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PRIME SELECTORS AND TORSION CLASSES OF LATTICE ORDERED GROUPS

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In this paper, the relations between torsion classes ([13] and [15]) and prime selectors [16] of lattice ordered groups are investigated. The existence of the largest presentations of torsion classes by prime selectors will be examined. We prove that if X is a hereditary torsion class, then X has no largest presentation by prime selectors. Some problems concerning torsion classes and prime selectors proposed in [16] will be solved.

1. PRELIMINARIES

Several classes of lattice ordered groups that have been thoroughly studied are not definable by identities, i.e., they fail to be varieties. As examples we can mention here (i) the class of all complete lattice ordered groups, (ii) the class of all archimedean lattice ordered groups, (iii) the class of all lattice ordered groups such that every bounded disjoint subset is finite or (iv) the class of all cardinal sums of linearly ordered groups. Thus, for classifying types of lattice ordered groups we need more general notions than the notion of a variety.

The notions of a torsion class and a hereditary torsion class were introduced by J. Martinez in [13], [15], and they have been dealt with in [3], [5], [6], [8], [11], [12], [14]; for the notion of a radical class cf. [9], [10] (the definitions are given below). All the above mentioned examples are radical classes; (iii) and (iv) are hereditary torsion classes ((i) and (ii) fail to be torsion classes).

One of the methods for studying torsion classes consists in using value selectors [15] or prime selectors [16] (rules that assign to each lattice ordered group G a system of values of G or prime subgroups of G, respectively, such that certain conditions are fulfilled).

In this paper we shall apply the standard denotations for lattice ordered groups (cf. Conrad [1] and Fuchs [4]). The group operation will be written additively.

The system of all convex l-subgroups of a lattice ordered group G will be denoted by $c(G)$; this system is partially ordered by inclusion. Then $c(G)$ is a complete lattice; the lattice operations in $c(G)$ will be denoted by $\wedge$ and $\vee$.

Let $\mathscr{G}$ be the class of all lattice ordered groups and let A be a nonempty subclass of $\mathscr{G}$. Consider the following conditions for A:

(a) If $G \in \mathscr{G}$ and $\{H_i\}_{i\in I} \subseteq A \cap c(G)$, then $\bigvee_{i\in I} H_i \in A$.

(b) If $G \in A$ and $H \in c(G)$, then $H \in A$.

(c) A is closed with respect to homomorphisms.

(d) A is closed with respect to isomorphisms.

The class A is said to be a torsion class if it satisfies (a) and (c); if, moreover, A fulfils also (b), then it is called a hereditary torsion class (cf. [16]; a different terminology has been used in [3], [5], [13], [14]). Every variety of lattice ordered groups is a hereditary torsion class (Holland [5]).

If the class A fulfils the conditions (a), (b) and (d), then it is called a radical class [9] (such classes were considered (under another terminology) already in [7]).

Let $\mathscr{T}$ and $\mathscr{H}$ be the class of all torsion classes or hereditary torsion classes, respectively. Both $\mathscr{T}$ and $\mathscr{H}$ are partially ordered by inclusion. Then $\mathscr{T}$ and $\mathscr{H}$ are complete lattices (cf. [13] and [16]). For $G \in \mathscr{G}$ and $A \in \mathscr{T}$ we denote by $A(G)$ the join of all convex l-subgroups of G belonging to A.

The following hereditary torsion classes were examined by Conrad [3]:

Ar – the class of all hyperarchimedean lattice ordered groups;

Fb – the class of all lattice ordered groups G such that each bounded disjoint subset of G is finite;

Fv – the class of all finite valued lattice ordered groups;

Dc – the class of all lattice ordered groups whose regular subgroups satisfy the descending chain condition;

Os – the class of all cardinal sums of linearly ordered groups;

Rs – the class of all cardinal sums of archimedean linearly ordered groups;

Bp – the class of all lattice ordered groups G such that each prime of G exceeds a unique minimal prime.

We shal use the following notation: N_0, Q and R are the sets of all positive integers, all rational numbers and all reals, respectively; each of these sets is linearly ordered in the natural way. Q and R are also considered as additive groups (and hence as linearly ordered groups).

2. PRIME SELECTORS

The notion of a prime selector was introduced in [16]. Let us recall some definitions and results concerning prime selectors, which we shall need in the sequel.

Let $G \in \mathcal{G}$, $0 \neq x \in G$. A convex l-subgroup of G maximal with respect to the property of noncontaining x is called a value of x in G. A convex l-subgroup of G is said to be a value (or a regular subgroup) if it is a value of an element of G.

A convex l-subgroup N of G is called a prime subgroup of G, if, whenever $a, b \in G$ and $a \wedge b \in N$, then $a \in N$ or $b \in N$. Every value of G is a prime subgroup of G. Let $M^0(G)$ and $P(G)$ be the set of all values of G and the set of all proper prime subgroups of G, respectively.

A prime selector is a function M which assigns to every lattice ordered roup G a subset $M(G)$ of $P(G)$ such that the following conditions are fulfilled:

(1) If φ is a homomorphism of G onto a lattice ordered group H and if $N \in M(G)$, $N \supseteq \operatorname{Ker}(\varphi)$, then $\varphi(N) \in M(H)$.

(2) If $C \in c(G)$ and $N \in P(G)$, then $N \cap C \in M(C)$ implies that $N \in M(G)$.

Let M_1 and M_2 be prime selectors. We put $M_1 \leqq M_2$ if $M_1(G) \subseteq M_2(G)$ is valid for each $G \in \mathcal{G}$. Let $\{M_i\}_{i \in I}$ be a family of prime selectors; we put $M_1(G) = \bigcap_{i \in I} M_i(G)$ and $M_2(G) = \bigcup_{i \in I} M_i(G)$ for each $G \in \mathcal{G}$. Then M_1 and M_2 are prime selectors and, moreover, $M_1 = \bigwedge_{i \in I} M_i$, $M_2 = \bigvee_{i \in I} M_i$.

If M is a prime selector we define $\mathsf{TOR}(M)$ to be the class of all lattice ordered groups G with $M(G) = P(G)$. For each torsion class T and each $G \in \mathcal{G}$ we put

$$h(T)(G) = \{N \in P(G) : T(G) \nsubseteq N\} .$$

Then we have (cf. [16], Propos. 1 and 2):

2.1. Proposition. *For every prime selector M, $\mathsf{TOR}(M)$ is a torsion class.*

2.2. Proposition. *For every torsion class T, $h(T)$ is a prime selector and $\mathsf{TOR}(h(T)) = T$. Moreover, if M is a prime selector with $\mathsf{TOR}(M) = T$, then $h(T) \leqq M$.*

A prime selector is called hereditary, if it fulfils the condition

(2′) whenever $G \in \mathcal{G}$, $C \in c(G)$ and $N \in P(G)$, then we have $N \cap C \in M(C)$ if and only if $N \in M(G)$ and $N \nsupseteq C$.

2.3. Proposition. (Cf. [16].) *If M is a hereditary prime selector, then $\mathsf{TOR}(M)$ is a hereditary torsion class. If T is a hereditary torsion class, then $h(T)$ is a hereditary prime selector.*

Let M be a hereditary prime selector such that for each $G \in \mathcal{G}$, all elements of $M(G)$ are values in G; then M is called a value selector.

If T is a torsion class and M is a prime selector with $\mathsf{TOR}(M) = T$, then we say that M presents T or that M is a presentation of T. For each torsion class T we denote by $M_0(T)$ the class of all prime selectors presenting T.

The following open questions 2.4, 2.6, 2.8 and 2.9 on the relations between torsion classes and prime selectors have been formulated by J. Martinez [16].

In [16] it is remarked that the map TOR preserves arbitrary meets, and the following question is proposed:

2.4. Question. *It would be useful to know whether* TOR *preserves joins. If so, then every torsion class has a largest presentation.*

A lattice ordered group is said to be epiarchimedean if each of its homomorphic images is archimedean. Let Ar be the class of all epiarchimedean lattice ordered groups; Ar is a hereditary torsion class (cf. [3], [16]). Let us denote by M_0 the prime selector of minimal primes.

2.5. Lemma. (Cf. [16].) $\mathsf{TOR}(M_0) = \mathrm{Ar}$.

2.6. Question. *Is M_0 the largest presentation of* Ar?

A value H in G is called special if there exists $g \in G$ such that H is the only value of g in G. A lattice ordered group G is said to be finite valued if each element $x \in G$, $x \neq 0$, has only a finite number of values. Let Fv be the class of all finite valued lattice ordered groups. Further, let F be the prime selector that picks all non-values and every special value.

2.7. Lemma. (Cf. [16].) $\mathsf{TOR}(F) = \mathrm{Fv}$.

2.8. Question. *Is F the largest presentation of* Fv?

It is also remarked in [16] that if T_i $(i \in I)$ are torsion classes and if $T = \bigvee_{i\in I} T_i$, then $h(T) = \bigvee_{i\in I} h(T_i)$, and that for each pair of hereditary torsion classes T_1, T_2 we have $h(T_1 \wedge T_2) = h(T_1) \wedge h(T_2)$.

2.9. Question. *In general, however, it is unknown whether h preserve finite meets.*

The question analogous to 2.4 concerning value selectors has been proposed by Martinez [15] and dealt with in the author's paper [12]. By modifying the construction from [12] we obtain here three types of constructions showing that the answer to all the above questions is 'No'. The intersections of the result implied by these constructions are nonempty. A further investigation using these (or analogous) constructions might perhaps enable one to shed light on the properties of partially ordered classes $M_0(X)$, where X is a torsion class, and, in particular, on the characterization of those torsion classes X which have a largest presentation (if such torsion classes X do exist).

Let $G \in \mathscr{G}$ and $Y \subseteq G$. We denote $Y^\delta = \{g \in G : |g| \wedge |y| = 0 \text{ for each } y \in Y\}$. It is well-known that Y^δ is a convex l-subgroup of G.

Let I be a linearly ordered set and for each $i \in I$ let G_i be a lattice ordered group such that G_i is linearly ordered whenever i fails to be the least element of I. We denote by $\Gamma_{i \in I} G_i$ the lexicographic product of the lattice ordered groups G_i (cf. Fuchs [4]). If all G_i are linearly ordered, then $\Gamma_{i \in I} G_i$ is linearly ordered as well.

Let R_1, R_2 be dense subsets of Q such that $R_2 = Q \setminus R_1$. Let f be a one-to-one mapping of Q onto N_0. Let $P = \{p_n : n \in N_0\}$ be the set of all primes. For each $x \in Q$ let K_x be the additive group of all rational numbers y which can be expressed as $y = zp_n^{-m}$, where $n = f(x)$, $m \in N_0$ and z is any integer; the group K_x is linearly ordered in the natural way. We denote by H_0 the class of all lattice ordered groups H which can be written in the form

$$H = \Gamma_{i \in I} H_i, \tag{3}$$

where (i) I is a convex subset of Q, and (ii) for each $i \in I$, H_i is isomorphic with K_i. Further, let H_0' be the class of all $H \in H_0$ such that the set I in (3) is a one-element set.

Because each H_i in (3) is lexicographically indecomposable and since for $x, y \in Q$, $x \neq y$, the linearly ordered groups K_x and K_y are not isomorphic, it follows from Mal'cev-Fuchs Theorem (cf. [4], Chap. II, Thm. 9) that for a given $H \in H_0$ the set I in (3) is uniquely determined and that the corresponding lattice ordered groups H_i are determined up to isomorphisms.

Let H be as in (3) and let $0 < x \in H$. Let i_0 be the least element of I with $x(i_0) \neq 0$. If $i_0 \in R_i$ $(i = 1, 2)$, then x is said to be of type R_i; let $R_i(H)$ be the set of all elements of H of type R_i. A homomorphism φ of a lattice ordered group G_1 into a lattice ordered group G_2 is called convex if $\varphi(G_1)$ is a convex subset of G_2. Let $H, H' \in H_0$ and let φ be a convex homomorphism of H into H' with $\varphi(H) \neq \{0\}$. Then it follows from the structure of lattice ordered groups belonging to H_0 that there is a dual ideal I_1 of I such that (under the notation as in (3)) $\varphi(H)$ is isomorphic with $\Gamma_{i \in I_1} H_i$ and $\operatorname{Ker}(\varphi) = \Gamma_{i \in I \setminus I_1} H_i$.

Now let G be any lattice ordered group, $i \in \{1, 2\}$. We denote by $R_i(G)$ the set of all elements $g \in G$ which have the following property: there exist $H \in H_0$ and a convex isomorphism φ of H into G such that $x \in \varphi(H)$ and $\varphi^{-1}(x) \in R_i(H)$. Further, let $R_0(G)$ be the set of those $g \in R_1(G) \cup R_2(G)$ for which there exists H with the just mentioned property, such that H belongs to H_0'.

3.1. Lemma. *For each $g \in R_1(G) \cup R_2(G)$, the element g has a unique value in G.*

This follows from [12], Lemma 3.2.

If $g \in R_1(G) \cup R_2(G)$, then the value of g in G will be denoted by $v_G(g)$. We put

$s_0(G) = \{v_G(x) : x \in R_0(G)\}$, $s_i(G) = \{v_G(x) : x \in R_i(G)\}$ $(i = 1, 2)$,
$s_i'(G) = s_i(G) \cup s_0(G)$ $(i = 1, 2)$.

3.2. Lemma. (Cf. [12], Lemma 4.2.) *s'_1 and s'_2 are value selectors.*

For $G \in \mathscr{G}$ let $t_0(G)$ be the set of all primes N in G such that N fails to be a value in G.

3.3. Lemma. *t_0 is a prime selector.*

Proof. The condition (1) is obviously fulfilled. The validity of (2) follows from the fact that if N is a value in G such that $N \not\supseteq C$, then $N \cap C$ is a value in C (cf. [16]; cf. also 4.1 and 4.2 below).

Let A_0 be the class of all lattice ordered groups G such that either $G = \{0\}$ or G is a direct sum of lattice ordered groups belonging to H'_0. It is easy to verify that A_0 is a hereditary torsion class. (This also follows immediately from Thm. 2.6 in [11].)

Put $t_i = s'_i \vee t_0$ $(i = 1, 2)$.

3.4. Lemma. $\mathsf{TOR}(t_1) = \mathsf{TOR}(t_2) = A_0$.

Proof. Let $G \in A_0$. Then each prime in G is a value in G. Thus from (5) in [12] we obtain $A_0 \subseteq \mathsf{TOR}(s'_1)$. In view of $s'_1 \leqq t_1$ we get $\mathsf{TOR}(s'_1) \subseteq \mathsf{TOR}(t_1)$ implying $A_0 \subseteq \mathsf{TOR}(t_1)$.

Conversely, let $G \in \mathsf{TOR}(t_1)$ and let H be a value in G. Then $H \in t_1(G) = s'_1(G) \cup \cup t_0(G)$, hence $H \in s'_1(G)$. This and Lemma 4.6 in [12] imply $G \in A_0$, thus $\mathsf{TOR}(t_1) \subseteq \subseteq A_0$.

The proof for t_2 is analogous.

Under the notation as above let $H^0 = \Gamma_{i \in Q} H_i$. From the definition of t_1 and t_2 we obtain

(4) $$H^0 \in \mathsf{TOR}(t_1 \vee t_2),$$

and obviously $A_0 \cap (H_0 \setminus H'_0) = \emptyset$; hence in particular

(5) $$H^0 \notin A_0 .$$

In view of 3.4, (4) and (5) we have

$$\mathsf{TOR}(t_1 \vee t_2) \neq \mathsf{TOR}(t_1) \vee \mathsf{TOR}(t_2) = A_0 .$$

Thus the mapping TOR need not preserve joins. Also, according to 3.4, t_1 and t_2 are presentations of A_0, but in view of (4) and (5) $t_1 \vee t_2$ fails to be a presentation of A_0; hence A_0 has no largest presentation. Therefore Question 2.4 is answered negatively.

The above result can be expressed as follows:

3.5. Proposition. *The class $M_0(A_0)$ is not upper directed, hence it has no greatest element.*

The value selectors s'_1 and s'_2 have been defined by means of the mapping f; let us write now $s'_1(f)$ and $s'_2(f)$ instead of s'_1 and s'_2. Obviously there exists an infinite set of mappings f_j $(j \in J)$ such that (i) every f_j is a one-to-one mapping of Q onto N_0, and (ii) if $j, k \in J$ and $j \neq k$, then $s_{i_1}(f_j) = s_{i_2}(f_k)$ is valid for $i_1, i_2 \in \{1, 2\}$. Thus 3.5 can be sharpened in the following way:

3.6. Proposition. *There exists an infinite set of pairs* (s'_{1j}, s'_{2j}) $(j \in J)$ *such that for each* $j \in J$ *we have* (i) s'_{1j} *and* s'_{2j} *are value selectors*; (ii) $s'_{1j} \vee t_0$ *and* $s'_{2j} \vee t_0$ *belong to* $M_0(A_0)$; (iii) *the set* $\{s'_{1j} \vee t_0, s'_{2j} \vee t_0\}$ *has no upper bound in* $M_0(A_0)$.

4. THE MAPPINGS t'_1 AND t'_2

For the following two lemmas cf. Conrad [3] and Martinez [15], [16].

4.1. Lemma. *If* $G, H \in \mathcal{G}$ *and if* $\Phi : G \to H$ *is an epimorphism, then the map* $N \to N\Phi^{-1}$ *is a one-to-one correspondence between* $P(H)$ *and the proper primes of* G *that contain* $\mathrm{Ker}(\Phi)$.

4.2. Lemma. *Let* $G \in \mathcal{G}$, $C \in c(G)$. *The map* $N \to N \cap C$ *is a one-to-one correspondence between the primes of* G *that do not contain* C *and* $P(C)$.

Moreover, each of the above correspondences can be restricted to the appropriate sets of values (cf. [16]), and hence also to the appropriate sets of primes which fail to be values.

Let C be a lattice ordered group. Consider the following condition for C:

(α) If $G \in \mathcal{G}$, $C \in c(G)$, $0 < g_1 \in G$ and if there exists $c_1 \in C$ with $c_1 \not\leqq g_1$, then there are elements $0 \leqq c_2 \in C$, $0 \leqq g_2 \in C^\delta$ such that $g_1 = c_2 + g_2$.

It is easy to verify that every linearly ordered group fulfils the condition (α).

4.3. Lemma. *Let* C *be a lattice ordered group fulfilling the condition* (α). *Let* $G \in \mathcal{G}$, $C \in c(G)$, $N_1 \in P(C)$. *Then* $N = N_1 + C^\delta$ *belongs to* $P(G)$.

Proof. Let $0 \leqq x \in G$, $0 \leqq y \in G$ and suppose that $x \wedge y \in N$. Thus there are elements $0 \leqq x_1 \in N_1$, $0 \leqq y_1 \in C^\delta$ with $x \wedge y = x_1 + y_1 = x_1 \vee y_1$. Now $N_1 \in P(C)$ implies $C \neq \{0\}$, hence there is $0 < x_2 \in C$ such that $x_2 > x_1$. Hence we infer that $x \wedge y \not\geqq x_2$; therefore either x or y does not exceed C. We may suppose that y does not exceed C. Hence there are elements $0 \leqq c_1 \in C$, $0 \leqq g_1 \in C^\delta$ with $y = c_1 + g_1$. Clearly $y = c_1 \vee g_1$. Thus

$$x \wedge y = x \wedge (c_1 \vee g_1) = (x \wedge c_1) \vee (x \wedge g_1), \quad x \wedge c_1 \in C, \quad x \wedge g_1 \in C^\delta .$$

From this and from $x \wedge y = x_1 \vee y_1$ we easily obtain $x_1 = x \wedge c_1$, $y_1 = x \wedge g_1$.

Assume that x exceeds C. Then $x_1 = c_1 \in N_1$, hence $y = x_1 + g_1 \in N$. Now assume that x does not exceed C. Hence there are elements $0 \leqq c_2 \in C$, $0 \leqq g_2 \in C^\delta$ with $x = c_2 + g_2 = c_2 \vee g_2$. This implies

$$x \wedge y = (c_2 \vee g_2) \wedge (c_1 \vee g_1) = (c_1 \wedge c_2) \vee (g_1 \wedge g_2) = x_1 \vee y_1 ,$$

whence $x_1 = c_1 \wedge c_2$. Since N_1 is a prime in C, we infer that either c_1 or c_2 belongs to N_1. Therefore either x or y belongs to N.

From 4.2 and 4.3 we obtain:

4.4. Lemma. *Let $G \in \mathcal{G}$ and $C \in c(G)$. Suppose that C fulfils the condition (α). Let N be a convex l-subgroup of G such that $N \nsupseteq C$. Then N is prime in G if and only if it fulfils the following conditions:*

(i) *$N \cap C$ is a prime in C;*
(ii) *$N = (N \cap C) + C^\delta$.*

Let $\mathcal{H}$ be a class of lattice ordered groups and for each $H \in \mathcal{H}$ let $h(H)$ be a subset of $P(H)$. For each $G \in \mathcal{G}$ we define $h'(G)$ to be the set of all primes N of G which have the following property: there exist $H \in \mathcal{H}$, a convex homomorphism φ of H into G and a prime $H_1 \in h(H)$ such that $\varphi(H) \neq \{0\}$, $\varphi(H_1) = \varphi(H) \cap N$ and $\operatorname{Ker}(\varphi) \subseteq H_1$.

4,5. Lemma. *h' is a prime selector.*

Proof. From 4.1 and 4.2 it follows that h' fulfils the conditions (1) and (2).
For every positive integer n we put

$$H_{1,2n} = H_{2,2n-1} = Q ,$$
$$H_{1,2n-1} = H_{2,2n} = R .$$

Next we denote

$$H_1 = \Gamma_{i \in N_0} H_{1i} , \quad H_2 = \Gamma_{i \in N_0} H_{2i} .$$

Let $0 < x \in H_j$ $(j \in \{1, 2\})$ and i_0 be the least element of N_0 with $x(i_0) \neq 0$. The element x is said to be of type Q or type R, if $H_{j,i_0} = Q$ or $H_{j,i_0} = R$, respectively. Let $\mathcal{H} = \{H_1, H_2\}$ and for $j \in \{1, 2\}$ let $t_1(H_j)$ and $t_2(H_j)$ be the set of all values $v_{H_j}(x)$ in H_j such that x is of type Q or of type R, respectively. From 4.5 we obtain:

4.6. Lemma. *t_1' and t_2' are prime selectors.*

Each homomorphic image of H_j $(j \in \{1, 2\})$ distinct from $\{0\}$ is isomorphic either to H_1 or to H_2; thus in the definition of t_1' and t_2' it suffices to consider convex isomorphisms of H_1 or H_2 instead of convex homomorphisms. Hence 4.4 and the fact that each prime subgroup in $H_j \in \mathcal{H}$ is a value in H_j yield the following lemma:

4.7. Lemma. *Let* $G \in \mathscr{G}$ *and let* $N \in t_j'(G)$ $(j = 1$ *resp.* $j = 2)$. *Then there exist* $H \in \mathscr{H}$, *a convex isomorphism* φ *of* H *into* G *and* $0 < x \in G$ *such that* (i) $x \in \varphi(H)$ *and* $\varphi^{-1}(x)$ *is of type* Q *resp.* R; (ii) $N = M_x + (\varphi(H))^\delta$, *where* M_x *is the value of* x *in* $[x]$.

4.8. Lemma. *Let* $G \in \mathscr{G}$. *Then* $t_1'(G) \cap t_2'(G) = \emptyset$.

Proof. By way of contradiction, assume that $N \in t_1'(G) \cap t_2'(G)$. Hence there exists $x \in G$ with the properties as in 4.7 such that x is of type Q; further, there exists $0 < y \in G$ having analogous properties as x with the distinction that y is of type R. Because $\varphi(H)$ is linearly ordered and $0 < x \in \varphi(H)$, we have $(\varphi(H))^\delta = \{x\}^\delta$. If x and y are comparable, e.g., if $x < y$, then $x \in M_y \subseteq M_y + \{y\}^\delta = N = M_x + \{x\}^\delta$, which is a contradiction. If x and y are incomparable, then $x \wedge y = 0$, whence $x \in \{y\}^\delta \subseteq N$, which is impossible.

4.9. Lemma. *Let* X *be a torsion class. Then* $\mathsf{TOR}(t_1' \vee h(X)) = X$.

Proof. According to 2.2 we have $X = \mathsf{TOR}(h(X)) \subseteq \mathsf{TOR}(t_1' \vee h(X))$. Assume that there exists $G \in \mathsf{TOR}(t_1' \vee h(X))$ such that $G \notin X$. Thus there is $N \in P(G)$ with $N \notin h(X)(G)$. Hence $N \supseteq X(G)$. Since $G \in \mathsf{TOR}(t_1' \vee h(X))$, we must have $N \in t_1'(G)$. Let H, φ and x be as in 4.7; then $N = M_x + \{x\}^\delta$. There exists $y \in \varphi(H)$ with $x < y$ such that $\varphi^{-1}(y)$ is of type R. Put $N' = M_y + \{y\}^\delta = M_y + (\varphi(H))^\delta$. According to 4.4, $N' \in P(G)$. Moreover, $M_y \in t_2(H)$ and therefore N' belongs to $t_2'(G)$. From $x, y \in \varphi(H)$ it follows that $\{y\}^\delta = \{x\}^\delta$. Clearly $M_x \subset M_y$. We infer that $N' \supset N$ and hence $N' \supset X(G)$. Thus $N' \notin h(X)(G)$ and this implies that $N' \in t_1'(G)$; with regard to 4.8 we have a contradiction.

Analogously we can prove

4.9'. Lemma. *Let* X *be a torsion class. Then* $\mathsf{TOR}(t_2' \vee h(X)) = X$.

4.10. Proposition. *Let* X *be a torsion class such that* $\{H_1, H_2\} \nsubseteq X$. *Then* $M_0(X)$ *is not upper-directed*; *hence* X *has no largest presentation.*

Proof. According to 4.9 and 4.9', both $t_1' \vee h(X)$ and $t_2' \vee h(X)$ are presentations of X. Let s be a prime selector with $t_i' \vee h(X) \leqq s$ $(i = 1, 2)$. Then $t_1' \vee t_2' \leqq s$. The definition of t_1' and t_2' immediately yields $\{H_1, H_2\} \subseteq \mathsf{TOR}(t_1' \vee t_2')$, whence $\{H_1, H_2\} \subseteq \mathsf{TOR}(s)$. Therefore $\mathsf{TOR}(s) \neq X$.

We obviously have $H_i \notin \mathrm{Ar}$ and $H_i \notin \mathrm{Rs}$ for $i = 1, 2$. Thus 4.10 implies the following corollary (answering Question 2.6):

4.11. Corollary. *Let* X *be a torsion class,* $Y \in \{\mathrm{Ar}, \mathrm{Rs}\}$, $X \subseteq Y$. *Then* X *has no largest presentation.*

Let us remark that instead of R and Q we can take in the above consideration any pair of non-isomorphic l-subgroups of R distinct from $\{0\}$. Further, for each $G \in \mathscr{G}$, all primes belonging to $t_i'(G)$ are values in G, but t_1' and t_2' fail to be hereditary, hence they are not value selectors.

Let $G \in \mathscr{G}$ and $G_1 \in c(G)$, $G_1 \neq G$. Assume that for each $0 < g \in G \setminus G_1$ and each $g_1 \in G_1$ we have $g > g_1$. Then G is said to be a lexico extension of G_1 and we write $G = \langle G_1 \rangle$. A lattice ordered group G' is called a lexico extension if there exists $G_1' \in c(G)$ such that $G' = \langle G_1' \rangle$.

For the following two lemmas cf. Conrad [2].

5.1. Lemma. *Every lexico extension fulfils the condition* (α).

It is easy to verify that if H is a lexico extension and if $H' \neq \{0\}$ is a homomorphic image of H, then H' is a lexico extension as well.

If $G \in \mathscr{G}$ and if there exist lattice ordered groups $G_1 \neq \{0\}$, G_2 such that $G = G_1 \circ G_2$ (the symbol $\circ$ denoting the operation of the lexicographic product), then G is a lexico extension.

Let $I = Q$ and for each $i \in I$ let H_i be the linearly ordered group isomorphic with R. Put $G_1 = \Gamma_{i \in I} H_i$, $G_2 = \prod_{i \in I} H_i$, $H = G_1 \circ G_2$. Denote

$$r_1(H) = M^0(H), \quad r_2(H) = P(H) \setminus M^0(H).$$

Put $\mathscr{H} = \{H\}$. Let r_1' and r_2' have analogous meanings as t_1' and t_2' in § 4 with the distinction that we take r_1 and r_2 instead of t_1 and t_2. According to 4.5, r_1' and r_2' are prime selectors.

5.2. Lemma. *Let $G \in \mathscr{G}$. Then $r_1'(G) \cap r_2'(G) = \emptyset$.*

Proof. Let $N \in r_1'(G)$. Then assertions analogous to 4.1 and 4.2 that concern the values of a lattice ordered group imply that N is a value in G. Similarly, if $N' \in r_2'(G)$, then N' is not a value. Thus $r_1'(G) \cap r_2'(G) = \emptyset$.

5.3. Lemma. *Let X be a torsion class. Then* $\mathrm{TOR}(r_1' \vee h(X)) = X$, $\mathrm{TOR}(r_2' \vee h(X)) = X$.

Proof. We proceed analogously as in 4.9. Clearly $X \subseteq \mathrm{TOR}(r_1' \vee h(X))$. Assume that there exists $G \in \mathrm{TOR}(r_1' \vee h(X))$ such that $G \notin X$. Hence there is $N \in P(G)$ with $N \notin h(X)(G)$; thus $N \supseteq X(G)$ and $N \in r_1'(G)$. There exist $N_1 \in M^0(H)$ and a convex homomorphism φ of H into G such that $\mathrm{Ker}(\varphi) \subseteq N_1$ and $N \cap \varphi(H) = \varphi(N_1)$.

There is $N_2 \in P(H) \setminus M^0(H)$ with $N_2 \supset N_1$. Put

$$N' = \varphi(N_2) + (\varphi(H))^\delta .$$

Then according to 4.4 and 5.1, N' is a prime in G. Moreover, from 4.1, 4.2 and from the analogous results concerning values we obtain that N' fails to be a value in G.

Thus $N' \in r'_2(G)$. In view of 4.4 we have also

$$N = \varphi(N_1) + (\varphi(H))^\delta ,$$

hence $N' \supseteq N$. Therefore $N' \supseteq X(G)$, implying $N' \notin h(X)(G)$. From this and from 5.2 we obtain $N' \notin (r'_1 \vee h(X))(G)$, whence $G \notin \mathsf{TOR}(r'_1 \vee h(X))$, which is a contradiction.

The proof of the second assertion is analogous.

According to the definition of H we have $H \in \mathsf{TOR}(r'_1 \vee r'_2) \subseteq \mathsf{TOR}(r'_1 \vee r'_2 \vee \vee h(X))$ for every torsion class X. Thus in view of 5.3 we infer:

5.4. Proposition. *Let X be a torsion class such that $H \notin X$. Then $M_0(X)$ is not upper directed, hence X has no largest presentation.*

5.5. Lemma. (Cf. [3], § 4.) *Let $G \in \mathscr{G}$. The following conditions are equivalent*:

(i) $G \in \mathrm{Bp}$.

(ii) *Each pair of incomparable primes in G generates G.*

There exist incomparable primes N_1, N_2 in H with $N_1, N_2 \subseteq G_2$, whence H does not fulfil the condition (ii) from 5.5; thus H does not belong to Bp. If $X \in \{\mathrm{Fb}, \mathrm{Fv}, \mathrm{Dc}, \mathrm{Os}\}$, then clearly $H \notin X$. Hence 5.4 implies the following corollary which answers Question 2.8:

5.6. Corollary. *Let X be a torsion class and let $Y \in \{\mathrm{Fb}, \mathrm{Fv}, \mathrm{Dc}, \mathrm{Os}, \mathrm{Bp}\}$. If $X \subseteq Y$, then X has no largest presentation.*

6. THE MAPPINGS $q_1[K]$, $q_2[K]$ AND $w[K]$

Let K be a lattice ordered group and let G_1 be as in § 5. We put $H' = G_1 \circ K$, $\mathscr{H} = \{H'\}$,

$$q_1[K] = M^0(H') , \quad q_2[K] = P(H') \setminus M^0(H') ,$$
$$(q_i[K])' = q'_i[K] \ (i = 1, 2)$$

(under analogous notation as in § 4). The proofs of the following Lemmas 6.1 and 6.2 are analogous to those of 5.2 and 5.3.

6.1. Lemma. *Let $G \in \mathscr{G}$. Then $q'_1[K](G) \cap q'_2[K](G) = \emptyset$.*

6.2. Lemma. *Let X be a torsion class. Then $\mathsf{TOR}(q'_i[K] \vee h(X)) = X$ $(i = 1, 2)$.*

The definition of $q'_i[K]$ $(i = 1, 2)$ yields $H' \in \mathsf{TOR}(q'_1[K] \vee q'_2[K])$. Thus in view of 6.2 we have

6.3. Proposition. *Let X be a torsion class such that $H' \notin X$. Then $M_0(X)$ is not upper directed, hence X has no largest presentation.*

6.4. Corollary. *Let X be a torsion class such that there is $G \in \mathcal{G} \setminus X$ with $G_1 \circ G \notin X$. Then the torsion class X has no largest presentation.*

6.5. Corollary. *Let X be a hereditary torsion class, $X \neq \mathcal{G}$. Then X has no largest presentation.*

7. THE MAPPING h

Let us consider the mapping $T \to h(T)$ from § 2. If T_1, T_2 are torsion classes, then we obviously have $T_1 \wedge T_2 = T_1 \cap T_2$. In [16] it is remarked that h preserves the meet of T_1 and T_2 if and only if

$$(*) \qquad (T_1 \cap T_2)(G) = T_1(G) \cap T_2(G)$$

is valid for each lattice ordered group G. The following example shows that $(*)$ need not hold.

Let α, β be distinct cardinals, $\alpha \neq 0 \neq \beta$. Let T_α be the class of all lattice ordered groups G_1 such that either $G_1 = \{0\}$ or G_1 can be expressed as a direct sum (= discrete direct product) of lattice ordered groups isomorphic to $\Gamma_{i<\omega_\alpha} H_i$, where each H_i is isomorphic to R. The class T_β is defined analogously (with α replaced by β). Then T_α and T_β are (non-hereditary) torsion classes and we have

$$T_\alpha \cap T_\beta = \{\{0\}\} .$$

Denote $\{\{0\}\} = \bar{0}$.

Suppose that $\alpha < \beta$ and put $G = \Gamma_{i<\omega_\beta} H_i$. There exists a uniquely determined convex l-subgroup G' of G such that G' is isomorphic with $\Gamma_{i<\omega_\alpha} H_i$. Then

$$T_\beta(G) = G \,, \quad T_\alpha(G) = G' \,,$$

whence

$$(T_\alpha \cap T_\beta)(G) = \bar{0}(G) = \{0\} \,,$$

$$T_\alpha(G) \cap T_\beta(G) = G' \neq \{0\} \,.$$

We have proved that there exists a proper class of pairs of torsion classes T_1, T_2 such that, for some $G = G(T_1, T_2)$, the relation $(*)$ fails to hold. Hence the answer to Question 2.9 is 'No'.

Since the mapping h preserves finite meets of hereditary torsion classes, we can ask whether h preserves also infinite meets of hereditary torsion classes. Let us consider the following example (showing that the answer is negative).

Let $P = \{p_n : n \in N_0\}$ be as in § 3. For each $n \in N_0$ let H_n be the subgroup of Q generated by the set $\{p_n^{-m}\}_{m \in N_0}$; H_n is linearly ordered in the natural way. Let T_n be the class of all lattice ordered groups G such that either (i) $G = \{0\}$, or (ii) G can be expressed as a lexicographic product of linearly ordered groups K_j $(j \in J(G))$ such that for each $j \in J(G)$, K_j is isomorphic to some H_m with $m \geqq n$.

From Thm. 2.6 in [11] it follows that every T_n is a hereditary torsion class. Put

$$G = \Gamma_{m \in N_0} H_m , \quad G_n = \Gamma_{m \in N_0^n} H_m ,$$

where $N_0^n = \{m \in N_0 : m \geqq n\}$. Further, put $N = \{0\}$, $T = \bigwedge_{n \in N_0} T_n$.

We have $T = \{\{0\}\} = \bar{0}$, $T(G) = \{0\} = N$, hence $N \in h(T)(G)$. On the other hand, if $n \in N_0$, then $T_n(G) = G_n \supset N$, whence $N \notin h(T_n)(G)$ and so $N \notin \bigcap(h(T_n)(G))$.

References

[1] *P. Conrad:* Lattice ordered groups, Tulane University, 1970.

[2] *P. Conrad:* Some structure theorems for lattice ordered groups. Trans. Amer. Soc. *99* (1961), 1—29.

[3] *P. Conrad:* Torsion radicals of lattice ordered groups. Symposia Math. *31* (1977), 479—513,

[4] *Л. Фукс:* Частично упорядоченные алгебраические системы, Москва 1965.

[5] *C. Holland:* Varieties of l-groups are torsion classes. Czech. Math. J. *29* (1979), 11—12.

[6] *C. Holland, J. Martinez:* Accessibility of torsion classes. Algebra universalis *9* (1979), 199—206.

[7] *J. Jakubík:* Distributivity in lattice ordered groups. Czech. Math. J. *22* (1972), 108—125.

[8] *J. Jakubík:* Products of torsion classes of lattice ordered groups. Czech. Math. J. *25* (1975), 576—585.

[9] *J. Jakubík:* Radical classes and radical mappings of lattice ordered groups. Symposia Math. *31* (1977), 451—477.

[10] *J. Jakubík:* Products of radical classes of lattice ordered groups. Acta Math. Univ. Comen. *39* (1980), 31—34.

[11] *J. Jakubík:* Torsion radicals of lattice ordered groups. Czech. Math. J. (to appear).

[12] *J. Jakubík:* On value selectors and torsion classes of lattice ordered groups. Czech. Math. J. *31* (1981), 306—313.

[13] *J. Martinez:* Torsion theory for lattice ordered groups. Czech. Math. J. *25* (1975), 284—299.

[14] *J. Martinez:* Torsion theory for lattice ordered groups, Part II: Homogeneous l-groups. Czech. Math. J. *26* (1976), 93—100.

[15] *J. Martinez:* Is the lattice of torsion classes algebraic? Proc. Amer. Math. Soc. *63* (1977), 9—14.

[16] *J. Martinez:* Prime selectors in lattice ordered groups. Czech. Math. J. (to appear).

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