}_j(t)$, $\dot{\omega}_{j+1}(t), \ldots$ coincide with some of those $\dot{u}(t)$ considered in (iii), we evidently have

$$\dot{q}(t) \in \bigcap_{j=1}^{\infty} \overline{\operatorname{conv}} \bigcup_{i=j}^{\infty} H(t, u_i(t)) \tag{1.11}$$

for $t \in T_0$; moreover,

$$\lim_{i\to\infty} u_i(t) = q(t)\,. \tag{1.12}$$

Let $\varepsilon > 0$, $t \in T_0$. By (1.12)

$$H(t, u_i(t)) \subset \Omega(H(t, q(t)), \varepsilon)$$

for i sufficiently large, where $\Omega(M, \varepsilon)$ denotes the ε-neighbourhood of the set M. Consequently,

$$\overline{\text{conv}} \bigcup_{i=j}^{\infty} H(t, u_i(t)) \subset \bar{\Omega}(H(t, q(t)), \varepsilon)$$

for j sufficiently large and

$$\bigcap_{j=1}^{\infty} \overline{\text{conv}} \bigcup_{i=j}^{\infty} H(t, u_i(t)) \subset H(t, q(t)) .$$

In view of (1.11), this proves $\dot{q}(t) \in H(t, q(t))$ for $t \in T_0$, which contradicts the existence of the set T_2 introduced above. Consequently, the inclusion $\dot{q}(t) \in H(t, q(t))$ holds for almost all $t \in [\alpha, \beta]$, i.e. $q \in \text{Sol}\, H$ and (1.7) immediately follows.

1.4. Theorem. *Let U be a set of functions $u : I_u \to \mathbb{R}^n$ which satisfy* (1.8), (1.9). *Further, let us assume that*

(1.13) if $u_i \in U$, $i = 1, 2$, $I_{u_1} \cap I_{u_2} \neq \emptyset$ *and* $u_1(t) = u_2(t)$ *provided* $t \in I_{u_1} \cap I_{u_2}$, *then the function* $u^* : I_{u_1} \cup I_{u_2} \to \mathbb{R}^n$ *defined by* $u^*(t) = u_i(t)$ *for* $t \in I_{u_i}$, $i = 1, 2$, *belongs to U as well.*

Then there exists $Q : \mathbb{R}^{n+1} \to \mathscr{K}_0^n$ which satisfies (1.4), (1.5) with H replaced by Q and

$$\text{Sol}\, Q = \Theta(U) . \tag{1.14}$$

1.5. Corollary. *A set U satisfying* (1.8), (1.9) *and* (1.13) *is the set of all solutions of a differential relation* (*i.e. there exists $H : \mathbb{R}^{n+1} \to \mathscr{K}_0^n$ satisfying* (1.4), (1.5) *and $U = \text{Sol}\, H$*) *if and only if $U = \Theta(U)$.*

1.6. Proof of Theorem 1.4. As was proved in [1], the set U has an at most countable subset, say

$$V = \{v_1, v_2, \ldots\} \subset U ,$$

which is dense with respect to the metric $\varkappa$ introduced as follows: If $u_i \in U$, $I_{u_i} = [a_i, b_i]$, $i = 1, 2$, we set

$$\tilde{u}_i(t) = \begin{cases} u_i(t) & \text{for } t \in I_{u_i} , \\ u_i(a_i) & \text{for } t < a_i , \\ u_i(b_i) & \text{for } t > b_i \end{cases}$$

and define

$$\varkappa(u_1, u_2) = \max_t |\tilde{u}_1(t) - \tilde{u}_2(t)| + |a_1 - a_2| + |b_1 - b_2| .$$

Let us define

$$Q(t, x) = \bigcap_{i=1}^{\infty} \overline{\text{conv}} \{\dot{v}_j(t);\ v_j(t) \in B(x, i^{-1})\} ; \tag{1.15}$$

we shall prove that Q is the mapping whose existence is claimed in Theorem 1.4. The proof of (1.4) and (1.5) is easy; moreover, it was proved in [1] that $U \subset \operatorname{Sol} Q$, hence by Theorem 1.2

$$\Theta(U) \subset \operatorname{Sol} Q .$$

In order to prove (1.14) it remains to establish the converse inclusion.

Let $q : I_q \to \mathbb{R}^n$, $q \in \operatorname{Sol} Q$, $[\alpha, \beta] \subset I_q$. According to (1.3) we have to construct a sequence of functions $q_k \in \mathscr{W}(U)$ which converge to q in the sense described by (1.3).

Set

$$L_0 = \{t \in [\alpha, \beta];\ t \text{ is a Lebesgue point of } \dot{q} \text{ and } \dot{q}(t) \in Q(t, q(t))\} ,$$

$$L_j = \{t \in [\alpha, \beta];\ \text{if } t \in I_{v_j} \text{ then } t \text{ is a Lebesgue point of } \dot{v}_j\} ,$$

$$j = 1, 2, \dots ,$$

$$L = \bigcap_{j=0}^{\infty} L_j .$$

Then $m(L) = \beta - \alpha$.

Now let $\tau \in L$ and $\xi > 0$. By (1.15) there exists a finite number $m = m(\tau, \xi)$ of functions $u_1, u_2, \dots, u_m$ from V (i.e. $u_1 = v_{j_1}$, $u_2 = v_{j_2}, \dots, u_m = v_{j_m}$), $\tilde{\lambda} > 0$ and positive numbers $\alpha_1, \alpha_2, \dots, \alpha_m$ with $\alpha_1 + \alpha_2 + \dots + \alpha_m = 1$ such that

$$\left|\dot{q}(\tau) - \sum_{j=1}^{m} \alpha_j \dot{u}_j(\tau)\right| < \xi \tag{1.16}$$

and

$$|q(t) - u_j(t)| < \xi \tag{1.17}$$

for $j = 1, 2, \dots, m$ and $\tau \leqq t \leqq \tau + \tilde{\lambda}$. Further, since $\tau \in L$, there exists $\lambda = \lambda(\tau, \xi)$, $0 < \lambda \leqq \tilde{\lambda}$ such that

$$\int_{\tau}^{\tau+\delta} |\dot{q}(\sigma) - \dot{q}(\tau)| \, d\sigma < \xi\delta , \tag{1.18}$$

$$\int_{\tau}^{\tau+\delta} |\dot{u}_j(\sigma) - \dot{u}_j(\tau)| \, d\sigma < \frac{\xi\delta}{m}$$

for $j = 1, 2, \dots, m$, $0 < \delta \leqq \lambda$.

Moreover, we choose λ so that

$$\int_{\tau}^{\tau+\lambda} \varrho(\sigma) \, d\sigma < \xi . \tag{1.19}$$

Let us now consider the family of all intervals of the form $[\tau, \tau + \lambda\iota^{-1}]$ where $\tau \in L$, $\iota = 1, 2, \dots$ and λ corresponds to τ in the way just described so that (1.16)–(1.19) hold. This family of intervals covers the set L in the sense of Vitali. Thus there exists an at most countable disjoint subfamily which covers the set L, and hence also the interval $[\alpha, \beta]$, except for a set of zero measure. Let us denote the intervals of this countable family by $\Lambda_i = [\tau_i, \tau_i + \lambda_i]$, $i = 1, 2, \ldots$.

Let $\nu > 0$ be such that $\int_A \varrho(\sigma)\, d\sigma < \xi$ provided $A \subset [\alpha, \beta]$, $m(A) < \nu$. There exists a positive integer $p = p(\nu)$ such that

$$m(\bigcup_{i=1}^{p} \Lambda_i) > \beta - \alpha - \nu\,.$$

We may assume without loss of generality that

$$\alpha \leqq \tau_1 < \tau_1 + \lambda_1 < \ldots < \tau_p < \tau_p + \lambda_p \leqq \beta\,.$$

Denoting $M_0 = [\alpha, \tau_1)$, $M_p = (\tau_p + \lambda_p, \beta]$, $M_i = (\tau_i + \lambda_i, \tau_{i+1})$ for $i = 1, 2, \ldots, p - 1$ we have

$$m(\bigcup_{i=0}^{p} M_i) < \nu\,, \tag{1.20}$$

$$\bigcup_{i=1}^{p} \Lambda_i \cup \bigcup_{i=0}^{p} M_i = [\alpha, \beta]$$

and the union on the left hand side of the last identity is disjoint.

Now we can define the sequence of functions $q_k \in \mathscr{W}(U)$ such that $q_k \to q$, $Jq_k \to 0$ both uniformly on $[\alpha, \beta]$. To this end, let $\xi_1, \xi_2, \ldots$ be a sequence of positive reals, $\xi_k \to 0$, and put $\xi = \xi_k$ with k arbitrary but fixed in the above considerations.

Let us first define q_k on Λ_i, $i = 1, 2, \ldots, p$. Let $\tau_i = \sigma_0^{(i)} < \sigma_1^{(i)} < \ldots < \sigma_m^{(i)} = \tau_i + \lambda_i$, $\sigma_j^{(i)} - \sigma_{j-1}^{(i)} = \alpha_j \lambda_i$ for $j = 1, 2, \ldots, m$ (cf. (1.16)) and define

$$q_k(t) = u_j(t) \quad \text{for} \quad t \in (\sigma_{j-1}^{(i)}, \sigma_j^{(i)}] \tag{1.21}$$

for $j = 1, 2, \ldots, m$ (more precisely, $u_j(t) = u_j^{(i)}(t)$). Further, we define

$$\begin{aligned} q_k(t) &= u_1(\tau_1) &&\text{for}\quad t \in M_0 \quad \text{or} \quad t = \tau_1\,,\\ q_k(t) &= q_k(\tau_i + \lambda_i) &&\text{for}\quad t \in M_i \quad \text{or} \quad t = \tau_{i+1}\,,\\ &i = 1, \ldots, p\,. \end{aligned} \tag{1.22}$$

In this way, q_k is defined on the whole interval $[\alpha, \beta]$ and obviously $q_k \in \mathscr{W}(U)$.

Let us first estimate the difference $|q_k(t) - q(t)|$. If $t \in \Lambda_i$, $i = 1, 2, \ldots, p$, we have (1.17). If $t \in M_i$, $i = 0, 1, \ldots, p$, then

$$\begin{aligned} |q_k(t) - q(t)| &= |q_k(\tau_i + \lambda_i) - q(t)| \leqq \\ &\leqq |q_k(\tau_i + \lambda_i) - q(\tau_i + \lambda_i)| + |q(\tau_i + \lambda_i) - q(t)| < \\ &< \xi_k + \int_{\tau_i + \lambda_i}^{t} \varrho(\sigma)\, d\sigma \end{aligned}$$

by (1.17) and (1.5); moreover, $0 \leqq t - (\tau_i + \lambda_i) \leqq \nu$ in virtue of (1.20). Hence

$$|q_k(t) - q(t)| < 2\xi_k \quad \text{for} \quad t \in [\alpha, \beta]$$

(cf. p. 559, line 4), which guarantees the uniform convergence $q_k \to q$.

Secondly, we have to estimate the value of the "jump function" Jq_k. To this end we need an estimate for the expression

$$\Delta_i q_k = q(\tau_i + \lambda_i) - q(\tau_i) - \sum_{j=1}^{m} [u_j(\sigma_j^{(i)}) - u_j(\sigma_{j-1}^{(i)})] .$$

We have (omitting for the moment the index i)

$$|\Delta q_k| = \left| \int_{\tau}^{\tau+\lambda} \dot{q}(\sigma) \, d\sigma - \sum_{j=1}^{m} \int_{\sigma_{j-1}}^{\sigma_j} \dot{u}_j(\sigma) \, d\sigma \right| =$$

$$= \left| \int_{\tau}^{\tau+\lambda} [\dot{q}(\sigma) - \dot{q}(\tau)] \, d\sigma - \sum_{j=1}^{m} \int_{\sigma_{j-1}}^{\sigma_j} [\dot{u}_j(\sigma) - \dot{u}_j(\tau)] \, d\sigma + \right.$$

$$\left. + \lambda[\dot{q}(\tau) - \sum_{j=1}^{m} \alpha_j \dot{u}_j(\tau)] \right| .$$

Taking into account (1.18) and (1.16) we obtain

$$|\Delta q_k| < \lambda\xi_k + \frac{\xi_k}{m} \sum_{j=1}^{m} (\sigma_j - \tau) + \lambda\xi_k \leqq 3\lambda\xi_k .$$

We proceed by estimating $Jq_k(\tau_l)$:

$$J q_k(\tau_l) = \sum_{i=2}^{l-1} [q_k(\tau_i+) - q_k(\tau_i-)] +$$

$$+ \sum_{i=1}^{l-1} \sum_{j=1}^{m-1} [q_k(\sigma_j^{(i)}+) - q_k(\sigma_j^{(i)}-)] = \sum_{i=2}^{l-1} [q_k(\tau_i+) - q_k(\tau_{i-1} + \lambda_{i-1})] +$$

$$+ \sum_{i=1}^{l-1} [\Delta_i q_k - q(\tau_i + \lambda_i) + q(\tau_i) + q_k(\tau_i + \lambda_i) - q_k(\tau_i+)] =$$

$$= \sum_{i=1}^{l-1} \Delta_i q_k + \sum_{i=2}^{l-1} [q(\tau_i) - q(\tau_{i-1} + \lambda_{i-1})] +$$

$$+ q(\tau_1) - q_k(\tau_1+) + q_k(\tau_{l-1} + \lambda_{l-1}) - q(\tau_{i-1} + \lambda_{l-1}) .$$

Taking into account the above estimate of $\Delta_i q_k$, the choice of ν together with (1.20) and the inequality (1.17), we conclude that

$$|Jq_k(\tau_l)| \leqq 3\xi_k \sum_{i=1}^{l-1} \lambda_i + \sum_{i=1}^{l-2} \int_{M_i} \varrho(\sigma) \, d\sigma + 2\xi_k \leqq 3\xi_k(\beta - \alpha) + \xi_k + 2\xi_k .$$

If $t \in (\tau_l, \tau_{l+1})$, then

$$|Jq_k(t)| \leqq |Jq_k(\tau_l)| + |d_l| ,$$

where d_l represents the contribution of jumps on $[\tau_l, t]$. We can estimate it similarly as $\Delta_i q_k$. As the function q_k has no jumps on $M_l = (\tau_l + \lambda_l, \tau_{l+1})$ we may assume without loss of generality that $t \in \Lambda_l$; hence there is a positive integer s such that $\sigma_s < t \leqq \sigma_{s+1}$. We have

$$|d_l| = \left|\sum_{j=0}^{s} [q_k(\sigma_j+) - q_k(\sigma_j-)]\right| =$$

$$= \left|\sum_{j=0}^{s-1} [u_{j+1}(\sigma_j) - u_{j+1}(\sigma_{j+1})] + u_{s+1}(\sigma_s) - q_k(\tau_l-)\right| \leqq$$

$$\leqq \left|\sum_{j=0}^{s-1} \int_{\sigma_j}^{\sigma_{j+1}} \dot{u}_{j+1}(\sigma)\, d\sigma\right| + |u_{s+1}(\sigma_s) - q(\sigma_s)| +$$

$$+ |q(\sigma_s) - q(\tau_l)| + |q(\tau_l) - q(\tau_{l-1} + \lambda_{l-1})| +$$

$$+ |q(\tau_{l-1} + \lambda_{l-1}) - q_k(\tau_{l-1} + \lambda_{l-1}-)| + |q_k(\tau_{l-1} + \lambda_{l-1}-) - q_k(\tau_l-)| .$$

Now the first and the third term in absolute value is estimated by (1.19), the second and the fifth is estimated by (1.17), while the fourth and the last is estimated by $\int_A \varrho(\sigma)\, d\sigma$ where the set A in both cases has a measure less than ν (cf. (1.20)). Hence we conclude

$$|d_l| < 6\xi ,$$

$$|Jq_k(t)| < [3(\beta - \alpha) + 9]\, \xi .$$

Thus $Jq_k \to 0$ uniformly and $q \in \Theta(U)$, which completes the proof of Theorem 1.4.

1.7. Remark. As was proved in [1], the map Q defined by (1.15) has the minimal property mentioned in Introduction. This implies that Q is unique and does not depend on the choice of the dense countable subset V of the set U (see the beginning of 1.6).

1.8. Let us assume that the set U has, in addition to the assumptions of Theorem 1.4, the following property:

(1.23) for every $t_0 \in \mathbb{R}$, $x_0 \in \mathbb{R}^n$ there is $\delta > 0$ and a function $u_0 : (t_0 - \delta, t_0 + \delta) \to \mathbb{R}^n$, such that $u_0 \in U$, $u(t_0) = x_0$.

Then the functions q_k from the proof of Theorem 1.4 (see (1.21), (1.22)) can be defined on M_i in the following way: For every $t \in M_i$ we find $\delta = \delta(t)$ from (1.23); the intervals $(t - \delta(t), t + \delta(t))$ form an open covering of the compact set M_i, hence we can select a finite covering. Diminishing the intervals of this finite covering, we can achieve that they are disjoint and still cover the set M_i (except for a finite number of their endpoints). On each interval thus obtained we define q_k to be equal to a function from U (such functions exist in virtue of (1.23)). After this modification, the functions q_k coincide with certain functions from the set U at every point t. Obviously, q_k may have a finite number of discontinuities at the endpoints of the intervals of the covering. However, due to (1.9) and the fact that $m(M) < \nu$ (cf. (1.20)) these discontinuities contribute to the "jump functions" Jq_k only by $2\int_M \varrho(t)\,dt < 2\xi$.

This suggests that we could have introduced a set $\mathscr{W}^0(U)$ analogously to $\mathscr{W}(U)$ with the only change that $\sigma_i = \tau_{i+1}$, $i = 0, \ldots, k$ (which essentially corresponds to the condition $\eta(w) = 0$ for $w \in \mathscr{W}^0(U)$, cf. (1.3) and above) and define $\Theta^0(U)$ again by (1.3) with $\mathscr{W}(U)$ replaced by $\mathscr{W}^0(U)$ (the condition $\eta \to 0$ is then automatically satisfied). Thus we obtain the following modification of Theorem 1.4, which can be proved in the same way as Theorem 1.4 with the above mentioned change in the definition of q_k on the sets M_i.

1.9. Theorem. *Let U be a set which satisfies the assumptions of Theorem 1.4 and, moreover, (1.23). Then there exists $Q : \mathbb{R}^{n+1} \to \mathscr{K}_0^n$ which satisfies (1.4), (1.5) with H replaced by Q and*

$$\operatorname{Sol} Q = \Theta^0(U).$$

1.10. Example. The following example shows that Theorem 1.9 is not valid if the assumption (1.23) (or a similar one) is omitted. Let U be the family of all functions u defined by one of the following relations:

$$u(t) = \text{const} > 0, \quad t \in [2^{-i}, 2^{-i+1}], \quad i \text{ odd};$$

$$u(t) = \text{const} < 0, \quad t \in [2^{-i}, 2^{-i+1}], \quad i \text{ even};$$

$$u(t) = \text{const}, \quad t \in (-\infty, 0].$$

Then, defining $Q(t, x)$ by (1.15), we evidently have $0 \in Q(t, 0)$ for $t \in (-\infty, 1]$ and hence $\psi \in \operatorname{Sol} Q$, where $\psi(t) = 0$ for $t \in (-\infty, 1]$. However, $\psi_0 \notin \Theta^0(U)$ provided ψ_0 is the restriction of ψ to any interval containing a neighbourhood of zero.

2

2.1. Given a differential relation (1.1), we denote

$$\Phi(y, \tau, t) = \{u(t);\ u \in \operatorname{Sol} F,\ u(\tau) = y\} \tag{2.1}$$

for $y \in \mathbb{R}^n$, $t \in \mathbb{R}$.

Let us assume that the mapping $F : \mathbb{R}^{n+1} \to \mathscr{K}_0^n$ satisfies conditions of Theorem 1.2, i.e. $F(t, .)$ is upper semicontinuous for almost every t and there exists a locally integrable function $\varrho : \mathbb{R} \to \mathbb{R}^+$ such that $F(t, x) \subset B(0, \varrho(t))$ for $(t, x) \in \mathbb{R}^{n+1}$. The following lemma establishes the main properties of Φ.

2.2. Lemma. *The set $\Phi(y, \tau, t)$ has the following properties:*

(2.2) $\Phi(y, \tau, t)$ is closed, $\Phi(y, \tau, t) \subset B(y, |\int_\tau^t \varrho(\sigma)\,d\sigma|)$;

$$\Phi(y, \tau, t) = \bigcup_{z \in \Phi(y,\tau,s)} \Phi(z, s, t) \tag{2.3}$$

provided $s \in [\tau, t]$;

(2.4) for every y, τ, t and $\varepsilon > 0$ there exists $\delta > 0$ such that the following implication holds: if k is a positive integer, $s \in [\tau, t]$, $\sigma_i \in \mathbb{R}$, $0 \leqq \mu_i \in \mathbb{R}$, $z_i \in \mathbb{R}^n$, $i = 0, 1, \ldots, k$ satisfy

$$\tau = \sigma_0 \leqq \sigma_0 + \mu_0 < \sigma_1 \leqq \sigma_1 + \mu_1 < \ldots < \sigma_k \leqq \sigma_k + \mu_k = s\,,$$

$$\sum_{i=0}^{k} \mu_i < \delta\,, \quad \left|\sum_{i=0}^{j} z_i\right| < \delta \quad \text{for} \quad j = 0, 1, \ldots, k\,,$$

then, introducing the notation $Z_1 = \Phi(y + z_0, \sigma_0 + \mu_0, \sigma_1)$,

$$Z_i = \bigcup_{u_{i-1} \in Z_{i-1}} \Phi(u_{i-1} + z_{i-1}, \sigma_{i-1} + \mu_{i-1}, \sigma_i)$$

for $i = 2, \ldots, k$, we have

(2.4$_1$) $$Z_k \subset \Omega(\Phi(y, \tau, s), \varepsilon)\,.$$

2.3. Remark. Note that the condition (2.4) implies upper semicontinuity of $\Phi(\cdot, \tau, t)$. Indeed, putting $\mu_0 = \mu_1 = 0$, $k = 1$ in (2.4) we obtain

(2.5) for every y, τ, t and $\varepsilon > 0$ there exists $\delta > 0$ such that if $s \in [\tau, t]$, $z \in \mathbb{R}^n$, $|z| < \delta$, then

(2.5$_1$) $$\Phi(y + z, \tau, s) \subset \Omega(\Phi(y, \tau, s), \varepsilon)\,.$$

2.4. Proof of Lemma 2.2. The properties (2.2) and (2.3) are evident. Let us prove (2.4). Assume on the contrary that there is $\varepsilon > 0$ such that for every j there are $\sigma_i^j, \mu_i^j, z_i^j$ satisfying the conditions of (2.4) with $\delta = j^{-1}$, $k = k_j$, and that (2.4$_1$) is not fulfilled, that is, there is $\omega_j \in Z_{k_j}$,

(2.6) $$\omega_j \notin \Omega(\Phi(y, \tau, s), \varepsilon)\,.$$

According to the definition of $\Phi(y, \tau, t)$ (see (2.1)), for every $i = 1, 2, \ldots, k_j$ we find a function $v_i^j : [\sigma_{i-1}^j + \mu_{i-1}^j, \sigma_i^j] \to \mathbb{R}^n$, $v_i^j \in \text{Sol}\, F$, $v_i^j(\sigma_{i-1}^j + \mu_{i-1}^j) = v_{i-1}^j(\sigma_{i-1}^j) + z_{i-1}$, $v_{k_j}^j(\sigma_{k_j}^j) = \omega_j$. Then $u_j \in \mathscr{W}(\text{Sol}\, F)$ provided $u_j : [\tau, s] \to \mathbb{R}^n$ is defined by

$$u_j(\sigma) = \begin{cases} v_1^j(\sigma_0^j + \mu_0^j) & \text{for} \quad \sigma \in [\sigma_0^j, \sigma_0^j + \mu_0^j]\,, \\ v_i^j(\sigma) & \text{for} \quad \sigma \in (\sigma_{i-1}^j + \mu_{i-1}^j, \sigma_i^j)\,, \quad i = 1, \ldots, k_j\,, \\ v_i^j(\sigma_i^j) & \text{for} \quad \sigma \in [\sigma_i^j, \sigma_i^j + \mu_i^j]\,, \quad i = 1, \ldots, k_j\,. \end{cases}$$

The equicontinuity of the functions $u_j - Ju_j$ implies that we can find a uniformly convergent subsequence (denoted again by $\{u_j\}$):

$$\lim_{j \to \infty} (u_j - Ju_j) = w$$

uniformly on $[\tau, s]$. Since $\lim_{j \to \infty} Ju_j = 0$ uniformly on $[\tau, s]$, we have by Theorem 1.2

$$w \in \Theta(\text{Sol}\, F) \subset \text{Sol}\, F$$

and hence

$$w(s) \in \Phi(y, \tau, s)$$

by (2.1). However, this contradicts (2.6) since $w(s) = \lim_{j\to\infty} \omega_j$. Lemma 2.2 is proved.

2.5. Lemma. *Let $\Phi : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} \to \mathscr{P}(\mathbb{R}^n)$ (the family of all subsets of $\mathbb{R}^n$) satisfy (2.2), (2.3) and (2.5). Let $\tau, t \in \mathbb{R}$, $y, q \in \mathbb{R}^n$, $q \in \Phi(y, \tau, t)$. Then there exists $u : [\tau, t] \to \mathbb{R}^n$ with $u(\tau) = y$, $u(t) = q$, such that*

(2.7) *u is absolutely continuous and $|\dot{u}(\sigma)| \leqq \varrho(\sigma)$ for almost all $\sigma \in [\tau, t]$;*

(2.8) *if $\vartheta, s \in [\tau, t]$, $\vartheta \leqq s$, then $u(s) \in \Phi(u(\vartheta), \vartheta, s)$.*

2.6. Proof. Without loss of generality let us put $\tau = 0$, $t = 1$. We shall construct the function u successively at the points $r\,2^{-j}$, $j = 1, 2, \ldots, r$ integer, $0 \leqq r \leqq 2^j$. Put $u(0) = y$, $u(1) = q$.

According to (2.3) there is $z \in \Phi(y, 0, \frac{1}{2})$ such that $q \in \Phi(z, \frac{1}{2}, 1)$. We define $u(\frac{1}{2}) = z$.

Assume that $u(\sigma)$ has been defined for $\sigma = r\,2^{-j}$ of the form described above with a fixed j. Let p be an odd integer, $0 \leqq p \leqq 2^{j+1}$. Again according to (2.3) there is $z = z(p, j + 1)$ such that

$$z \in \Phi\left(u\left(\frac{p-1}{2^{j+1}}\right), \frac{p-1}{2^{j+1}}, \frac{p}{2^{j+1}}\right)$$

and

$$u\left(\frac{p+1}{2^{j+1}}\right) \in \Phi\left(z, \frac{p}{2^{j+1}}, \frac{p+1}{2^{j+1}}\right).$$

We define $u(p/2^{j+1}) = z = z(p, j + 1)$. In this way we define u on a dense subset of $[0, 1]$. By (2.2) we have

$$\left|u\left(\frac{p}{2^j}\right) - u\left(\frac{q}{2^j}\right)\right| \leqq \int_{p/2^j}^{q/2^j} \varrho(\sigma)\, d\sigma,$$

hence u is uniformly continuous on its definition domain and consequently, it can be extended in a unique way as a uniformly continuous function to the whole interval $[0, 1]$. It is clear that (2.7) holds.

Let us pass to the proof of (2.8). Recall that $0 \leqq \vartheta \leqq s \leqq 1$ as we have put $\tau = 0$, $t = 1$. Let p, q, j, k be integers to be fixed later, $0 \leqq p \leqq 2^j$, $0 \leqq q \leqq 2^k$, $j > 0$, $k > 0$, and assume $q\,2^{-k} \leqq \vartheta \leqq s \leqq p\,2^{-j}$. Then by construction

$$u\left(\frac{p}{2^j}\right) \in \Phi\left(u\left(\frac{q}{2^k}\right), \frac{q}{2^k}, \frac{p}{2^j}\right).$$

Hence by (2.3)

$$u\left(\frac{p}{2^j}\right) \in \bigcup_z \Phi\left(z, \vartheta, \frac{p}{2^j}\right), \tag{2.9}$$

where the union is taken over $z \in \Phi(u(q/2^k), q/2^k, \vartheta)$.

Let $\varepsilon > 0$, $z \in \Phi(u(q/2^k), q/2^k, \vartheta)$. Then (2.5) implies that there is $\delta > 0$ such that

$$\Phi\left(z, \vartheta, \frac{p}{2^j}\right) \subset \Omega\left(\Phi\left(u(\vartheta), \vartheta, \frac{p}{2^j}\right), \varepsilon\right) \tag{2.10}$$

provided $|z - u(\vartheta)| < \delta$. In view of (2.7) – the absolute continuity of u – there exists $\delta_1 > 0$ such that $|u(\vartheta) - u(q/2^k)| < \delta/2$ if $|\vartheta - q/2^k| < \delta_1$. On the other hand, by virtue of the inclusion in (2.2) there is $\delta_2 > 0$ such that

$$|z - u(q/2^k)| < \delta/2 \quad \text{if} \quad |\vartheta - q/2^k| < \delta_2 .$$

(Replace y, τ, t in the inclusion by $u(q/2^k)$, $q/2^k$, ϑ, respectively.)

Consequently, if $2^{-k} < \min(\delta_1, \delta_2)$ then we can find an integer q so that $0 \leqq \vartheta - q/2^k < \min(\delta_1, \delta_2)$. Then $|z - u(\vartheta)| < \delta$ for $z \in \Phi(u(q/2^k), q/2^k, \vartheta)$ and (2.10) holds. Since $\varepsilon > 0$ was arbitrary, this together with (2.9) yields

$$u\left(\frac{p}{2^j}\right) \in \Phi\left(u(\vartheta), \vartheta, \frac{p}{2^j}\right).$$

By the same argument as above, (2.3) yields

$$u\left(\frac{p}{2^j}\right) \in \bigcup_z \Phi\left(z, s, \frac{p}{2^j}\right), \tag{2.11}$$

the union being taken over $z \in \Phi(u(\vartheta), \vartheta, s)$.

Let $\varepsilon > 0$. Then by the uniform continuity of u there is $\delta_3 > 0$ such that

$$\left|u(s) - u\left(\frac{p}{2^j}\right)\right| < \varepsilon/2 \quad \text{if} \quad \left|s - \frac{p}{2^j}\right| < \delta_3 .$$

Thus (2.11) yields

$$u(s) \in \Omega(\bigcup_z \Phi(z, s, p/2^j), \varepsilon/2) \tag{2.12}$$

provided $|s - p/2^j| < \delta_3$. Further, there is $\delta_4 > 0$ such that

$$\Phi(z, s, p/2^j) \subset B(z, \varepsilon/2) \quad \text{if} \quad |s - p/2^j| < \delta_4 \tag{2.13}$$

(cf. the inclusion in (2.2)). Consequently, if we assume $2^{-j} < \min(\delta_3, \delta_4)$ then we can find an integer p so that $0 \leqq p/2^j - s < \min(\delta_3, \delta_4)$ and (2.12) together with (2.13) yields $u(s) \in \Omega(\Omega(\Phi(u(\vartheta), \vartheta, s), \varepsilon/2), \varepsilon/2)$, which immediately implies (2.8). Lemma 2.5 is proved.

Now we shall prove a theorem which actually represents a conversion of Lemma 2.2.

2.7. Theorem. *Let* $\Phi : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} \to \mathscr{P}(\mathbb{R}^n)$ *satisfy* (2.2)–(2.4).

Then there exists $F : \mathbb{R}^{n+1} \to \mathscr{K}^n_0$ *such that* (2.1) *holds. Moreover, the map* F *has the following properties*:

(i) *for almost every* $t \in \mathbb{R}$, $F(t, .)$ *is upper semicontinuous*;

(ii) *there exists a locally integrable function* $\varrho : \mathbb{R} \to \mathbb{R}^+$ *such that* $F(t, x) \subset \subset B(0, \varrho(t))$ *for* $(t, x) \in \mathbb{R}^{n+1}$.

2.8. Proof. Let U be the set of all functions $u : [\tau, t] \to \mathbb{R}^n$ such that (2.7) and (2.8) hold. The crucial point of the proof consists in establishing the identity

$$\Theta(U) = U \tag{2.14}$$

(cf. (1.3)). Let $q \in \Theta(U)$, $[\alpha, \beta] \subset I_q$. Let $w_j \in \mathscr{W}(U)$ be the function from (1.3) so that $w_j \to q$, $Jw_j \to 0$ both uniformly on $[\alpha, \beta]$. In virtue of (2.4) this means: for every $\varepsilon > 0$ there is j^* such that

$$w_j(s) \in \Omega(\Phi(q(\vartheta), \vartheta, s), \varepsilon) \tag{2.15}$$

for every ϑ, $s \in [\alpha, \beta]$ and every $j > j^*$. However, this evidently implies

$$q(s) \in \Phi(q(\vartheta), \vartheta, s)$$

for every ϑ, $s \in [\alpha, \beta]$. Thus q satisfies (2.8) with α, β, q instead of τ, t, u, respectively. As (2.7) with the same change of notation is obviously fulfilled, we conclude that $q \in U$, which proves the identity (2.14) since the inclusion $\Theta(U) \supset U$ is evident. By Theorem 1.4 there exists $F : \mathbb{R}^{n+1} \to \mathscr{K}^n$ such that $\operatorname{Sol} F = U$ and F has the properties (i), (ii) from Theorem 2.7. The validity of (2.1) is easily verified.

Given $M \subset \mathbb{R}^n$, let us denote by ∂M the boundary of M.

2.9. Definition. Let $x : [\tau, t] \to \mathbb{R}^n$, $x \in \operatorname{Sol} F$. If

$$x(s) \in \partial\Phi(x(\tau), \tau, s)$$

for all $s \in [\tau, t]$, then x is called *a Fukuhara solution*. Let us denote by Fuk F the family of all Fukuhara solutions of (1.1).

2.10. Theorem. *If* $F : \mathbb{R}^{n+1} \to \mathscr{K}^n_0$ *satisfies* (i), (ii) *from Theorem* 2.7, *then*

$$\Theta(\operatorname{Fuk} F) = \operatorname{Sol} F\,. \tag{2.16}$$

2.11. Corollary. *If* $F, G : \mathbb{R}^{n+1} \to \mathscr{K}^n$ *and* Fuk $F \subset$ Sol G, *then* Sol $F \subset$ Sol G.
This corollary is a consequence of Theorem 1.2.

2.12. Proof of Theorem 2.10. The inclusion $\Theta(\operatorname{Fuk} F) \subset \operatorname{Sol} F$ is obvious (cf.

Theorem 1.2). Thus, let $w \in \operatorname{Sol} F$, $w : [\alpha, \beta] \to \mathbb{R}^n$. For every positive integer k we define a function $w_k : [\alpha, \beta] \to \mathbb{R}^n$ in the following way: First, we find a decomposition

$$\alpha = \tau_0 < \tau_1 < \ldots < \tau_{j-1} < \tau_j = \beta ,$$

$$\tau_i - \tau_{i-1} = (\beta - \alpha)/j \quad \text{for} \quad i = 1, \ldots, j ,$$

such that

$$\Phi(w(\tau_{i-1}), \tau_{i-1}, (\beta - \alpha)/j) \subset B(w(\tau_{i-1}), 1/k) ; \tag{2.17}$$

this is possible in virtue of (2.2).

Find a Fukuhara solution f_k^1 of (1.1), $f_k^1 : [\tau_0, \tau_1] \to \mathbb{R}^n$. such that $f_k^1(\tau_0) = w(\tau_0)$ and put

$$w_k(s) = f_k^1(s) \quad \text{for} \quad s \in [\tau_0, \tau_1] .$$

Evidently

$$|w_k(\tau_1) - w(\tau_1)| \leqq \frac{1}{k}$$

in virtue of (2.17).

If $w(\tau_1) = w_k(\tau_1)$, put

$$w_k(s) = f_k^2(s) \quad \text{for} \quad s \in (\tau_1, \tau_2] ,$$

where f_k^2 is an arbitrarily chosen Fukuhara solution satisfying $f_k^2(\tau_1) = w(\tau_1)$.

If $w(\tau_1) \neq w_k(\tau_1)$, find the point z_2 in $\partial\Phi(w(\tau_1), \tau_1, \tau_2)$ which has the form $z_2 = w(\tau_2) + \gamma_2[w(\tau_1) - w_k(\tau_1)]$ with $\gamma_2 \geqq 0$, and put

$$w_k(s) = f_k^2(s) \quad \text{for} \quad s \in (\tau_1, \tau_2] ,$$

where f_k^2 is the Fukuhara solution satisfying

$$f_k^2(\tau_1) = w(\tau_1) , \quad f_k^2(\tau_2) = z_2 .$$

Then in both cases

$$J w_k(\tau_2) = w(\tau_1) - w_k(\tau_1) ,$$

hence

$$|J w_k(\tau_2)| \leqq \frac{1}{k} .$$

Moreover, $w(\tau_2) - w_k(\tau_2) = w(\tau_2) - z_2 = -\gamma_2 \, J w_k(\tau_2)$ with $\gamma_2 \geqq 0$ and

$$|w(\tau_2) - w_k(\tau_2)| \leqq 1/k .$$

Now assume that w_k is defined on $[\tau_0, \tau_{i-1}]$, $i > 2$ and that

$$|J w_k(\tau_{i-1})| \leqq \frac{1}{k} ,$$

$$w(\tau_{i-1}) - w_k(\tau_{i-1}) = -\gamma_{i-1} \, J w_k(\tau_{i-1}) \quad \text{with} \quad \gamma_{i-1} \geqq 0$$

and

$$|w(\tau_{i-1}) - w_k(\tau_{i-1})| \leqq \frac{1}{k}.$$

If $Jw_k(\tau_{i-1}) = 0$, choose any Fukuhara solution f_k^i such that $f_k^i(\tau_{i-1}) = w(\tau_{i-1})$ and set

$$w_k(s) = f_k^i(s) \quad \text{for} \quad s \in (\tau_{i-1}, \tau_i].$$

If $Jw_k(\tau_{i-1}) \neq 0$, then find the point z_i in $\partial\Phi(w(\tau_{i-1}), \tau_{i-1}, \tau_i)$ which has the form $z_i = w(\tau_i) + \gamma_i Jw_k(\tau_i)$ with $\gamma_i \geqq 0$, and put

$$w_k(s) = f_k^i(s) \quad \text{for} \quad s \in (\tau_{i-1}, \tau_i],$$

where f_k^i is the Fukuhara solution satisfying

$$f_k^i(\tau_{i-1}) = w(\tau_{i-1}), \quad f_k^i(\tau_i) = z_i.$$

Then

$$Jw_k(\tau_i) = Jw_k(\tau_{i-1}) + [w(\tau_{i-1}) - z_{i-1}].$$

Since the two vectors on the right hand side have opposite directions while their magnitudes are less or equal to $1/k$, we in both cases have $|Jw_k(\tau_i)| \leqq 1/k$.

Taking into account the fact that w_k is continuous on each (τ_{i-1}, τ_i), we conclude by induction that $|Jw_k(s)| \leqq 1/k$ for $s \in [\alpha, \beta]$. Evidently, $|w_k(s) - w(s)| \leqq 1/k$ for $s \in [\alpha, b]$. This easily yields

$$w \in \Theta(\text{Fuk } F),$$

which again implies (2.16), thus completing the proof of Theorem 2.10.

References

[1] *J. Jarník:* Constructing the minimal differential relations with prescribed solutions, Časopis pěst. mat. *105* (1980), 311—315.

[2] *P. Krbec* and *J. Kurzweil:* Kneser's theorem for multivalued differential delay equations, Časopis pěst. mat. *104* (1979), 1—8.

Authors' address: 115 67 Praha 1, Žitná 25, ČSSR (Matematický ústav ČSAV).