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DIMENSION OF THE SUM OF TWO COPIES OF A GRAPH

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The dimension of a graph G (see [2], [3], [4], [5]) is the minimum number of complete graphs the product of which contains G as a spanned subgraph. This paper describes some methods how to estimate the dimension of the sum of two copies of a graph. Some theorems also hold for more than two copies but this will be the subject of another paper as there are some other problems connected. There are two basic estimation methods: one involving the maximum degree; the other involving the chromatic number. These estimations are of interest because recently it has been proved that solving the question of the dimension in general is an NP-complete problem [3], [1].

1. PRELIMINARIES

1.1. Conventions and notation. The word *graph* is used for a symmetric graph without loops. If G is a graph, we will denote by $V(G)$ and $E(G)$ the sets of vertices and edges of G, respectively. For convenience, the symbol $E(G)$ will be used also for the associated binary relation on $V(G)$. For $V(G)$ we will usually take some suitable set of natural numbers.

A *homomorphism* $f : G \to H$ is a mapping $f : V(G) \to V(H)$ such that $\{f(x), f(y)\} \in E(H)$ whenever $\{x, y\} \in E(G)$.

A *spanned subgraph* of G is a graph H with $V(H) \subset V(G)$ and $E(H) = E(G) \cap (V(H) \times V(H))$.

The degree of a vertex x in G will be denoted by $d_G(x)$. Further, we put $\Delta(G) = \sup \{d_G(x) : x \in V(G)\}$.

The *cardinality* of G, denoted $|G|$, is understood to be the cardinality of $V(G)$.

The complete graph with n vertices is denoted by K_n.

Let X be a set. Then we use the notation $P(X) = \{Y : Y \subset X\}$ and $P_{\leqq d}(X) = \{Y \subset X : |Y| \leqq d\}$.

Let $\Gamma : X \to P(X)$. Then $\bar{\Gamma} : X \to P(X)$ is defined as follows:

$$y \in \bar{\Gamma}(x) \Leftrightarrow x \in \Gamma(y) .$$

Obviously, Γ corresponds to a graph iff $\Gamma = \bar{\Gamma}$ and $x \notin \Gamma(x)$. Such Γ will be sometimes denoted by Γ_G, where G is the associated graph, and will be called a *graph-mapping*.

The symbol $I(N \times d)$ denotes a matrix $N \times d$, the i-th row of which is the vector $ii \ldots i$. This vector will be sometimes denoted by $I_i(d)$.

1.2. Some constructions. Let G, H be graphs. We assume them disjoint; if they are not, we make them so formally e.g. by replacing the vertices x of the first one by $(x, 0)$ and the vertices y of the second one by $(y, 1)$. The graph $G + H$ is defined by

$$V(G + H) = V(G) \cup V(H), \quad E(G + H) = E(G) \cup E(H)$$

and is referred to as the *sum* of G and H.

Graph $G \square H$ is defined by

$$V(G \square H) = \{(i, j) : i \in V(G),\ j \in V(H)\},$$

$$E(G \square H) = \{\{(i_1, j_1), (i_2, j_2)\} : \text{either } i_1 = i_2 \text{ and } \{j_1, j_2\} \in E(H), \text{ or } \{i_1, i_2\} \in E(G) \text{ and } j_1 = j_2\}$$

and is referred to as the *cartesian product* of G and H.

Let G_i, $i \in J$, be graphs. The (categorial) *product* of this system, denoted by $\bigtimes_{i \in J} G_i$, is defined by

$$V(\bigtimes G_i) = \bigtimes V(G_i),$$

$$E(\bigtimes G_i) = \{\{(x_i)_J, (y_i)_J\} : \forall i (x_i, y_i\} \in E(G_i)\}.$$

The homomorphisms $p_j : \bigtimes_J G_i \to G_j$ sending $(x_i)_{i \in J}$ to x_j will be sometimes called *projections*. If $J = \{1, 2\}$ we denote $\bigtimes_J G_i$, as usual, by $G_1 \times G_2$. If $G_i = G$ for all i, the symbol G^n is used.

1.3. Encodings. The dimension of a finite graph is the least natural n such that G can be embedded into $\mathbb{N}^n$ (where $\mathbb{N}^n$ is the n-th categorial power of the complete graph whose vertices are all the natural numbers, i.e. $V(\mathbb{N}^n)$ is the n-th power of the set of all natural numbers and $E(\mathbb{N}^n) = \{\{(x_i)_{i \in J}, (y_i)_{i \in J}\} : \forall_i\ x_i \neq y_i\}$, J being the set $1, \ldots, n$).

Thus, the inequality $\dim G \leqq n$ can be proved by associating the vertices $x \in V(G)$ with distinct vectors $v(x) = (v_1(x), \ldots, v_n(x))$ in natural numbers so that for $\{x, y\} \in E(G)$ the vectors $v(x)$ and $v(y)$ differ in all the coordinates (we say that the vectors $v(x)$ and $v(y)$ do not meet), and for $\{x, y\} \notin E(G)$ they agree in at least one coordinate (the vectors $v(x)$ and $v(y)$ meet). The vectors will be written simply as words in the coordinates (i.e., e.g. 0102 stands for (0, 1, 0, 2)). A particular choice of the vectors above will be referred to as an *encoding*.

1.4. We shall need the following easy proposition, which is taken from [2].

Proposition. *For* $n \geqq 2$, $\dim (K_n + K_1) = n$. □

2. THE BASIC FACTS

2.1. Theorem. *Let G, H be graphs, $\varphi : G \to H$, φ a homomorphism.*

a) *Then* $\dim(G + G) \leqq \dim G + \dim(H + H)$.

b) *Let there exist an encoding of the sum $H + H$ in m coordinates* ($m \geqq \dim(H + H)$) *such that the first* $\dim H$ *coordinates constitute two identical encodings of H. Then*

$$\dim(G + G) \leqq \dim G + m - \dim H .$$

Proof. Let $v(t)$, $i \in V(H + H)$ be the encoding of $H + H$, $u(j)$, $j \in V(G)$ the encoding of G and $v'(i)$, $i \in V(H + H)$ the encoding of $H + H$ after removing the first $\dim H$ coordinates (the second proposition of the theorem). Define $\varphi + \varphi : G + + G \to H + H$ in the obvious way. For $x \in V(G + G)$ we denote $y = (\varphi + \varphi)(x)$ and define $z(x) = u(x)\, v(y)$ (resp. $z(x) = u(x)\, v'(y)$). We shall prove that $z(x)$ constitute an encoding of $G + G$.

Let x_1 and x_2 be connected (i.e. $u(x_1)$ and $u(x_2)$ do not meet). Then y_1 and y_2 are also connected and thus $v(y_1)$ and $v(y_2)$ ($v'(y_1)$ and $v'(y_2)$) do not meet. Hence $z(x_1)$ and $z(x_2)$ cannot meet.

Let x_1 and x_2 be not connected. Then one of the following three cases occurs:

a) x_1, x_2 are in one copy of G: then $u(x_1)$ and $u(x_2)$ do meet.

b) x_1, x_2 are in distinct copies and the corresponding vertices in one copy are not connected: then $u(x_1)$ and $u(x_2)$ obviously meet again.

c) x_1, x_2 are in distinct copies and the corresponding vertices in one copy are connected. Then also y_1 and y_2 are in distinct copies of H and the corresponding vertices in one copy of H are connected, thus $v(y_1)$ and $v(y_2)$ meet in the part v'. □

2.2. Lemma. *Let $\Gamma : X \to P_{\leqq d}(X)$ be such that $\bar{\Gamma} : X \to P_{\leqq d}(X)$ again. Then there exist d permutations $p_1, \ldots, p_d$ of the set X such that*

$$\Gamma(x) \subset \{p_1(x), \ldots, p_d(x)\}$$

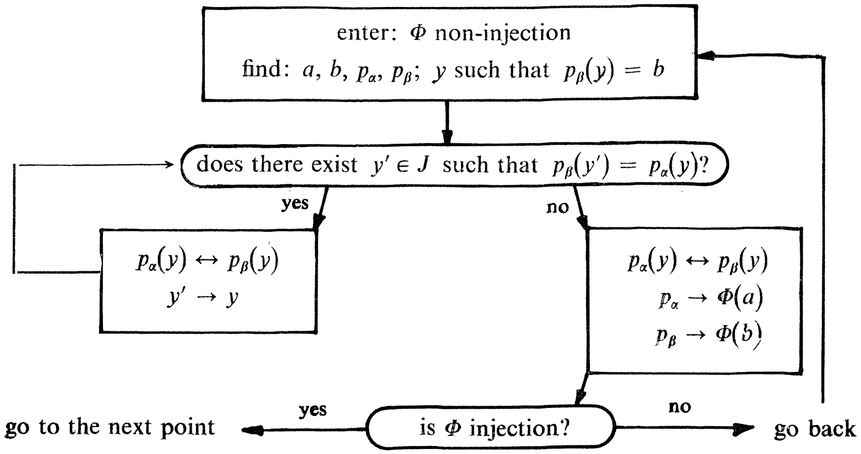
for all $x \in X$. The matrix $|X| \times d$ with columns $p_1, \ldots, p_d$ will be called the Generalized Latin Rectangle corresponding to Γ (further GLR Γ*).*

Proof. Let us define arbitrarily $P_{t_1}(1) = \{p_1(1), \ldots, p_{t_1}(1)\} = \Gamma(1)$, where $t_1 = = |\Gamma(1)|$. Let $P_{t_i}(i)$ be defined for $i \in \{1, \ldots, j - 1\} = J$ in such a way that $p_\alpha(x) \neq \neq p_\alpha(y)$ whenever both the expressions are defined, $x \neq y$ and $x, y \leqq j - 1$.

For all $x \in \Gamma(j)$ there exists an $\alpha \leqq d$ such that $x \notin p_\alpha(J) = \{p_\alpha(1), \ldots, p_\alpha(j - 1)\}$ (otherwise $|\Gamma^{-1}(x)| > d - a$ contradiction). Now, if there exists an injection $\Phi : \Gamma(j) \to P = \{p_1, \ldots, p_d\}$ such that $\Phi(x) = p_\alpha \Rightarrow x \notin p_\alpha(J)$, it suffices to define Pt_j as the image of $\Gamma(j)$, $p_\alpha(j) = x = \Phi^{-1}(p_\alpha)$.

If such an injection does not exist, we take any Φ with $\Phi(x) = p_\alpha \Rightarrow x \notin p_\alpha(J)$.

Then there exist $a, b \in \Gamma(j)$ and $\alpha, \beta \leqq d$ such that $\Phi(a) = \Phi(b) = p_\alpha$ and $\Phi^{-1}(p_\beta) = 0$ (as for all $x \in V(G)$ the inequality $|\Gamma(x)| \leqq |P|$ holds), with $a, b \in p_\beta(J)$. Thus there exists $y \in J$ such that $p_\beta(y) = b$. The continuation is clear from the following diagram:



Throughout this diagram p_α is always an injection and at last in the $(j - 1)$-th step we find that y' does not exist, thus $p_\beta(J)$ is an injection (after commutation) and does not contain b.

This procedure applied to the vertices $2, \ldots, |X|$ will generate partial mappings (injections) $p_\alpha : X \to X$ and it suffices to complete them (in any way) to permutations. □

2.3. Theorem. $\dim (G + G) \leqq \dim G + \Delta(G)$.

Proof. Let us take GLR Γ_G and let its rows be $u(i)$, $i \in V(G)$ $(\Gamma(i) \subset u(i))$ and the respective encoding $v(i)$, $i \in V(G)$. Putting

$$w(i, 0) = v(i)\, I_i(\Delta(G)) \quad \text{and} \quad w(i, 1) = v(i)\, u(i)$$

we have an encoding of the sum $G + G$:

The vectors $w(i, j)$ for fixed j form an encoding of G. Let $w(a, 0)$, $w(b, 1)$ be an arbitrary pair. If $\{a, b\} \notin E(G)$, the vectors meet in the part v, if $\{a, b\} \in E(G)$, then $a \in \Gamma(b)$ and, due to $\Gamma(b) \subset u(b)$, the vectors meet in the part u. □

2.4. Corollary. $\dim (K_n + K_n) = n$. □

2.5. Theorem. *Let G be a λ-chromatic graph. Then*

$$\dim (G + G) \leqq \dim G + \lambda - 1 .$$

Proof. G is λ-chromatic means that there exists a homomorphism $\varphi : G \to K_\lambda$. According to 2.3 (and 2.4) there exists such an encoding of $K_\lambda + K_\lambda$ that its length is λ and the first coordinates constitute two identical encodings of K_λ. From 2.1 b) we immediately obtain the theorem. □

2.6. Corollary. *Let there exists a homomorphism $\varphi : G \to H$, let λ_H be the chromatic number of H. Then*

$$\dim (G + G) \leqq \dim G + \min (\Delta(H), \lambda_H - 1) .$$

Proof. According to 2.3 and 2.5 we have an encoding of $H + H$ of length $\dim H + \Delta(H)$ (resp. $\dim H + \lambda_H - 1$) such that the first $\dim H$ coordinates constitute identical encodings of H. By 2.1 b) we get the rest. □

3. SOME FURTHER APPLICATIONS

3.1. Theorem. (about the dimension of categorial products):

Let $G_1, \ldots, G_n$ be graphs and $\dim (G_i + G_i) = h_i$. *Then*:

a) $$\dim \left(\mathop{\times}_{i=1}^{n} G_i + \mathop{\times}_{i=1}^{n} G_i \right) \leqq \dim (\times G_i) + \min h_i .$$

b) *If we put* $p_i = \min (\Delta(G_i), \lambda_i - 1)$, *where λ_i is the chromatic number of G_i, then*

$$\dim (\times G_i + \times G_i) \leqq \dim (\times G_i) + \min p_i .$$

Proof. Recalling that the projections $p_j : \times G_i \to G_j$ are homomorphisms, we see that the proposition a) follows from 2.1 and b) from 2.1 and 2.6. □

3.2. For comparison, we present here an analogous theorem for cartesian products:

Theorem. *Let $G_1, \ldots, G_n$ be graphs and* $\dim (G_i + G_i) = h_i$. *Then*

a) $$\dim \left(\prod_{i=1}^{n} G_i + \prod_{i=1}^{n} G_i \right) \leqq \dim (\prod G_i) + \sum_{i=1}^{n} h_i .$$

b) *If we put* $p_i = \min (\Delta(G_i), \lambda_i - 1)$, *where λ_i is the chromatic number of G_i, then*

$$\dim (\prod G_i + \prod G_i) \leqq \dim (\prod G_i) + \sum_{i=1}^{n} p_i .$$

Proof. For the sake of simplicity we shall prove the case $n = 2$. The proof for $n > 2$ follows easily by induction.

Let $u_{i0}(j), u_{i1}(j), j \in V(G_i)$ be the corresponding encoding of $G_i + G_i$. As each vector u_{i0} meets each vector u_{i1} (for fixed i), we can assume that either the set of the symbols in the j-th coordinate of the vectors u_{i0} is included in the set of the symbols in the same coordinate of the vectors u_{i1} or vice versa, and both of them are subsets of $V(G_i)$.

Let us define vectors $u_{i0}(k, m)$, $k \in V(G_1)$, $m \in V(G_2)$ as follows: if $u_{10}(k) = a_1 \dots a_{h_1}$, then $u_{10}(k, m) = \bar{a}_1 \dots \bar{a}_{h_1}$, where $\bar{a}_j = a_j + (m - 1) . |G_1|$, if $u_{20}(m) = b_1 \dots b_{h_2}$, then $u_{20}(k, m) = \bar{b}_1 \dots \bar{b}_{h_2}$, where $\bar{b}_j = b_j + (k - 1) . |G_2|$.

The vectors $u_{i1}(k, m)$ are defined in the same way (again using the vectors $u_{i1}(k)$, resp. $u_{i1}(m)$). For m fixed, the vectors $u_{10}(k, m)$ and $u_{11}(k, m)$ constitute an encoding of $G_1 + G_1$ and for k fixed, the vectors $u_{20}(k, m)$ and $u_{21}(k, m)$ constitute an encoding of $G_2 + G_2$. Now, let $v(k, m)$ be the corresponding encoding of $G_1 \square G_2$ and let us put

$$w(k, m, i) = v(k, m)\, u_{1i}(k, m)\, u_{2i}(k, m)\,, \quad i = 0, 1\,.$$

For i fixed, the vectors $w(k, m, i)$ form an encoding of $G_1 \square G_2$: Let vertices (k_1, m_1) and (k_2, m_2) be not connected; then the vectors w meet in the part v. Let the vertices be connected; then either $k_1 = k_2$ and m_1 and m_2 are connected in G_2 and they meet neither in the part u_2, nor in the part u_1 (since the elements of the vector $u_{i1}(k_1, m_1)$ are between $(m_1 - 1) . |G_1|$ and $m_1 . |G_1|$ but the elements of $u_{i1}(k_1, m_2)$ are between $(m_2 - 1) . |G_1|$ and $m_2 . |G_1|$), or k_1 is connected with k_2 in G_1 and $m_1 = m_2$; for analogous reasons the vectors w meet neither in the part u_1, nor in the part u_2.

Now, let us take $w(k_1, m_1, 0)$ and $w(k_2, m_2. 1)$ arbitrarily. Either the vertices (k_1, m_1) and (k_2, m_2) are not connected and then the vectors w meet in the part v, or $k_1 = k_2$ and m_1 and m_2 are connected in G_2 and then the vectors w meet in the part u_2, or finally $m_1 = m_2$ and k_1 and k_2 are connected in G_1 and then the vectors w meet in the part u_1. Thus the vectors w really do constitute an encoding of $G_1 \square G_2 + G_1 \square G_2$.

To prove the second proposition of the theorem we shall prove the following proposition: If there exists an encoding of $G_i + G_i$ of length h_i such that the first k_i coordinates constitute encodings of G_i, then

$$\dim (G_1 \square G_2 + G_1 \square G_2) \leqq \dim (G_1 \square G_2) + \sum_{i=1}^{n} (h_i - k_i)\,.$$

To prove this proposition it suffices to take vectors $\bar{u}_i$ obtained from u_i by removing the first k_i coordinates and to make the same operations with these vectors as we did in the first part of this proof with u_i. By 2.4 and 2.5 we immediately get the second proposition of the theorem. □

3.3. Corollary. *Let there exist a homomorphism* $\varphi : G \to \prod_{i=1}^{n} H_i$, *let* $\dim (H_i + H_i) = h_i$, *then*

$$\dim (G + G) \leqq \dim G + \sum_{i=1}^{n} h_i\,.$$

If we put $p_i = \min(\Delta(H_i), \lambda_i - 1)$, *then*

$$\dim(G + G) \leqq \dim G + \sum_{i=1}^{n} p_i . \quad \square$$

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