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HYPERGRAPHS AND INTERVALS

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1. In the present paper by a hypergraph (or a graph) we mean an ordered pair $(V, \mathscr{E})$, where V is a finite nonempty set and $\mathscr{E}$ is a set of subsets of V with the property that if $E \in \mathscr{E}$, then $|E| \geqq 1$ (or $|E| = 2$, respectively); elements of V or $\mathscr{E}$ are called *vertices* or *edges*, respectively. If $H = (V, \mathscr{E})$ is a hypergraph, then we shall write $V(H) = V$ and $\mathscr{E}(H) = \mathscr{E}$. Obviously, every graph is a hypergraph. Our concept of a graph is the same as that in [1] or [6], and our concept of a hypergraph is relative to the concept of a simple hypergraph in [2]. (Graph theoretical terms which we shall use without definitions can be found in [1] or [6]).

If $\mathscr{E}_0$ is a finite nonempty set of a finite nonempty sets, then we denote by $\langle \mathscr{E}_0 \rangle$ the hypergraph H_0 with

$$V(H_0) = \bigcup_{E_0 \in \mathscr{E}_0} E_0$$

and $\mathscr{E}(H_0) = \mathscr{E}_0$.

In the present paper the letters $g, h, i, j, k, m,$ and n mean integers. We shall say that a sequence $(v_1, \ldots, v_n)$, $n \geqq 1$, is an arrangement, if for any g and h, $1 \leqq g < h \leqq n$, we have that $v_g \neq v_h$. An arrangement $(v_1, \ldots, v_n)$ is referred to as an arrangement on a set V if $V = \{v_1, \ldots, v_n\}$. Let $\alpha = (v_1, \ldots, v_n)$ be an arrangement on a set V; we say that $X \subseteq V$ is an interval set in α if there exist i and k, $1 \leqq i \leqq k \leqq n$, such that $X = \{v_j;\ i \leqq j \leqq k\}$; the set of all interval sets in α will be denoted by $\operatorname{Int}(\alpha)$.

Let V be a finite nonempty set, and let A be a nonempty set of arrangements on V; then we denote

$$\operatorname{Int}(A) = \bigcap_{\alpha \in A} \operatorname{Int}(\alpha)\,.$$

Let H be a hypergraph. Following [9] we shall say that an arrangement π on $V(H)$ is a projectoidic arrangement on H if $\mathscr{E}(H) \subseteq \operatorname{Int}(\pi)$. We shall say that H is a *projectoid* if there exists a projectoidic arrangement on H. Note that the terms "projectoidic" and "projectoid" have a relation to the term "projective" in the sense in which it is used in mathematical linguistics (see, for example, [8]).

Projectoids were investigated in [3], [4], [9], [10] and [11], but in [4] and [11] projectoids were studied by means of the matrix theory (as a class of (0, 1)-matrices), and in [11] also as bipartite graphs. Various characterizations of projectoids are known. In this section the characterization given by the present author in [9] will be recorded.

Let H be a hypergraph. Following [9] we denote by $\Omega(H)$ the set of all sets $R \subseteq V(H)$ such that at least one of the following conditions holds:

(1) there exists $v \in V(H)$ with the property that $R = \{v\}$;
(2) $R \in \mathscr{E}(H)$;
(3) there exist $R_1, R_2 \in \Omega(H)$ such that $R_1 \cap R_2 \neq \emptyset$ and $R = R_1 \cup R_2$.

The following theorem was proved in [9]:

Theorem 1. *A hypergraph H is a projectoid if and only if for every three $R_1, R_2, R_3 \in \Omega(H)$, the hypergraph $\langle\{R_1, R_2, R_3\}\rangle$ is a projectoid.*

2. Let H be a hypergraph. We denote by $\Pi(H)$ the set of projectoidic arrangements on H. This means that H is a projectoid if and only if $\Pi(H) \neq \emptyset$.

Let H be a projectoid. According to the definition, $\mathscr{E}(H) \subseteq \operatorname{Int}(\Pi(H))$. In this section a characterization of the set $\operatorname{Int}(\Pi(H))$ will be given.

Let H be a hypergraph. Consider arbitrary $X, Y \subseteq V(H)$; we shall write $X \sim Y$ if at least one of the sets $X \cap Y$, $X - Y$ and $Y - X$ is empty; on the other hand, we shall write $X \nsim Y$ if the sets $X \cap Y$, $X - Y$ and $Y - X$ are nonempty. We denote by $\Sigma(H)$ the set of all sets $S \subseteq V(H)$ such that at least one of the following conditions holds:

(0) $S = V(H)$;
(1) there exists $v \in V(H)$ with the property that $S = \{v\}$;
(2) $S \in \mathscr{E}(H)$;
(3) there exist $S_1, S_2 \in \Sigma(H)$ such that $S_1 \cap S_2 \neq \emptyset$ and $S \in \{S_1 \cap S_2, S_1 \cup S_2\}$;
(4) there exist $S', S'' \in \Sigma(H)$ such that $S' \nsim S''$ and $S = S' - S''$.

Obviously, $\Omega(H) \subseteq \Sigma(H)$. We shall prove that if $\Pi(H) \neq \emptyset$, then $\Sigma(H) = \operatorname{Int}(\Pi(H))$.

Let H be a hypergraph, and let $U \subseteq V(H)$. We say that U is free in H if for every $E \in \mathscr{E}(H)$, $E \sim U$.

Lemma 1. *Let H be a projectoid, $|V(H)| = n \geqq 2$, let $(v_1, \ldots, v_n)$ be a projectoidic arrangement of H, and let $1 \leqq j < n$. Then there exist i and k, $1 \leqq i \leqq j < k \leqq n$, such that $\{v_i, \ldots, v_j\}, \{v_{j+1}, \ldots, v_k\} \in \Sigma(H)$, and $\{v_i, \ldots, v_k\}$ is free in H.*

Proof. Denote

$$B = \{i_0;\ 1 \leqq i_0 \leqq j,\ \{v_{i_0}, \ldots, v_j\} \in \Sigma(H)\} \quad \text{and}$$
$$B' = \{k_0;\ j + 1 \leqq k_0 \leqq n,\ \{v_{j+1}, \ldots, v_{k_0}\} \in \Sigma(H)\}\,.$$

Since $\{v_j\}, \{v_{j+1}\} \in \Sigma(H)$, we have that $B \neq \emptyset \neq B'$. We denote by i and k the minimum integer in B and the maximum integer in B', respectively. We wish to show that $\{v_i, \ldots, v_k\}$ is free in H.

Assume, on the contrary, that $\{v_i, \ldots, v_k\}$ is not free in H. Then there exists $E \in \mathscr{E}(H)$ such that $E \nsim \{v_i, \ldots, v_k\}$. Since $(v_1, \ldots, v_n)$ is a projectoidic arrangement on H, there exist f and g, $1 \leqq f < g \leqq n$, such that $E = \{v_f, \ldots, v_g\}$. Since $\{v_f, \ldots, v_g\} \nsim \{v_i, \ldots, v_k\}$, we have that either $f < i \leqq g < k$ or $i < f \leqq k < g$. We shall assume that $f < i \leqq g < k$ (the latter case would be proved analogously).

First, let $g \leqq j$; then $E \cap \{v_i, \ldots, v_j\} \neq \emptyset$; we have that $\{v_f, \ldots, v_j\} = E \cup \{v_i, \ldots, v_j\}$. Next, let $g > j$; then $E \nsim \{v_{j+1}, \ldots, v_h\}$; we have that $\{v_f, \ldots, v_j\} = E - \{v_{j+1}, \ldots, v_h\}$. In both cases $\{v_f, \ldots, v_j\} \in \Sigma(H)$ and thus $f = i$, which is a contradiction.

Thus $\{v_i, \ldots, v_k\}$ is free in H, which completes the proof.

Lemma 2. *Let H be a projectoid, let π be a projectoidic arrangement of H, and let X be an interval set in π such that $X \notin \Sigma(H)$. Then there exists an interval set Z in π such that Z is free in H and $X \nsim Z$.*

Proof. Denote $\pi = (v_1, \ldots, v_n)$. Since $X \in \operatorname{Int}(\pi)$, there exist g and j, $1 \leqq g \leqq j \leqq n$, such that $X = \{v_g, \ldots, v_j\}$. Since $X \notin \Sigma(H)$, we have that $|X| \neq n$. Without loss of generality we assume that $j < n$. According to Lemma 1, there exist i and k, $1 \leqq i \leqq j < k \leqq n$, such that $\{v_i, \ldots, v_k\}$ is free in H and $\{v_i, \ldots, v_j\} \in \Sigma(H)$. Since $X \notin \Sigma(H)$, we have that $i \neq g$. If $i > g$, then we put $Z = \{v_i, \ldots, v_k\}$, and the lemma is proved.

Now, assume that $i < g$. Then $1 < g$. According to Lemma 1, there exist f and h, $1 \leqq f \leqq g - 1 < h \leqq n$, such that $\{v_f, \ldots, v_h\}$ is free in H and $\{v_g, \ldots, v_h\} \in \Sigma(H)$. Hence, $h \neq j$. If $h > j$, then $\{v_g, \ldots, v_h\} \nsim \{v_i, \ldots, v_j\}$ and thus $\{v_g, \ldots, v_j\} = \{v_g, \ldots, v_h\} \cap \{v_i, \ldots, v_j\} \in \Sigma(H)$, which is a contradiction. This means that $h < j$. We put $Z = \{v_f, \ldots, v_h\}$ and the lemma is proved.

Theorem 2. *Let H be a projectoid. Then* $\operatorname{Int}(\Pi(H)) = \Sigma(H)$.

Proof. Since H is a projectoid, we have that $\mathscr{E}(H) \subseteq \operatorname{Int}(\Pi(H))$. It follows from the definition of $\Sigma(H)$ that $\Sigma(H) \subseteq \operatorname{Int}(\Pi(H))$.

We now wish to prove that $\operatorname{Int}(\Pi(H)) \subseteq \Sigma(H)$. Assume, on the contrary, that there exists $X \in \operatorname{Int}(\Pi(H)) - \Sigma(H)$. Consider an arbitrary projectoidic arrangement $\pi = (v_1, \ldots, v_n)$ on H. Since $X \in \operatorname{Int}(\pi)$, there exist i and j, $1 \leqq i \leqq j \leqq n$, such that $X = \{v_i, \ldots, v_j\}$. Since $X \notin \Sigma(H)$, it follows from Lemma 2 that there exist g and h, $1 \leqq g \leqq h \leqq n$, such that $\{v_g, \ldots, v_h\}$ is free in H and $\{v_g, \ldots, v_h\} \nsim \{v_i, \ldots, v_j\}$. Without loss of generality we assume that $g \leqq i$. Then $g < i \leqq h < j$. We denote by π' the sequence $(v'_1, \ldots, v'_n)$, where $v'_m = v_m$ for $1 \leqq m < g$ or $h < m \leqq n$, and the subsequence $(v'_g, \ldots, v'_h)$ is identical with the sequence $(v_h, \ldots, v_g)$. Since $\{v_g, \ldots, v_h)$

is free in H, we have that π' is a projectoidic arrangement on H. Obviously, $X \notin \operatorname{Int}(\pi')$, which is a contradiction.

Thus the theorem is proved.

Corollary. *Let H_1 and H_2 be projectoids with $V(H_1) = V(H_2)$. Then $\Pi(H_1) \subseteq \Pi(H_2)$ if and only if $\Sigma(H_2) \subseteq \Sigma(H_1)$.*

Proof. If $\Pi(H_1) \subseteq \Pi(H_2)$, then $\operatorname{Int}(\Pi(H_2)) \subseteq \operatorname{Int}(\Pi(H_1))$ and therefore, according to Theorem 2, $\Sigma(H_2) \subseteq \Sigma(H_1)$.

Let $\Sigma(H_2) \subseteq \Sigma(H_1)$. Since $\mathscr{E}(H_2) \subseteq \Sigma(H_2)$ and $\Sigma(H_1) = \operatorname{Int}(\Pi(H_1))$, we have that $\mathscr{E}(H_2) \subseteq \operatorname{Int}(\pi)$ for each $\pi \in \Pi(H_1)$. Hence, $\Pi(H_1) \subseteq \Pi(H_2)$, which completes the proof.

3. A graph isomorphic to the intersection graph (in the sense of [6]) of a finite nonempty family of intervals in the real line is called an interval graph. Various characterizations of interval graphs can be found in [4], [5] and [7]. The following characterization of interval graphs is due to Lekkerkerker and Boland [7]:

(LB) A graph G is an interval graph if and only if (a) G contains no induced cycle of length $\geqq 4$, and (b) for any three vertices v_1, v_2 and v_3 of G, there exist distinct $i, j, k \in \{1, 2, 3\}$ with the property that every $v_i - v_k$ path P in G contains a vertex v_P such that $\{v_j, v_P\} \in \mathscr{E}(G)$.

It is clear that a graph G is an interval graph if and only if there exists a projectoid H such that G is isomorphic to the intersection graph of $\mathscr{E}(H)$. Therefore, a necessary condition for a hypergraph H to be a projectoid is that the intersection graph of $\mathscr{E}(H)$ be an interval graph. However, examples showing that this condition is not sufficient can be easily found. On the other hand, the concept of projectoid can serve as a tool for characterizing interval graphs. In the language of the matrix theory such an approach was adopted by Fulkerson and Gross [4]. In the present section another approach will be suggested.

Let D be a digraph in the sense of [1] or [6]. We denote by $V(D)$ and $A(D)$ its vertex set and arc set, respectively. For every $v \in V(D)$, we denote

$$N(v, D) = \{w \in V(D);\ (v, w) \in A(D)\} \quad \text{and} \quad M(v, D) = N(v, D) \cup \{v\}\ .$$

We denote by M_D the hypergraph with $V(M_D) = V(D)$ and

$$\mathscr{E}(M_D) = \{M(v, D);\ v \in V(D)\}\ .$$

A hypergraph H is said to be an M-hypergraph of G is there exists a digraph D such that the underlying graph of D (see [1]) is identical with G, and M_D is identical with H.

Lemma 3. *Let G be an interval graph. Then at least one of its M-hypergraphs is a projectoid.*

Proof. Obviously, there exists a projectoid H such that G is isomorphic to the intersection graph of $\mathscr{E}(H)$. Consider a projectoidic arrangement $(v_1, \ldots, v_n)$ of H. For every $E \in \mathscr{E}(H)$, we denote $i(E) = \min \{i; v_i \in E\}$. Moreover, we denote by D the digraph with $V(D) = \mathscr{E}(H)$, such that $(E', E'') \in A(D)$ if and only if $v_{i(E'')} \in E'$ for any distinct $E', E'' \in \mathscr{E}(H)$. It is easy to see that the underlying graph of D is isomorphic to G. Denote $m = |\mathscr{E}(H)|$. Consider such an arrangement $(E_1, \ldots, E_m)$ that for any $j, k \in \{1, \ldots, m\}$, $i(E_j) < i(E_k)$ implies $j < k$. It is clear that $(E_1, \ldots, E_m)$ is a projectoidic arrangement of M_D. Hence, M_D is a projectoid, which completes the proof.

Remark. Lemma 3 gives a necessary condition for a graph to be an interval graph, but this condition is not sufficient. For example, let D be a digraph with $n \geqq 4$ vertices $v_1, \ldots, v_n$ such that

$$A(D) = \{(v_1, v_2), (v_1, v_3), \ldots, (v_1, v_{n-1}) (v_n, v_2), (v_n, v_3), \ldots, (v_n, v_{n-1})\} .$$

Since $(v_1, \ldots, v_n)$ is a projectoidic arrangement of M_D, we have that M_D is a projectoid. Obviously, the underlying graph of D is $K(2, n - 2)$, which is not an interval graph.

Lemma 4. *Let G be a graph with no induced cycle of length four. Assume that at least one of the M-hypergraphs of G is a projectoid. Then G is an interval graph.*

Proof. It follows from the assumption that there exists a digraph D such that the underlying graph of D is identical with G and M_D is a projectoid. Consider a projectoidic arrangement $(v_1, \ldots, v_n)$ of M_D.

First, let G contain an induced cycle C of length $\geqq 5$. Then there exist i, j and k, $1 \leqq i < j < k \leqq n$, such that $v_i, v_j, v_k \in V(C)$, $\{v_i, v_k\} \in \mathscr{E}(C)$, and $\{v_i, v_j\}, \{v_j v_k\} \notin \mathscr{E}(C)$. Without loss of generality we assume that $(v_i, v_k) \in A(D)$. Since C is an induced subgraph of G, we have that $\{v_i, v_j\} \notin \mathscr{E}(G)$. This implies that $v_j \notin M(v_i, D)$. Since $v_i, v_k \in M(v_i, D)$ and $i < j < k$, we have that $(v_1, \ldots, v_n)$ is not a projectoidic arrangement of M_D, which is a contradiction. This means that G contains no induced cycle of length > 3.

Assume that G is not an interval graph. Since G contains no induced cycle of length > 3, it follows from (LB) that there exist integers i, j, k, $1 \leqq i < j < k \leqq n$, and a $v_i - v_k$ path P in G such that for no $v \in V(P)$, $\{v_j, v\} \in \mathscr{E}(G)$. It is clear that there exist integers g and h, $1 \leqq g < j < h \leqq n$, such that $\{v_g, v_h\} \in \mathscr{E}(P)$. Without loss of generality we assume that $(v_g, v_h) \in A(D)$. It is obvious that $v_j \notin M(v_g, D)$. Therefore, $(v_1, \ldots, v_n)$ is not a projectoidic arrangement of M_D, which is a contradiction. This means that G is an interval graph, which completes the proof.

According to (LB), an interval graph contains no induced cycle of length four. Combining this observation with Lemmas 3 and 4, we find the following

Theorem 3. *A graph G is an interval graph if and only if G contains no cycle of length four and at least one of the M-hypergraphs of G is a projectoid.*

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