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Valter Šeda

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ANTITONE OPERATORS AND ORDINARY DIFFERENTIAL EQUATIONS

VALTER ŠEDA, Bratislava

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Dedicated to Professor M. ŠVEC *on his 60th birthday anniversary*

A solution of the functional equation $x = T(x)$ is often sought in a partially ordered set E with the operator $T: E \to E$ showing some monotonicity properties. The theory has been mostly developed on the assumption that T is isotone (see [1], [2]). In this paper the fixed points of T are investigated when T is antitone and completely continuous. The results are applied to the study of initial- and boundary value problems for ordinary differential equations. Some results obtained by L. Erbe, A. Ja. Chochrjakov are improved here.

1. FIXED POINTS OF ANTITONE OPERATORS

Let $(E, \leqq)$ be a partially ordered set, $T: E \to E$. T is called *isotone* (*antitone*) if for any two elements $x, y \in E$, $x \leqq y$ implies that $T(x) \leqq T(y)$ $(T(x) \geqq T(y))$. If $x_0 \leqq y_0$ are two points of E, then the subset $[x_0, y_0] = \{z \in E : x_0 \leqq z \leqq y_0\}$ is called an *order interval*. If E is a real Banach space (or more generally a real Hausdorff topological vector space) and $P \subset E$ is a *cone* (i.e. P is closed, $P + P \subset \subset P$, $cP \subset P$ for each $c > 0$ and $P \cap (-P) = \{0\}$), then a *partial ordering* $\leqq$ can be induced in E by the rule $x \leqq y$ iff $y - x \in P$. E with this ordering is called an *ordered Banach space* (OBS) *with positive cone* P and it is denoted by (E, P). The positive cone is called *normal* if every order interval is bounded. This happens iff there exists a constant $\delta > 0$ such that $0 \leqq x \leqq y$ implies $\|x\| \leqq \delta\|y\|$ (Theorem 1.5, [1, p. 627]). Finally, a map from a Banach space into itself is called *completely continuous* if it is continuous and maps bounded sets into compact sets.

The investigation of the fixed points (for short f.p.) of an antitone operator starts with the lemma which asserts that any two different f.p. of such a mapping are incomparable.

Lemma 1. *If T is an antitone operator which maps a partially ordered set $(E, \leqq)$ into itself and $x \leqq y$ are two of its f.p., then $x = y$. Moreover, if the set S_1 of all* *f.p. of T shows the property that any two-point subset of S_1 has an upper bound or a lower bound (in S_1) (S_1 is up- or down-directed), then there exists at most one f.p. of T.*

Proof. If $x, y \in S_1$ and $x \leqq y$, then $x = T(x) \geqq T(y) = y$ and the first part of the lemma is proved. If $u, v \in S_1$ and x (y) is an upper bound (a lower bound) of the pair $\{u, v\}$ lying in S_1, then, by what has been already proved, $u = x = v$ $(u = y = v)$. The proof of the lemma is complete.

The existence of f.p. of an antitone operator is a more delicate question than that of their uniqueness. The following considerations will be of use.

If T is antitone, then T^2 is isotone. Denote by $S_1(S_2)$ the set of all f.p. of $T(T^2)$. Clearly $S_1 \subset S_2$ and, if $a \in S_2 - S_1$, then there exists a unique $b \in S_2 - S_1$, $b \neq a$, such that $T(a) = b$, $T(b) = a$. Further, we see that T is the identity mapping on S_1, $T(S_2) = S_2$ and T is an injection on S_2. This implies that if $c = \sup S_2 \in S_2$ (c is the maximal f.p. of T^2) or if $c = \inf S_2 \in S_2$ (c is the minimal f.p. of T^2), then $T(c) = \inf S_2 \in S_2$ or $T(c) = \sup S_2 \in S_2$, respectively. Further, if S_2 has an odd number of elements, then $S_1 \neq \emptyset$.

The following lemma is an application of the Tarski theorem. Here as usual a *conditionally complete lattice* means such a lattice that each of its bounded subsets has the sup and the inf.

Lemma 2. *Let $(E, \leqq)$ be a conditionally complete lattice, $x_0 \leqq y_0$ two elements of E. Let $T: [x_0, y_0] \to E$ be antitone and such that*

$$x_0 \leqq T(y_0) \leqq T(x_0) \leqq y_0 \,. \tag{1}$$

Let there be no such $x \in [x_0, y_0]$ that $x = T^2(x) < T(x)$. Then T has a unique f.p.

Proof. Since E is a conditionally complete lattice, $[x_0, y_0]$ is a complete lattice. In view of (1), the isotone operator T^2 maps $[x_0, y_0]$ into itself. By the Tarski theorem ([12]) there exists the least u and the greatest v f.p. of T^2. As $u = T^2(u) \leqq T(u) = v$, by the assumption $u = v$ and hence u is the unique f.p. of T^2 as well as of T.

More interesting results can be obtained in the case of a completely continuous operator.

Lemma 3. *Let (E, P) be an OBS with a normal positive cone and let $T: E \to E$ be a completely continuous antitone mapping. Further, let there exist two points $x_0, y_0 \in E$ such that (1) is true. Then T has at least one f.p. in the order interval $[x_0, y_0]$.*

Proof. (1) implies that the antitone operator T maps $[x_0, y_0]$ into itself. Since $[x_0, y_0] = (x_0 + P) \wedge (y_0 - P)$ is a closed, convex and bounded set and T is completely continuous, by the Schauder f.p. theorem the assertion follows.

The uniqueness of the f.p. of T and its construction is guaranteed by the next lemma.

Lemma 4. *Let all assumptions of Lemma 3 be fulfilled and suppose further that there is no such x in the order interval $[x_0, y_0]$ that $x = T^2(x) < T(x)$. Then T has a unique f.p. u in $[x_0, y_0]$ and the sequences $\{x_p\}_{p=0}^{\infty}$, $\{y_p\}_{p=0}^{\infty}$ defined by*

$$x_{p+1} = T(x_p), \quad y_{p+1} = T(y_p) \quad (p = 0, 1, 2, \ldots)$$

converge to u and

$$x_0 \leqq y_1 \leqq x_2 \leqq y_3 \leqq \ldots \leqq x_{2p} \leqq y_{2p+1} \leqq \ldots \leqq u \leqq \ldots \tag{2}$$

$$\ldots \leqq x_{2p+1} \leqq y_{2p} \leqq \ldots \leqq x_3 \leqq y_2 \leqq x_1 \leqq y_1 .$$

Proof. Consider T^2. From (1) it follows that $x_0 \leqq T(y_0) \leqq T^2(x_0) \leqq T^2(y_0) \leqq \leqq T(x_0) \leqq y_0$ and further, T^2 is isotone and completely continuous. By Corollary 2.2 ([2, p. 369]) there exists a minimal u and a maximal v f.p. of T^2 in $[x_0, y_0]$. Similarly as in the proof of Lemma 2 we get that $u = v$ and hence u is the unique f.p. of T^2 as well as of T in $[x_0, y_0]$. Moreover, both sequences $\{x_{2p}\}_{p=0}^{\infty}$, $\{y_{2p}\}_{p=0}^{\infty}$ converge to u. By induction we get (2) and due to the normality of the positive cone P and (2) the convergence of the whole sequences $\{x_p\}_{p=0}^{\infty}$, $\{y_p\}_{p=0}^{\infty}$ to u as $p \to \infty$ is guaranteed.

With help of Lemmas 3 and 4 the next theorem and its corollary will be proved.

Theorem 1. *Let (E, P) be an OBS with a normal positive cone, let $T: E \to E$ be a completely continuous antitone mapping. Let there exist an $a \in E$ such that*

$$T(x) \geqq a \quad (T(x) \leqq a) \quad (x \in E) . \tag{3}$$

Then the set S_1 of all f.p. of T is nonempty, $S_1 \subset [a, T(a)]$ ($S_1 \subset [T(a), a]$) and S_1 is compact.

Proof. Denote $x_0 = a$, $y_0 = T(a)$ ($x_0 = T(a)$, $y_0 = a$). Then $x_0 \leqq y_0$ and in view of (3) and of the fact that T is antitone, $x_0 \leqq T(y_0) \leqq T(x_0) \leqq y_0$ ($x_0 \leqq \leqq T(y_0) \leqq T(x_0) \leqq y_0$). Hence (1) is satisfied and by Lemma 3, $S_1 \neq \emptyset$. If $u = T(u)$, then again using (3) and the monotonicity of T we have $a \leqq T(u) = u \leqq T(a)$ ($T(a) \leqq T(u) = u \leqq a$). Hence $S_1 \subset [a, T(a)]$ ($S_1 \subset [T(a), a]$). Thus S_1 is bounded and $T(S_1) = S_1$ lies in a compact subset of E. S_1 is closed and this gives that S_1 is compact. The proof is complete.

Corollary. *If all assumptions of Theorem 1 are satisfied and there is no such x in $[a, T(a)]$ ($[T(a), a]$) that $x = T^2(x) < T(x)$, then T has a unique f.p. u and the sequence $\{x_p\}_{p=0}^{\infty}$ ($\{y_p\}_{p=0}^{\infty}$) defined by $x_0 = a$, $x_{p+1} = T(x_p)$ ($y_0 = a$, $y_{p+1} =$*

$= T(y_p))$ $(p = 0, 1, 2, \ldots)$ *converges to* u *and*

$$x_0 \leqq x_2 \leqq \ldots \leqq x_{2p} \leqq \ldots \leqq u \leqq \ldots \leqq x_{2p+1} \leqq \ldots \leqq x_3 \leqq x_1$$

$$(y_1 \leqq y_3 \leqq \ldots \leqq y_{2p+1} \leqq \ldots \leqq u \leqq \ldots \leqq y_{2p} \leqq \ldots \leqq y_2 \leqq y_0).$$

Proof. The result follows from Lemma 4 and Theorem 1.

Remark. If all assumptions of Theorem 1 are fulfilled except (3) which is replaced by the hypothesis

$$x \geqq a \text{ implies } T(x) \geqq a \quad (x \leqq a \text{ implies } T(x) \leqq a),$$

then T has at least one f.p. in $[a, T(a)]$ $([T(a), a])$. The proof proceeds as that of the first part of Theorem 1.

2. INITIAL-VALUE PROBLEMS

First, by means of Lemma 1 the Peano uniqueness theorem will be extended to the case when in the differential (for short d.) equation

$$x' = f(t. x) \tag{4}$$

f satisfies Carathéodory conditions. f as well as all functions which will be considered throughout the paper will be supposed to be real.

Theorem 2. *Suppose* $t_0, x_0, a > 0, b > 0$ *are reals,* f *is a function defined on* $Q = \{(t, x) \in R^2 : |t - t_0| \leqq a, |x - x_0| \leqq b\}$ *which satisfies Carathéodory conditions, i.e.*

(i) $f(\cdot, x)$ *is measurable in* $[t_0 - a, t_0 + a]$ *for each fixed* $x \in [x_0 - b, x_0 + b]$,
(ii) $f(t, \cdot)$ *is continuous in* $[x_0 - b, x_0 + b]$ *for each fixed* $t \in [t_0 - a, t_0 + a]$,
(iii) *there is an* $m \in L([t_0 - a, t_0 + a])$ *such that*

$$|f(t, x)| \leqq m(t) \quad ((t, x) \in Q). \tag{5}$$

Suppose further that f *is nondecreasing (nonincreasing) in* $x \in [x_0 - b, x_0 + b]$ *for each* $t \in [t_0 - a, t_0]$ $(t \in [t_0, t_0 + a])$.

Then there exists a unique solution in the extended sense of the initial-value problem (for short IVP) (4),

$$x(t_0) = x_0 \tag{6}$$

on an interval $[t_0 - a_1, t_0 + a_1]$ *with* $0 < a_1 \leqq a$.

Proof. The existence of a solution to (4), (6) is guaranteed by the Carathéodory existence theorem ([4, p. 43]). In order to prove the uniqueness, put $M_1(t) = |\int_{t_0}^t m(s) \, ds|$ for $|t - t_0| \leqq a$. Then there exists an a_1, $0 < a_1 \leqq a$, such that

$M_1(t) \leqq b$ for all $t \in [t_0 - a_1, t_0 + a_1]$. Let $K = \{g \in C([t_0 - a_1, t_0 + a_1]) : : |g(t) - x_0| \leqq M_1(t),\ t_0 - a_1 \leqq t \leqq t_0 + a_1\}$. Then with respect to (5), the operator $T : K \to C([t_0 - a_1, t_0 + a_1])$ which is defined by

$$(7) \qquad T(g)(t) = x_0 + \int_{t_0}^{t} f[s, g(s)]\, ds \quad (t_0 - a_1 \leqq t \leqq t_0 + a_1)$$

maps K into itself. Moreover, each solution x of (4), (6) on an interval i satisfies $|x(t) - x_0| \leqq M_1(t)$ in $i \cap [t_0 - a_1, t_0 + a_1]$ and hence it can be extended to the interval $[t_0 - a_1, t_0 + a_1]$ (if necessary) and its reduction to that interval belongs to K. When K is ordered in the natural way, T is antitone. The existence of a maximum solution to the Cauchy problem (4), (6) as it is assured by Theorem 1.2 ([4, p. 45]), implies that Lemma 1 can be applied. By this lemma there exists at most one solution of the mentioned problem in K and hence in $C([t_0 - a_1, t_0 + a_1])$.

Lemma 2 yields an existence and uniqueness statement which is not based on the Carathéodory existence theorem. Roughly speaking, condition (ii) in Theorem 2 will be replaced by another one. First we recall the definition of the superpositionally measurable (integrable) function. The set of all real measurable functions on $[t_0 - a, t_0 + a]$ will be denoted by $M([t_0 - a, t_0 + a])$.

Definition. Let $f : Q \to R$, where Q is defined above. f will be called *superpositionally measurable* (*superpositionally integrable*) *on* Q if for each function $g \in \in M([t_0 - a, t_0 + a])$ $(g \in L([t_0 - a, t_0 + a]))$ such that $|g(t) - x_0| \leqq b$ $(t_0 - a \leqq \leqq t \leqq t_0 + a)$ the composite function $f[\cdot, g(\cdot)]$ is measurable (integrable) in $[t_0 - a, t_0 + a]$.

If f satisfies Carathéodory conditions in Q, then it is superpositionally integrable on Q. On the other hand, when f is superpositionally measurable, then it satisfies the condition (i) from Theorem 2. (ii) need not be fulfilled as the following example shows.

Example. Let

$$f(t, x) = \begin{cases} -1 & 0 \leqq x \leqq 1 \\ -\frac{1}{2} & x = 0 \\ -\frac{1}{4} & -1 \leqq x < 0 \quad \text{and} \quad 0 \leqq t \leqq 1 \,. \end{cases}$$

Then f is superpositionally integrable in $[0, 1] \times [-1, 1]$, yet (ii) is not satisfied.

Theorem 3. *Let f be a superpositionally measurable function on Q which satisfies all assumptions of Theorem 2 except* (ii). *Let the IVP*

$$(8) \qquad \begin{aligned} x' &= f(t, z)\,, \quad x(t_0) = x_0 \\ z' &= f(t, x)\,, \quad z(t_0) = x_0 \end{aligned}$$

have at most one solution. Then there exists a unique solution in the extended sense of (4), (6) *on an interval* $[t_0 - a_1, t_0 + a_1]$.

Proof. Keeping the notation from the proof of Theorem 2, we consider the set $K_1 = \{g \in L([t_0 - a_1, t_0 + a_1]) : |g(t) - x_0| \leqq M_1(t), t \in [t_0 - a_1, t_0 + a_1] \text{ a.e.}\}$. f is superpositionally integrable on $Q_1 = \{(t, x) \in R^2 : |t - t_0| \leqq a_1, |x - x_0| \leqq b\}$ and the operator T which is given by (7) maps K_1 into itself. Since $u \leqq v$ means in $L([t_0 - a_1, t_0 + a_1]$ that $u(t) \leqq v(t)$ for almost all $t \in [t_0 - a_1, t_0 + a_1]$, K_1 is a complete lattice and T is antitone. The functions $u(t) = x_0 - M_1(t)$, $v(t) = x_0 + M_1(t)$ $(t_0 - a_1 \leqq t \leqq t_0 + a_1)$ show the property $u \leqq T(v) \leqq T(u) \leqq v$. If $x = T^2(x)$, then denoting $z = T(x)$ we have $z(t) = x_0 + \int_{t_0}^t f[s, x(s)]\, ds$, $x(t) = x_0 + \int_{t_0}^t f[s, z(s)]\, ds$ $(t \in [t_0 - a_1, t_0 + a_1])$. This implies that the ordered pair (x, z) as well as (z, x) is a solution of (8). Since (8) has at most one solution, T^2 has at most one f.p.. By Lemma 2, there exists a unique solution of (4), (6) in K_1. Since any solution in $[t_0 - a_1, t_0 + a_1]$ of that problem has to lie in K_1, the proof is complete.

Remark. Both Theorems 2 and 3 remain to be true when the onesided problem is considered (to the right or to the left of the initial point t_0). When f_0 is the function from the example above, then by Theorem 3 the problem $x' = f_0(t, x)$, $x(0) = 0$ has a unique solution in an interval $[0, a_1]$.

3. BOUNDARY VALUE PROBLEMS

Boundary value problems (for short BVP-s) represent a rich source of problems with isotone and antitone operators. Consider the following class of problems (compare with [3, pp. 158–159]).

Let $n \geqq 2$ be a natural number, $a < b$ real numbers. Let, further, $p_j \in L([a, b])$ $(j = 1, \ldots, n)$ and $f : [a, b] \times R \to R$ satisfy locally Carathéodory conditions. Let $B_i(x) = \sum_{j=1}^n \alpha_{ij} x^{(j-1)}(a) + \sum_{j=1}^n \beta_{ij} x^{(j-1)}(b)$ $(i = 1, 2, \ldots, n)$ be a set of n linearly independent boundary conditions where α_{ij}, β_{ij} $(i, j = 1, \ldots, n)$ are real numbers. Let $L^{(n)}([a, b])$ be the class of all functions with $x^{(n)} \in L([a, b])$. Denote by L_0 the differential operator

$$L_0(x) = x^{(n)} + \sum_{j=1}^n p_j(t)\, x^{(n-j)} \quad (x \in L^{(n)}([a, b]))\,.$$

Definition. We shall say that the linear differential operator L_0 is *inverse monotone* (*inverse antimonotone*) *with respect to boundary conditions*

$$B_i(x) = 0 \quad (i = 1, 2, \ldots, n) \tag{9}$$

if for any $x \in L^{(n)}([a, b])$ satisfying (9) the following implication holds:

$L_0(x)(t) \geqq 0$ a.e. in $[a, b]$ implies $x(t) \geqq 0$ $(x(t) \leqq 0)$ in $[a, b]$.

Since the set of all solutions of the BVP (9),

$$L_0(x) = 0 \tag{10}$$

forms a linear subspace of $L^{(n)}([a, b])$, L_0 is inverse monotone (inverse antimonotone) iff (9), (10) has only the trivial solution and the Green function G of the problem (10), (9) satisfies the inequality

$$G(t, s) \geqq 0 \quad (G(t, s) \leqq 0) \quad \text{in} \quad [a, b] \times [a, b] .$$

If this is the case, the linear integral operator

$$U(x) = \int_a^b G(t, s)\, x(s)\, \mathrm{d}s \quad (x \in C([a, b])) \tag{11}$$

maps the OBS (C, P) into itself. Here and in what follows C will mean the Banach space $C([a, b])$ with the sup norm $\|\cdot\|$ and $P \subset C$ the normal positive cone of all nonnegative functions. P is generating, i.e. $C = P - P$ and U is completely continuous.

The eigenvalues of the homogeneous BVP (10), (9), i.e. the numbers λ (they may be complex) for which there exists a nontrivial solution u (the so called eigenfunction) of the problem

$$\pi: \qquad L_0(x) = \lambda x\,, \quad B_i(x) = 0 \quad (i = 1, 2, \ldots, n)$$

(if they exist) are different from zero. λ is an eigenvalue and u is the corresponding eigenfunction of (10), (9) iff $1/\lambda$ is an eigenvalue and u is the eigenfunction of U. Thus the one-to-one correspondence between the eigenvalues of (10), (9) and nonzero eigenvalues of U is established. As for the operator U we shall consider first the case that $G(t, s) \geqq 0$ in $[a, b] \times [a, b]$. Then U is positive and either its spectral radius $r(U) = 0$ or $r(U) > 0$. In the former case U has no eigenvalues different from zero. If $r(U) > 0$, by Lemma 5.2 ([7, p. 77]) $r(U)$ is an eigenvalue of U, and there exists an eigenfunction $u \in P$ corresponding to $r(U)$. All the other eigenvalues λ of U satisfy $|\lambda| \leqq r(U)$. The case $G(t, s) \leqq 0$ in $[a, b] \times [a, b]$ can be reduced to the above one by observing that the eigenvalues of $-U$ differ from those of U only by the sign and the corresponding eigenfunctions remain the same. Hence in this case $-r(U)$ is the eigenvalue of (11) with the greatest absolute value and the corresponding eigenfunction u again belongs to P. Coming back to the problem (10), (9) we see that the following alternative holds. Either this problem has no eigenvalues (this corresponds to the case $r(U) = 0$) or there is a positive (a negative) eigenvalue λ_0 and a nonnegative function u such that all eigenvalues λ of (10), (9) satisfy $|\lambda| \geqq \lambda_0$ $(|\lambda| \geqq -\lambda_0)$ and u is the eigenfunction of (10), (9) corresponding to λ_0.

By means of these remarks we shall prove the following fundamental lemma for inverse monotone (inverse antimonotone) operators.

Lemma 5. *Let L_0 be an inverse monotone (inverse antimonotone) operator. Let λ_0 be the positive (negative) eigenvalue of* (10), (9) *such that all eigenvalues λ of that problem (if they exist) satisfy $|\lambda| \geqq |\lambda_0|$. If* (10), (9) *has no eigenvalue at all, put $\lambda_0 = +\infty$ ($\lambda_0 = -\infty$). Let f satisfy the Lipschitz condition*

$$|f(t, x) - f(t, y)| \leqq L|x - y| \quad (a \leqq t \leqq b,\ x, y \in R) \tag{12}$$

with a constant $0 \leqq L < \lambda_0$ ($0 \leqq L < -\lambda_0$). Then for any real numbers C_i, ($i = 1, 2, \ldots, n$), there exists a unique solution z of the BVP

$$L_0(x) = f(t, x) \tag{13}$$

$$B_i(x) = C_i \quad (i = 1, 2, \ldots, n) \tag{14}$$

and for any function $x_0 \in C$ the sequence $\{x_p\}_{p=0}^{\infty}$ defined by

$$L_0(x_{p+1}) = f[t, x_p(t)] \tag{15}$$

$$B_i(x_{p+1}) = C_i \quad (i = 1, 2, \ldots, n) \quad (p = 0, 1, 2, \ldots)$$

is such that the sequence $\{x_p^{(j)}\}_{p=0}^{\infty}$ converges uniformly to $z^{(j)}$ ($j = 0, 1, \ldots, n - 1$) on $[a, b]$.

Proof. The problem (13), (14) is equivalent to the integral equation

$$x(t) = w(t) + \int_a^b G(t, s) f[s, x(s)]\, ds \quad (a \leqq t \leqq b), \tag{16}$$

where w is the unique solution of (10), (14), G is the Green function of (10), (9). Let $T: C \to C$ be the operator

$$T(x)(t) = w(t) + \int_a^b G(t, s) f[s, x(s)]\, ds \quad (x \in C,\ a \leqq t \leqq b). \tag{17}$$

In view of (12), for any $x, y \in C$ we have

$$|T(x)(t) - T(y)(t)| \leqq L \int_a^b |G(t, s)|\, |x(s) - y(s)|\, ds \quad (a \leqq t \leqq b). \tag{18}$$

By the remarks above, the spectral radius ϱ of the operator $V: C \to C$ which is defined by $V(x)(t) = L \int_a^b |G(t, s)|\, x(s)\, ds$ ($x \in C$, $a \leqq t \leqq b$) is $\varrho = L/|\lambda_0|$ (if $|\lambda_0| = +\infty$, then $\varrho = 0$). As $L < |\lambda_0|$, $\varrho < 1$. Hence there exists a norm $\|\cdot\|_1$ equivalent to the sup-norm $\|\cdot\|$ in $C([a, b])$ ([7, pp. 15–16]) such that $\|V\|_1 < L/|\lambda_0| + \varepsilon < 1$ with $\varepsilon > 0$ being sufficiently small. As V is a positive operator and the norm $\|\cdot\|$ is monotone, that is, $0 \leqq x \leqq y$ implies $\|x\| \leqq \|y\|$, the norm $\|\cdot\|_1$ is monotone, too. Hence (18) implies that $\|T(x) - T(y)\|_1 \leqq \|V\|_1\, \|x - y\|_1$. T is a contraction and by the Banach f.p. theorem we have that for any $x_0 \in C([a, b])$ the sequence given by $x_{p+1} = T(x_p)$ ($p = 0, 1, 2, \ldots$) and hence by (15) converges in the norm $\|\cdot\|_1$ as well as in $\|\cdot\|$ to the unique solution z of (16) and thus, of (13), (14). With respect to the meaning of the sup norm $\|\cdot\|$, the sequence $\{x_p\}_{p=0}^{\infty}$ uniformly converges to z on $[a, b]$ and it is uniformly bounded there. Let $j \in \{1, \ldots, n-1\}$. Then

$$x_{p+1}^{(j)}(t) = w^{(j)}(t) + \int_a^b \frac{\partial^j G(t,s)}{\partial t^j} f[s, x_p(s)]\, \mathrm{d}s \quad (a \leqq t \leqq b,\; p = 0, 1, 2, \ldots)$$

and as there exists an $m \in L(^2a, b])$ such that

$$|x_{p+1}^{(j)}(t)| \leqq |w^{(j)}(t)| + \int_a^b \left|\frac{\partial^j G(t,s)}{\partial t^j}\right| m(s)\, \mathrm{d}s\,, \quad (a \leqq t \leqq b,\; p = 0, 1, 2, \ldots)\,,$$

all sequences $\{x_p^{(j)}\}_{p=0}^{\infty}$ $(j = 1, \ldots, n-1)$ are uniformly bounded on $[a, b]$. From the uniform continuity of $\partial^{n-1}G/\partial t^{n-1}$ on the triangles $a \leqq s \leqq t \leqq b$ and $a \leqq t \leqq \leqq s \leqq b$ it follows that $\{x_p^{(n-1)}\}_{p=0}^{\infty}$ is equicontinuous on $[a, b]$. Hence $\{x_p\}_{p=0}^{\infty}$ has the property that each of its subsequences $\{x_{p(r)}\}_{r=0}^{\infty}$ has a subsequence $\{x_s\}_{s=0}^{\infty}$ such that $\{x_s^{(j)}\}_{s=0}^{\infty}$ uniformly converges to $z^{(j)}$ on $[a, b]$ $(j = 0, 1, \ldots, n-1)$. Then the whole sequence $\{x_p\}_{p=0}^{\infty}$ possesses the same property and the proof of the lemma is complete.

Remark. Lemma 5 strengthens the Uniqueness theorem III in ([13, p. 272]). Although this lemma does not concern the theory of antitone operators directly, because the operator T given by (17) is not antitone in general, the lemma has been given here because of its importants. In special cases when L is inverse monotone and $f(t, \cdot)$ in nonincreasing in R or when L is inverse antimonotone and $f(t, \cdot)$ is nondecreasing in R, T is antitone.

Suppose that k, $1 \leqq k \leqq n-1$, is a natural number. We shall deal with a special case of the boundary conditions (9), namely with the conditions

$$(19) \qquad x^{(i)}(a) = 0 \quad (i = 0, \ldots, k-1)\,, \qquad x^{(i)}(b) = 0 \quad (i = 0, \ldots, n-k-1)\,.$$

A sufficient condition for the operator L_0 to be inverse monotone or inverse antimonotone with respect to (19) is given by the following lemma.

Levin's lemma. ([8, pp. 80–81]). *If the differential equation* (10) *is disconjugate on* $[a, b]$, *then there exists the Green function* G *of the BVP* (10), (19), *its sign is determined by the inequality*

$$(20) \qquad G(t, s)\,(t - b)^{n-k} \geqq 0 \quad (a \leqq t,\; s \leqq b)$$

and $G(t, s) \neq 0$ *in* $a < t$, $s < b$.

By means of Lemma 1 we prove the next theorem.

Theorem 4. *Let the differential equation* (10) *be disconjugate in* $[a, b]$, *let* $(-1)^n \cdot f(t, \cdot)$ $(f(t, \cdot))$ *be nondecreasing in* R *for each fixed* $t \in [a, b]$. *Suppose further that the solutions of all IVP-s for the differential equation* (13) *at the point* b (a) *are unique. Then for any real numbers* A_i, B_i $(i = 0, \ldots, n-2)$ *there exists at* *most one solution of the BVP* (13),

$$(21) \qquad x(a) = A_0\,, \quad x^{(i)}(b) = B_i \quad (i = 0, \ldots, n-2)$$

$$((22) \qquad x^{(i)}(a) = A_i\,, \quad x(b) = B_0 \quad (i = 0, \ldots, n-2))\,.$$

Proof. Consider only the BVP (13), (21). The problem (13), (22) can be dealt with in a similar way. Suppose that there are two different solutions x_1, x_2 of (13), (21). Let $S = \{t \in [a, b] : x_1(t) = x_2(t)\}$. If b were a limit point of S, then together with $x_1^{(i)}(b) = x_2^{(i)}(b)$ $(i = 0, \ldots, n-2)$, $x_1^{(n-1)}(b) = x_2^{(n-1)}(b)$ would hold and by the last assumption of the theorem, $x_1 = x_2$. Therefore there exists a t_1, $a \leqq t_1 < b$, such that $x_1(t_1) = x_2(t_1) = \bar{A}_0$ and $x_1(t) \neq x_2(t)$ in (t_1, b), say $x_1(t) > x_2(t)$. Thus x_1, x_2 are two comparable solutions of the BVP (13),

$$(23) \qquad x(t_1) = \bar{A}_0\,, \quad x^{(i)}(b) = B_i \quad (i = 0, \ldots, n-2)\,.$$

By Levin's lemma there exists the Green function G of the corresponding homogeneous problem (10), (19) for $k = 1$ and (20) implies that $(-1)^n G(t, s) \leqq 0$ in the square $t_1 \leqq t, s \leqq b$. The problem (13), (23) is thus equivalent to the integral equation

$$(24) \qquad x(t) = y(t) + \int_{t_1}^{b} G(t, s) f[s, x(s)] \, \mathrm{d}s \quad (t \in [t_1, b])\,,$$

where y is the solution of (10) satisfying (23). The right-hand side of (24) defines an antitone operator T which maps $C([t_1, b])$ into itself. By Lemma 1, the existence of two comparable f.p. of T is impossible and this proves the theorem.

Corollary. *Suppose that* $p', q, r \in C((0, \infty))$, $f = f(t, x)$, $\partial f / \partial x \in C(D)$ *where* $D = (0, \infty) \times R$ *and*

$$\frac{\partial f(t, x)}{\partial x} \leqq 0 \left(\frac{\partial f(t, x)}{\partial x} \geqq 0 \right)$$

in D. *Suppose further that*

$$(25) \qquad x'' + \left(q(t) - \frac{p^2(t)}{4} - \frac{p'(t)}{2} \right) x = 0$$

is disconjugate in $(0, \infty)$. *Then for any real numbers* $0 < a < b$, A_0, A_1, B_0, B_1 *there exists at most one solution of the BVP*

$$(26) \qquad x''' + p(t)\, x'' + q(t)\, x' = f(t, x)\,,$$

$$(27) \qquad x(a) = A_0\,, \quad x^{(i)}(b) = B_i \quad (i = 0, 1)\,,$$

$$((28) \qquad x^{(i)}(a) = A_i \quad (i = 0, 1)\,, \quad x(b) = B_0)\,.$$

Proof. Since the substitution $y = x \exp\left(-\frac{1}{2} \int_a^t p(s) \, \mathrm{d}s\right)$ transforms (25) into the form $y'' + p(t)\, y' + q(t)\, y = 0$, the last equation is disconjugate, too. Using Rolle's theorem we can prove that the equation $x''' + p(t)\,x'' + q(t)\,x' = 0$ is disconjugate on $(0, \infty)$. Since all assumptions of Theorem 4 are satisfied in $[a, b]$, by this theorem the statement of Corollary follows.

Remark. If $x'' + (q(t) - \frac{1}{2}\,p'(t))\,x = 0$ is disconjugate in $(0, \infty)$, so is (25). Therefore the corollary stregthens the result of Theorem 4.6 in [5, p. 720].

The existence of a solution to the BVP (13), (21) and (13), (22), respectively, is ensured by the following theorem.

Theorem 5. *Let all assumptions of Theorem* 4 *be satisfied and further, let the solution of all IVP-s for the differential equation* (13) *at the point* b (a) *be extensible to the whole interval* $[a, b]$. *Then for any real numbers* A_i, B_i $(i = 0, \ldots, n - 2)$ *there exists a unique solution of the BVP* (13), (21) ((13), (22)).

Proof. Since the method of the proof in the case (13), (22) is very similar to that of (13), (21), only the latter will be given here. For any $c \in R$ we denote the solution of the IVP (13),

$$x^{(i)}(b) = B_i \quad (i = 0, \ldots, n - 2)\,, \quad x^{(n-1)}(b) = c\,, \tag{29}$$

by $x(\cdot, c)$. By the assumption of the theorem $x(\cdot, c)$ exists on $[a, b]$ and hence the mapping $F : R \to R$ with $F(c) = x(a, c)$ is well defined. Theorem 4 implies that F is one-to-one and in the case n is even (n is odd) F is decreasing (increasing). In case a is sufficiently close to b this follows by considering of the sign of derivative, and by Theorem 4 this is true in the general case, too. To make clear further properties of F, denote by $y(\cdot, c)$ the solution of (10), (29) and by $K = K(t, s)$ the Cauchy function for the d. equation (10), i.e. $K(\cdot, s)$ is the solution of (10) which is determined by the initial conditions

$$x^{(i)}(s) = 0 \quad (i = 0, \ldots, n - 2)\,, \quad x^{(n-1)}(s) = 1\,.$$

Suppose that n is even, $c > 0$. Then $K(t, s) \leqq 0$ for $a \leqq t \leqq s \leqq b$, $x(s, c) < x(s, 0)$ $(a \leqq s < b)$ and $f(t, \cdot)$ is nondecreasing in R, hence

$$\begin{aligned} F(c) = x(a, c) = y(a, c) + \int_b^a K(a, s)\, f[s, x(s, c)]\, \mathrm{d}s = y(a, 0) + \\ + c\,K(a, b) + \int_b^a K(a, s) f[s, x(s, c)]\, \mathrm{d}s \leqq y(a, 0) + \\ + c\,K(a, b) + \int_b^a K(a, s) f[s, x(s, 0)]\, \mathrm{d}s = \\ = c\,K(a, b) + x(a, 0) = \\ = c\,K(a, b) + F(0)\,. \end{aligned} \tag{30}$$

When $c < 0$, then we get

$$F(c) \geqq c\,K(a, b) + F(0)\,. \tag{31}$$

Similarly in the case n is odd, $c > 0$ $(c < 0)$, we come to the inequality

$$F(c) \geqq c\,K(a, b) + F(0) \tag{32}$$

$$\left(F(c) \leqq c\,K(a, b) + F(0)\right). \tag{(33)}$$

In (30), (31) $K(a, b) < 0$, while in (32), (33) $K(a, b) > 0$. Hence these four inequalities imply that F attains arbitrarily large positive and negative values. The proof of the theorem will be complete if we show that F is a continuous function. This will be done in the case n is even. Let $\{c_n\}$ be a nondecreasing sequence tending to c. Then $x(t, c_1) > x(t, c_2) > \ldots > x(t, c_n) > \ldots > x(t, c)$ for each t, $a \leqq t < b$. Denote $\lim_{n \to \infty} x(t, c_n) = z(t)$. As the theorem on continuous dependence of the solution on initial conditions is true for the linear d. equation (10), $\lim_{n \to \infty} y(t, c) = y(t, c)$ and from

$$x(t, c_n) = y(t, c_n) + \int_b^t K(t, s) f[s, x(s, c_n)]\, \mathrm{d}s \quad (t \in [a, b])$$

by the limit process we conclude that the integrable function z satisfies

$$z(t) = y(t, c) + \int_b^t K(t, s) f[s, z(s)]\, \mathrm{d}s \quad (t \in [a, b])$$

and thus, z is a solution of (13), (29). The uniqueness of this IVP guarantees that $z(t) = y(t, c)$ $(a \leqq t \leqq b)$ and F is continuous from the left. Similarly the right-hand side continuity of F can be proved.

Corollary. *Let the hypotheses of Corollary to Theorem* 4 *hold and suppose further that the solution of all IVP-s for* (26) *can be extended to the whole interval* $(0, \infty)$. *Then for any real numbers* $0 < a < b$, A_0, A_1, B_0, B_1 *there exists a unique solution of the BVP* (26), (27) *and* (26), (28).

The next theorem is a consequence of Theorems 4 and 5.

Theorem 6. *Let the following hypotheses be satisfied:*

1. $p_j \in L_{\mathrm{loc}}((0, \infty))$ $(j = 1, \ldots, n)$ *and* $f : H = (0, \infty) \times R \to R$ *satisfies locally Carathéodory conditions in* H.
2. *The differential equation* (10) *is disconjugate in* $(0, \infty)$.
3. $(-1)^n f(t, .)$ $(f(t, .))$ *is nondecreasing in* R *for each fixed* $t \in (0, \infty)$.
4. *The solutions of all IVP-s for* (13) *are unique and can be extended to the whole interval* $(0, \infty)$.

Then for any solution x_0 of (13) on $(0, \infty)$, any $T > 0$, any $A > A_0 \geqq 0$ where $\sum_{i=0}^{n-1} [x_0^{(i)}(T)]^2 = A_0^2$ there is a pair of solutions x, y of (13) on $(0, \infty)$ such that

$$\sum_{i=0}^{n-1} [x^{(i)}(T)]^2 = \sum_{i=0}^{n-1} [y^{(i)}(T)]^2 = A^2 \tag{34}$$

and

$$x(t) \geqq x_0(t) \geqq y(t) \quad (0 < t < \infty).$$

Moreover, if for any $s \in (0, \infty)$ the Cauchy function K for (10) is such that

$$\frac{\partial K(t, s)}{\partial t} \neq 0 \quad \text{for all } t, \quad 0 < t < s \quad (\text{for all } t, \ s < t < \infty), \tag{35}$$

then $x(t) > x_0(t) > y(t)$ $(0 < t < \infty)$ and $x'(t) - x_0'(t) \leqq 0$, $y'(t) - x_0'(t) \geqq 0$ in $(0, \infty)$ (and $x'(t) - x_0'(t) \geqq 0$, $y'(t) - x_0'(t) \leqq 0$ in $(0, \infty)$).

Proof. Consider the case $(-1)^n f(t, .)$ is nondecreasing in R. Suppose that a solution x_0 of (13), a $T > 0$ and an $A > A_0$ are given. Consider a sequence of BVP-s (13),

$$\sum_{i=0}^{n-1} [x^{(i)}(T)]^2 = A^2, \quad x^{(i)}(T + m) = x_0^{(i)}(T + m) \tag{36}$$
$$(i = 0, \ldots, n - 2, \ m = 1, 2, \ldots).$$

Fix an m and denote the solution of the IVP (13),

$$x^{(i)}(T + m) = x_0^{(i)}(T + m) \quad (i = 0, \ldots, n - 2), \tag{37}$$
$$x^{(n-1)}(T + m) = x_0^{(n-1)}(T + m) + c$$

by $x(., c)$. Clearly $x(t, 0) = x_0(t)$ in $(0, \infty)$ and by Theorem 5, there exist c_1, c_2 with $c_1 < 0 < c_2$ when n is even and $c_1 > 0 > c_2$ when n is odd such that $x(T, c_1) = A$, $x(T, c_2) = -A$, hence $\sum_{i=0}^{n-1} [x^{(i)}(T, c_k)]^2 \geqq A^2$ $(k = 1, 2)$. Using the Theorem on continuous dependence of solutions on the initial condition we get that there is a c_3 lying between 0 and c_1 such that $x(T, c_3) > x_0(T)$ and $\sum_{i=0}^{n-1} [x^{(i)}(T, c)]^2 = A^2$. Similarly there is a $c_4 \in (0, c_2]$ (if n is even) or $c_4 \in [c_2, 0)$ (if n is odd) with $x(T, c_4) < x_0(T)$ and $\sum_{i=0}^{n-1} [x^{(i)}(T, c_4)]^2 = A^2$. By Theorem 4, $x(t, c_3) \neq x_0(t)$, $x(t, c_4) \neq x_0(t)$ in $(0, T + m)$ and in view of the above inequalities $x(t, c_3) > x_0(t) > x(t, c_4)$ in that interval. Denote $x(., c_3)$ and $x(., c_4)$, by x_m and y_m, respectively. If there are more solutions $x(., c_3)$ and $x(., c_4)$ of (13), (36), then we choose the one with the smallest $|c_3|$ and the smallest $|c_4|$, respectively.

Consider the sequence $\{x_m\}_{m=1}^{\infty}$. Since all x_m satisfy (34), there is a subsequence $\{x_{m_k}\}$ and a solution x of (13) such that $x^{(i)}$ is a locally uniform limit of $x_{m_k}^{(i)}$ on $(0, \infty)$

$(i = 0, 1, \ldots, n-1)$. With respect to the inequality $x_{m_k}(t) > x_0(t)$ in $(0, T + m_k)$ we have $x(t) \geqq x_0(t)$ in $(0, \infty)$. Similarly there exists a subsequence $\{y_{m_l}\}$ and a limit solution y of (13) with the above mentioned properties which satisfies (34) and $y(t) \leqq \leqq x_0(t)$ in $(0, \infty)$.

If (35) is true, then in the case n is even (n is odd)

$$\frac{\partial K(t, s)}{\partial t} > 0 \left(\frac{\partial K(t, s)}{\partial t} < 0 \right)$$

for $0 < t < s < \infty$ and $c_3 < 0$ ($c_3 > 0$). x_m satisfies the integral equation $x_m(t) = y(t, c_3) + \int_{T+m}^t K(t, s) f[s, x_m(s)]\, ds$ $(0 < t \leqq T + m)$ while $x_0(t) = y(t, 0) + \int_{T+m}^t K(t, s) f[s, x_0(s)]\, ds$ $(0 < t \leqq T + m)$. Here $y(., c)$ is the solution of (10), (37). Hence $y(t, c_3) = y(t, 0) + c_3 K(t, T + m)$, $(0 < t < \infty)$. This implies that

$$x_m'(t) - x_0'(t) = c_3 \frac{\partial K(t, T + m)}{\partial t} + \int_{T+m}^t \frac{\partial K(t, s)}{\partial t} \{f[s, x_m(s)] - f[s, x_0(s)]\}\, ds$$

$$\leqq 0\, (0 < t \leqq T + m),$$

and by the limit process, $x'(t) - x_0'(t) \leqq 0$ in the whole interval $(0, \infty)$. Since $x(t) - x_0(t) \geqq 0$ and $x(t) \not\equiv x_0(t)$ in $(0, \infty)$, we have $x(t) - x_0(t) > 0$ in this interval. Similarly we obtain $y'(t) - x_0'(t) \geqq 0$ and, in view of $y(t) - x_0(t) \leqq 0$ we arrive at the inequality $y(t) - x_0(t) < 0$ in $(0, \infty)$.

In the case $f(t, .)$ is nondecreasing in R, instead of (13), (36) we consider the sequence of the BVP-s (13),

(38)
$$\sum_{i=0}^{n-1} [x^{(i)}(T)]^2 = A^2, \quad x^{(i)}\left(\frac{1}{m}\right) = x_0^{(i)}\left(\frac{1}{m}\right) \quad (i = 0, \ldots, n-2,\ m \geqq m_0)$$

with $1/m_0 < T$. Then the existence of a pair of solutions x_m, y_m of (13), (38) such that $x_m(t) > x_0(t) > y_m(t)$ in $(1/m, \infty)$ can be proved. Again there exist subsequences $\{x_{m_k}\}$ and $\{y_{m_k}\}$, the limit of which x and y, respectively, is the solution of (13) for which the statement of the theorem is true. If (35) is valid, then $\partial K(t, s)/\partial t > 0$ for $0 < s < t < \infty$ and from

$$x_m'(t) - x_0'(t) = c_5 \frac{\partial K(t, 1/m)}{\partial t} + \int_{1/m}^t \frac{\partial K(t, s)}{\partial t} \{f[s, x_m(s)] - f[s, x_0(s)]\}\, ds$$

with a $c_5 > 0$ we come to the conclusion that $x'(t) - x_0'(t) \geqq 0$ in $(0, \infty)$. This implies that $x(t) - x_0(t) > 0$ in the same interval. Similarly the statements $y(t) - x_0(t) < 0$, $y'(t) - x_0'(t) \leqq 0$ hold and the proof of Theorem 6 is complete.

Corollary. *Let the hypotheses of Corollary to Theorem 5 hold. Then for any solution* x_0 *of* (26), *any* $T > 0$, *and any* $A > A_0 \geqq 0$ *where* $\sum_{i=0}^{2} [x_0^{(i)}(T)]^2 = A_0^2$

there is a pair of solutions x, y *of* (26) *on* $(0, \infty)$ *such that* (34) *is true for* $n = 3$ *and*

$$x(t) > x_0(t) > y(t), \quad x'(t) - x_0'(t) \leqq 0, \quad y'(t) - x_0'(t) \geqq 0$$

in $(0, \infty)$,

$$(x(t) > x_0(t) > y(t), \; x'(t) - x_0'(t) \geqq 0, \; y'(t) - x_0'(t) \leqq 0$$

in $(0, \infty)$).

Proof. The assumptions 1–4 of Theorem 6 can be easily checked to be fulfilled. As to (35), since $y'' + p(t)\,y' + q(t)\,y = 0$ is disconjugate in $(0, \infty)$, there is no nontrivial solution x of $x''' + p(t)\,x'' + q(t)\,x' = 0$ with two different zeros of x'.

Remarks. 1. When n is even and all assumptions of Theorem 6 as well as (35) are fulfilled, to any solution x_0 of (13) there exist two "funnels" of solutions x, y of (13). The first of them consists of infinitely many pairs x, y satisfying $x(t) \geqq x_0(t) \geqq y(t)$ in $(0, \infty)$ and such that the differences $|x(t) - x_0(t)|$, $|y(t) - x_0(t)|$ decrease as $t \to \infty$. For the pairs of the second one these differences increase for $t \to \infty$.

2. The last corollary improves the result of Theorem 4.9 ([5, p. 721]) in the case when the hypotheses of Theorem 4.6 hold.

The next theorems bring existence statements for the BVP (13),

$$(39) \qquad x^{(i)}(a) = A_i \quad (i = 0, \ldots, k - 1) \qquad x^{(i)}(b) = B_i \quad (i = 0, \ldots, n - k - 1)$$

with A_i, B_i being given numbers. The proofs are based on the notion of a lower and an upper solution.

Suppose again that $p_j \in L([a, b])$ $(j = 1, \ldots, n)$, $f : [a, b] \times R \to R$ satisfies locally Carathéodory conditions, $1 \leqq k \leqq n - 1$ is a natural number. Following [10, p. 281] we shall call x a *lower solution* (y an *upper solution*) of the d. equation (13) if x $(y) \in L^{(n)}([a, b])$ and x (y) satisfies the d. inequality $L_0(x) \leqq f(t, x)$ $(L_0(y) \geqq f(t, y))$ a.e. on the whole interval $[a, b]$. In this definition a lower and an upper solution need not satisfy any further conditions. However, if a lower solution x and an upper solution y of the d. equation (13) satisfy

$$L_0(x) \leqq f(t, y) \leqq f(t, x) \leqq L_0(y) \quad \text{a.e. in} \quad [a, b],$$

then we shall say that they are conjugate to each other.

Finally, for a $y \in L^{(n)}([a, b])$ let v_y be the solution of the problem (10),

$$x^{(i)}(a) = y^{(i)}(a) \quad (i = 0, \ldots, k - 1),$$

$$x^{(i)}(b) = y^{(i)}(b) \quad (i = 0, \ldots, n - k - 1).$$

v_y will be called the *solution of* (10) *associated with the function* y *with respect to the BVP* (10), (39).

Lemma 6. *Let the following assumptions be satisfied:*

1. $n - k$ *is an even (odd) number.*
2. (10) *is disconjugate in* $[a, b]$.
3. $f(t, .)$ *is nonincreasing (nondecreasing) in* R *for each fixed* $t \in [a, b]$.
4. *There exist a lower and an upper solution* x_0, y_0, *respectively, of the equation* (13) *with the properties*

   a) $x_0(t) \leqq y_0(t)$ $(x_0(t) \geqq y_0(t))$ *for all* $t \in [a, b]$,

   b) $v_{x_0}(t) \leqq w(t) \leqq v_{y_0}(t)$ $(v_{x_0}(t) \geqq w(t) \geqq v_{y_0}(t))$ *for each* $t \in [a, b]$ *where* w *is the solution of the problem* (10), (39),

   c) x_0, y_0 *are conjugate each to other.*

Then there exists at least one solution x *of the problem* (13), (39) *which satisfies*

$$x_0(t) \leqq x(t) \leqq y_0(t) \quad (x_0(t) \geqq x(t) \geqq y_0(t)) \quad (a \leqq t \leqq b)\,.$$

Any two solutions x, y *of that problem are incomparable, i.e. there exist two distinct numbers* $t_1, t_2 \in (a, b)$ *such that* $(x(t_1) - y(t_1))(x(t_2) - y(t_2)) < 0$.

Proof. Similarly as in the proof of Theorem 4 the problem (13), (39) is equivalent to the integral equation

$$x(t) = w(t) + \int_a^b G(t, s) f[s, x(s)] \, ds \quad (a \leqq t \leqq b)\,, \tag{40}$$

where G is the Green function of the corresponding homogeneous problem (10), (19). In view of Levin's lemma, the right-hand side of (40) defines an antitone operator T which maps C into itself. T is completely continuous. By Lemma 2 the inequalities (1) have to be shown in order to complete the proof of the existence of a f.p. of T and thus of the existence of a solution to (13), (39).

If $n - k$ is an even number, then by the assumptions 3 and 4 we have

$$x_0(t) = v_{x_0}(t) + \int_a^b G(t, s)\, L_0(x_0)(s) \, ds \leqq w(t) + \int_a^b G(t, s) f[s, y_0(s)] \, ds \leqq$$

$$\leqq w(t) + \int_a^b G(t, s) f[s, x_0(s)] \, ds \leqq v_{y_0}(t) + \int_a^b G(t, s)\, L_0(y_0)(s) \, ds = y_0(t)\,,$$

which means that $x_0(t) \leqq T(y_0)(t) \leqq T(x_0)(t) \leqq y_0(t)$ in $[a, b]$. When $n - k$ is odd, the last inequalities change their sign. In both cases we get the existence of a f.p. of T in $[x_0, y_0]$ and $[y_0, x_0]$, respectively. If there are more, by Lemma 1 they must be incomparable.

The next theorem is an easy consequence of Theorem 1.

Theorem 7. *Let the assumptions* 1–3 *of Lemma* 6 *be satisfied. Further, let*

4'. *there exist a constant* c_1 *such that* $f(t, x) \geqq c_1$ $((t, x) \in [a, b] \times R$ *a.e.) or* $f(t, x) \leqq c_1$ $((t, x) \in [a, b] \times R$ *a.e.).*

Denote by u the solution of the BVP $L_0(x) = c_1$, (39), *and by z the solution of* $L_0(x) = f[t, u(t)]$, (39). *Let G be the Green function of* (10), (19). *Then the set of all solutions of the BVP* (13), (39) *is nonempty, compact in the* sup-*norm* $\|.\|$ *and each solution x of that BVP satisfies in the case* $G(t, s) \geqq 0$ $(n - k$ *is even*$)$, $f(t, x) \geqq c_1$ *and* $G(t, s) \leqq 0$ $(n - k$ *is odd*$)$, $f(t, x) \leqq c_1$ *the inequalities*

$$u(t) \leqq x(t) \leqq z(t) \quad (a \leqq t \leqq b)$$

while for $G(t, s) \geqq 0, f(t, x) \leqq c_1$ *and* $G(t, s) \leqq 0, f(t, x) \geqq c_1$ *the inequalities*

$$z(t) \leqq x(t) \leqq u(t) \quad (a \leqq t \leqq b).$$

Proof. Only the case $G(t, s) \geqq 0$ will be considered. The other case can be dealt with in a similar way. Keeping the notations from the proof of Lemma 6, if $f(t, x) \geqq \geqq c_1$ $(f(t, x) \leqq c_1)$, then for any $x \in C$, $T(x)(t) = w(t) + \int_a^b G(t, s) f[s, x(s)] \, ds \geqq \geqq w(t) + \int_a^b G(t, s) c_1 \, ds = u(t)$ $(T(x)(t) \leqq u(t))$ $(a \leqq t \leqq b)$. Hence the assumption (3) of Theorem 1 is satisfied and in this case $[a, T(a)]$ is the order interval $[u, z]$. The other assumptions of Theorem 1 have been shown in the proof of Lemma 6. Thus Theorem 1 implies the statement of Theorem 7.

In the next theorem the requirement that the lower and the upper solutions should be conjugate is replaced by the condition that f satisfies a local Lipschitz condition in x and that there exists a non negative (non positive) Green function for the BVP (19),

$$L_0(x) + Kx = 0, \tag{41}$$

where $K > 0$ $(K < 0)$ is an arbitrary constant.

Theorem 8. *Let the assumptions* 1–4a *of Lemma* 6 *be satisfied. Let further the following assumptions hold*:

4b′. x_0 *and* y_0 *satisfy the boundary conditions* (39).

5. *For any* $K > 0$ $(K < 0)$ *the linear differential operator* $L_0(x) + Kx$ *is inverse monotone (inverse antimonotone) with respect to the boundary conditions* (19), *i.e. the problem* (41), (19) *has only the trivial solution and its corresponding Green function* G_1 *satisfies*

$$G_1(t, s) \geqq 0 \quad (G_1(t, s) \leqq 0) \quad in \quad [a, b] \times [a, b].$$

6. *f satisfies a Lipschitz condition in the second variable from R locally on the set* $[a, b] \times R$.

Then there are two sequences $\{x_m\}_{m=0}^{\infty}, \{y_m\}_{m=0}^{\infty}$ *of lower and upper solutions of* (13), *respectively, which satisfy*

$$x_0(t) \leqq x_1(t) \leqq \ldots \leqq x_m(t) \leqq \ldots \leqq y_m(t) \leqq \ldots \leqq y_1(t) \leqq y_0(t) \tag{42}$$

$$(x_0(t) \geqq x_1(t) \geqq \ldots \geqq x_m(t) \geqq \ldots \geqq y_m(t) \geqq \ldots \geqq y_1(t) \geqq y_0(t)) \, (a \leqq t \leqq b)$$

and which converge uniformly to a solution of (13), (39).

Proof. Consider the case $n - k$ is even. Then by the assumption of the theorem there exists a lower solution x_0 and an upper y_0 of (13) such that $x_0(t) \leqq y_0(t)$ for all $t \in [a, b]$. By the assumption 6 there is a $K > 0$ such that

$$|f(t, x) - f(t, y)| \leqq K|x - y|$$

for all $t \in [a, b]$, $x_0(t) \leqq x \leqq y \leqq y_0(t)$.

Let $\alpha_0(t) = L_0(x_0)(t) - f[t, x_0(t)]$, $\beta_0(t) = L_0(y_0)(t) - f[t, y_0(t)]$ $(a \leqq t \leqq b)$. By the assumption 4, $\alpha_0(t) \leqq 0$, $\beta_0(t) \geqq 0$ a.e. in $[a, b]$.

Consider the BVP (19),

$$L_0(x) + Kx = -\alpha_0(t) \tag{43}$$

and the BVP (19),

$$L_0(x) + Kx = -\beta_0(t)\,. \tag{44}$$

Denote the solution of the former (of the latter) BVP as z_0 (u_0). Since $z_0(t) = -\int_a^b G_1(t, s)\,\alpha_0(s)\,\mathrm{d}s$, $u_0(t) = -\int_a^b G_1(t, s).\ \beta_0(s)\,\mathrm{d}s$ $(a \leqq t \leqq b)$, we have

$$z_0(t) \geqq 0\,, \quad u_0(t) \leqq 0 \quad \text{in} \quad [a, b]\,. \tag{45}$$

Further,

$$L_0(x_0 + z_0)(t) - f[t, x_0(t) + z_0(t)] = L_0(x_0)(t) - f[t, x_0(t)] +$$
$$+ L_0(z_0)(t) + f[t, x_0(t)] - f[t, x_0(t) + z_0(t)] \leqq \alpha_0(t) - K\,z_0(t) -$$
$$- \alpha_0(t) + K\,z_0(t) = 0 \quad \text{a.e. in} \quad [a, b]$$

and similarly $L_0(y_0 + u_0)(t) \geqq \beta_0(t) - K\,u_0(t) - \beta_0(t) - K|u_0(t)| = 0$ a.e. in $[a, b]$.

Clearly $x_0 + z_0$, $(y_0 + u_0) \in L^{(n)}([a, b])$ and hence, $x_1 = x_0 + z_0$ is a lower solution of (13), while $y_1 = y_0 + u_0$ is an upper solution of that equation. (45) implies that

$$x_0(t) \leqq x_1(t)\,, \quad y_1(t) \leqq y_0(t) \quad \text{in} \quad [a, b]\,.$$

Further, we have

$$L_0(y_0 - x_0)(t) = \beta_0(t) - \alpha_0(t) - |f[t, y_0(t)] - f[t, x_0(t)]| \geqq$$
$$\geqq \beta_0(t) - \alpha_0(t) - K|y_0(t) - x_0(t)| \quad \text{a.e. in} \quad [a, b]\,.$$

Hence

$$L_0(y_0 - x_0)(t) + K[y_0(t) - x_0(t)] \geqq \beta_0(t) - \alpha_0(t)\,. \tag{46}$$

On the other hand

$$L_0(z_0 - u_0)(t) + K[z_0(t) - u_0(t)] = \beta_0(t) - \alpha_0(t) \quad (\text{a.e. in } [a, b]) \tag{47}$$

Both functions $y_0 - x_0$ and $z_0 - u_0$ fulfil (19) and therefore, (46), (47) give that they satisfy the inequality

$$(48)\quad y_0(t) - x_0(t) \geqq \int_a^b G_1(t, s)\,[\beta_0(s) - \alpha_0(s)]\,\mathrm{d}s = z_0(t) - u_0(t) \quad (t \in [a, b])\,,$$

which yields $y_1(t) \geqq x_1(t)$ in $[a, b]$.

We can continue the process by constructing a sequence of lower solutions $\{x_m\}_{m=0}^{\infty}$ and a sequence of upper solutions $\{y_m\}_{m=0}^{\infty}$ of (13) satisfying (42) and the boundary conditions (39). Denote $x(t) = \lim_{m\to\infty} x_m(t)$, $y(t) = \lim_{m\to\infty} y_m(t)$ $(a \leqq t \leqq b)$. In order to show that x and y satisfy (13), (39) we have to investigate x_m and y_m in more detail. Similarly as above, let $\alpha_m(t) = L_0(x_m)(t) - f[t, x_m(t)]$, $\beta_m(t) = L_0(y_m)(t) - f[t, y_m(t)]$ (a.e. in $[a, b]$), let z_m (u_m) be the solution of (43) ((44)) with α_m and β_m instead of α_0 and β_0, respectively, which satisfies (19). Then $x_{m+1} = x_m + z_m$ and $y_{m+1} = y_m + u_m$. Hence $L_0(x_{m+1})(t) = L_0(x_m)(t) + L_0(z_m)(t) = f[t, x_m(t)] - K\, z_m(t)$ and

$$(49)\quad x_{m+1}(t) = w(t) + \int_a^b G(t, s)\,[-K\, z_m(s) + f[s, x_m(s)]\,\mathrm{d}s \quad (a \leqq t \leqq b)\,.$$

As $z_m(t) = x_{m+1}(t) - x_m(t) \to 0$ for $m \to \infty$ $(a \leqq t \leqq b)$, by the Lebesgue theorem we get

$$x(t) = w(t) + \int_a^b G(t, s)\, f[s, x(s)]\,\mathrm{d}s \quad (a \leqq t \leqq b)\,.$$

Here w is the solution of (10), (39). Similarly, y satisfies the same integral equation and hence problem (13), (39). As the convergence of x_m to x as well as that of y_m to y is monotone and all the functions involved are continuous, by Dini's theorem the convergence is uniform on $[a, b]$. x and y are comparable solutions of the BVP (13), (39) and thus f.p. of an antitone operator. Lemma 1 implies that $x = y$.

In the case $n - k$ is odd we proceed similarly as above. Instead of (43) we consider the equation $L_0(x) - Kx = -\alpha_0(t)$ and (44) is replaced by $L_0(x) - Kx = -\beta_0(t)$. Then z_0, u_0 are provided with opposite signs as in (45). Instead of (48) we come to the inequality $x_0(t) - y_0(t) \geqq u_0(t) - z_0(t)$ and hence the inequalities (42) reverse their order. (49) changes its form into $x_{m+1}(t) = w(t) + \int_a^b G(t, s)\,[z_m(s) + f[s, x_m(s)]]\,\mathrm{d}s$. The final result is the same as above and the theorem is proved.

Remark. By similar considerations as those made in the proof of Lemma 5 which use (49) as their starting point we can show that the sequences $\{x_m^{(j)}\}$, $\{y_m^{(j)}\}$ converge uniformly to $x^{(j)}$ $(j = 1, \ldots, n - 1)$.

Under the assumption that all functions standing in the d. equation (13) are continuous the hypothesis on the local Lipschitz continuity of f can be omitted.

Theorem 9. *Suppose that the following assumptions are satisfied:*

1. *$n - k$ is an even (odd) number.*
2. *All coefficients p_j $(j = 1, \ldots, n)$ in (10) are continuous in $[a, b]$ and (10) is disconjugate in $[a, b]$.*

3. *f is continuous in $[a, b] \times R$ and $f(t, .)$ is nonincreasing (nondecreasing) in R for each fixed $t \in [a, b]$.*
4. *For any $K > 0$ ($K < 0$) the linear differential operator $L_0(x) + Kx$ is inverse monotone (inverse antimonotone) with respect to the boundary conditions (19). Then there exists at least one solution x of the BVP (13), (39).*

Proof. Since the proof in the case $n - k$ is odd does not differ substantially from that in the case $n - k$ is even, only the latter case will be investigated.

First we show that there exist a strict lower solution x_0 and a strict upper one y_0 of the equation (13), i.e. $x_0, y_0 \in C_n([a, b])$ and x_0 (y_0) satisfies the strict d. inequality $L_0(x_0)(t) < f[t, x_0(t)]$ ($L_0(y_0)(t) > f[t, y_0(t)]$) in $[a, b]$. Further, x_0 and y_0 possess the following properties:

a) $x_0(t) \leqq y_0(t)$ (in the case $n - k$ is odd $x_0(t) \geqq y_0(t)$ would hold) in $[a, b]$.

b) x_0 and y_0 satisfy the boundary conditions (39).

Let $k > 0$ and let x_0 be the solution of the BVP $L_0(x) = -k$, (39) and y_0 the solution of $L_0(x) = k$, (39). If w is the solution of (10), (39) and G is the Green function of (10), (19), then $x_0(t) = w(t) - \int_a^b G(t, s)\, k\, ds \leqq w(t) \leqq w(t) + \int_a^b G(t, s)\, k\, ds = y_0(t)$. Further, $L_0(x_0)(t) = -k < f[t, w(t)] < k = L_0(y_0)(t)$ ($a \leqq t \leqq b$) for a sufficiently great k. This implies that for such a k $L_0(x_0)(t) < f[t, x_0(t)]$ and $L_0(y_0)(t) > f[t, y_0(t)]$ in $[a, b]$.

Denote $W = \{(t, x) \in R^2 : x_0(t) \leqq x \leqq y_0(t),\ a \leqq t \leqq b\}$. Let A be the set of all functions $g \in C(W)$ such that $g(t, .)$ is nonincreasing in $[x_0(t), y_0(t)]$ for each $t \in [a, b]$ and g satisfies a Lipschitz condition in the second variable on W. $A \neq \emptyset$ since for $g_1(t, x) = -x$ ($(t, x) \in W$), $g_1 \in A$. We shall show that A is a lattice of continuous functions on W with the property:

(α) For every pair $(t_1, x_1), (t_2, x_2)$ of distinct points of W there exists a function $g \in A$ such that $g(t_i, x_i) = f(t_i, x_i)$ ($i = 1, 2$).

Then by the Stone theorem ([9, p. 184]) there exists a sequence $\{g_m\}$ of functions $g_m \in A$ which uniformly converges to f on W.

First of all, similarly as in remark b) in [9, p. 183], we get that the maximum and the minimum of two continuous functions satisfying a Lipschitz condition in the second variable on W also enjoy this property. If both functions are nonincreasing in the second variable, then the same property is shared by their maximum and minimum. For, if $g_1, g_2 \in A$, $(t, x_i) \in W$ ($i = 1, 2$), $x_1 < x_2$ and $g_i(t, x_k) = g_{ik}$ ($i, k = 1, 2$), then $g_{i1} \geqq g_{i2}$ ($i = 1, 2$). Suppose that $g_{11} \leqq g_{21}$, $g_{12} \geqq g_{22}$. Then $\min(g_{12}, g_{22}) = g_{22} \leqq g_{12} \leqq g_{11} = \min(g_{11}, g_{21})$ and $\max(g_{12}, g_{22}) = g_{12} \leqq g_{11} \leqq g_{21} = \max(g_{11}, g_{21})$. The same result can be obtained in the other cases. Thus A is a lattice of continuous functions on W. Now to prove the property (α) of A, consider two arbitrary distinct points $(t_1, x_1), (t_2, x_2)$ of W. Two cases may occur. If $t_1 = t_2$, then by the linear interpolation and the constant extrapolation we obtain a function g_0 which is Lipschitz continuous in R, nonincreasing and such that $g_0(x_i) = f(t_1, x_i)$ ($i = 1, 2$). The function g defined in W by $g(t, x) = g_0(x)$

is the searched function in A mentioned in (α). If $t_1 \neq t_2$, then we define g_0 at straight lines $t = t_1$, $t = t_2$ $(x \in R)$ by $g_0(t_1, x) = f(t_1, x_1)$, $g_0(t_2, x) = f(t_2, x_2)$ and then by the linear interpolation and the constant extrapolation we extend g_0 to R^2. $g_0|_W$ is again the searched function in A. Thus the existence of a sequence $\{g_m\} \subset A$ which uniformly converges to f on W is proved.

The next step in the proof will consist in showing that x_0 is a lower solution of almost all equations $L_0(x) = g_m(t, x)$. In fact, $L_0(x_0)(t) < f[t, x_0(t)]$ in $[a, b]$. Since both sides of this inequality are continuous in $[a, b]$, there is an $\varepsilon > 0$ such that $L_0(x_0)(t) < f[t, x_0(t)] - \varepsilon$. But $\{g_m[\cdot, x_0(\cdot)]\}$ uniformly converges to $f[\cdot, x_0(\cdot)]$ in $[a, b]$ and hence there is an m_1 such that $L_0(x_0)(t) < f[t, x_0(t)] - \varepsilon < g_m[t, x_0(t)]$ $(a \leqq t \leqq b)$ for all $m \geqq m_1$. Similarly there exists an m_2 with the property: $m \geqq m_2$ implies that $L_0(y_0)(t) > g_m[t, y_0(t)]$ $(a \leqq t \leqq b)$. As g_m can be extended from W to the whole strip $[a, b] \times R$ preserving the properties shared by the functions from A, Theorem 8 can be applied to all BVP-s $L_0(x) = g_m(t, x)$ $(m \geqq m_3 = \max(m_1, m_2))$, (39). By this theorem there exists at least one solution to each of the mentioned BVP-s. Choose one of them (if necessary by the help of the axiom of choice) and denote it by x_m. x_m satisfies the integral equation

$$(50) \qquad x_m(t) = w(t) + \int_a^b G(t, s)\, g_m[s, x_m(s)]\, \mathrm{d}s \quad (a \leqq t \leqq b,\ m \geqq m_3)$$

from which we get

$$x_m^{(i)}(t) = w^{(i)}(t) + \int_a^b \frac{\partial^i G(t, s)}{\partial t^i}\, g_m[s, x_m(s)]\, \mathrm{d}s$$

$$(a \leqq t \leqq b,\ i = 1, \ldots, n-1, \quad m \geqq m_3).$$

Since

$$(51) \qquad x_0(t) \leqq x_m(t) \leqq y_0(t) \quad (a \leqq t \leqq b,\ m \geqq m_3)$$

and g_m is uniformly bounded in W, all sequences $\{x_m^{(i)}\}$ $(i = 0, 1, \ldots, n-1)$ are uniformly bounded in $[a, b]$. Using the d. equation $L_0(x_m) = g(t, x_m)$ we get that $\{x_m^{(n)}\}$ is uniformly bounded, too. By the Ascoli lemma there is a subsequence $\{x_{m_k}\}$ and a function x such that $\{x_{m_k}^{(i)}\}$ $(i = 0, 1, \ldots, n-1)$ are uniformly convergent on $[a, b]$ to $x^{(i)}$. The limiting process in (50) gives that x satisfies $x(t) = w(t) + \int_a^b G(t, s)\,.$ $.\, f[s, x(s)]\, \mathrm{d}s$ $(a \leqq t \leqq b)$ and thus x satisfies the BVP (13), (39).

Remarks. 1. By (51) we have got that the obtained solution x satisfies the inequalities $x_0(t) \leqq x(t) \leqq y_0(t)$ $(a \leqq t \leqq b)$.

2. Lemma 6 and Theorems 7–9 have guaranteed the existence of a solution to the BVP (13), (39). Looking through their proof we see that they also hold in the general case of the boundary conditions (14) when the linear d. operator L_0 is inverse monotone (inverse antimonotone) with respect to (9). In this case the solution v_y of (10) associated with the function y with respect to (10), (14) is determined by the boundary conditions $B_i(x) = B_i(y)$ $(i = 1, 2, \ldots, n)$. Lemma 6 has the following form.

Lemma 6'. *Let the following assumptions be satisfied:*

1. *The differential operator L_0 is inverse monotone (inverse antimonotone) with respect to the boundary conditions* (9).
2. *$f(t, .)$ is nonincreasing (nondecreasing) in R for each fixed $t \in [a, b]$.*
3. *There exist a lower and an upper solution x_0, y_0, respectively, of the equation* (13) *with the properties:*
   a) $x_0(t) \leqq y_0(t)$ $(x_0(t) \geqq y_0(t)$ *for all* $t \in [a, b])$.
   b) $v_{x_0}(t) \leqq w(t) \leqq v_{y_0}(t)$ $(v_{x_0}(t) \geqq w(t) \geqq v_{y_0}(t))$ *for each $t \in [a, b]$ where w is the solution of* (10), (14).
   c) *x_0, y_0 are conjugate to each other.*

Then there exists at least one solution x of the problem (13), (14) *which satisfies*

$$x_0(t) \leqq x(t) \leqq y_0(t) \quad (x_0(t) \geqq x(t) \geqq y_0(t)) \quad (a \leqq t \leqq b)\,.$$

Any two solutions of this problem are incomparable.

Theorems 7–9 can be generalized in the same way. By a modification of Theorem 7 we can prove the following theorem.

Theorem 10. *Let $g : R^2 \to R$ be continuous in (t, x), periodic in t with a period $T > 0$ and let there exist two numbers $0 < a \leqq \pi/T$ and c_1 such that $a^2x + g(t, x)$ is nonincreasing in $x \in R$ for each fixed $t \in R$ and either $a^2x + g(t, x) \geqq c_1$ $((t, x) \in R^2)$ or $a^2x + g(t, x) \leqq c_1$ $((t, x) \in R^2)$. Then the differential equation*

$$x'' = g(t, x)$$

has a periodic solution of the period T.

Proof. The searched periodic solution can be obtained as the solution of the BVP

$$x'' + a^2x = a^2x + g(t, x)\,,$$

$$x(0) = x(T)\,, \quad x'(0) = x'(T)\,.$$

Since the Green function of the corresponding homogeneous BVP is of the form

$$G(t, s) = \frac{1}{2a(1 - \cos aT)} \{\sin [a(T - t + s)] + \sin [a(t - s)]\} =$$

$$= \frac{1}{a(1 - \cos aT)} \sin \frac{aT}{2} \cos \left[\frac{aT}{2} + a(s - t)\right] \quad (0 \leqq s \leqq t \leqq T)$$

and $G(t, s) = G(s, t)$ for $0 \leqq t \leqq s \leqq T$ ([11, p. 168]), we have that $G(t, s) \geqq 0$ in $[a, b] \times [a, b]$ for $0 < \pi/T$. A modification of Theorem 7 gives the result.

Consider now the d. equation

$$x''' = g(t, x) \tag{52}$$

where g is continuous in R^2, periodic in t with a period $T > 0$. The existence of a T-periodic solution to this equation can be shown thanks to Theorem 2 ([6, p. 641]) which asserts that the operator $L_0(x) = x''' + a(t)\,x$ $(x \in C_3([0, T]))$ is inverse monotone (inverse antimonotone) with respect to the boundary conditions

$$x^{(i)}(0) = x^{(i)}(T) \quad (i = 0, 1, 2) \tag{53}$$

when $a \in C([0, T])$, $a(t) \not\equiv 0$ in $[0, T]$ and $a(t) \geqq 0$ $(a(t) \leqq 0)$ in that interval.

Theorem 11. *Let $g : R^2 \to R$ be continuous in (t, x), periodic in t with period $T > 0$ and let there exist a function $a \in C([0, T])$ such that $a(t) \not\equiv 0$, $a(t) \geqq 0$ $(a(t) \leqq 0)$ in $[0, T]$ and $a(t).\ x + g(t, x)$ is nonincreasing (nondecreasing) in $x \in R$ for each $t \in [0, T]$. Then there exists a T-periodic solution of* (52).

Proof. With the help of a modification of Theorem 9 the existence of a solution to the BVP $x''' + a(t)\,x = a(t)\,x + g(t, x)$, (53), can be proved, the periodic extension of which is the mentioned solution.

Remark. Theorem stregthens the existence statement of Theorem 5 in [6, p. 642].

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Author's address: 816 31 Bratislava, Mlynská dolina, ČSSR (Matematicko-fyzikálna fakulta Univerzity Komenskeho).