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ON SEGAL'S POSTULATES FOR GENERAL QUANTUM MECHANICS

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Segal's postulates [1] deal with real algebraïc systems. There are two groups of postulates.

I.1. The system $\mathfrak{A}$ is a real linear space.

2. In $\mathfrak{A}$ there exist an identity element I and for every $U \in \mathfrak{A}$ and a positive integer n an element U^n of $\mathfrak{A}$, such that usual rules for operations with polynomials in a single variable are valid: if f, g and h are polynomials with real coefficients, and if $f(g(\alpha)) = h(\alpha)$ for all real α, then $f(g(U)) = h(U)$; here $f(U) = \beta_0 I + \sum_{k=1}^{m} \beta_k U^k$ if $f(\alpha) = \sum_{k=0}^{m} \beta_k \alpha^k$.

II.1. $\mathfrak{A}$ is a real Banach space with the norm $\| \ \|$.

2. $\|U^2 - V^2\| \leqq \text{Max}\,[\|U^2\|, \|V^2\|]$.

3. $\|U^2\| = \|U\|^2$.

4. $\|\sum_{U \in \mathfrak{K}} U^2\| \leqq \|\sum_{U \in \mathfrak{S}} U^2\|$ if $\mathfrak{K} \subset \mathfrak{S}$ and $\mathfrak{S}$ is a finite subset of $\mathfrak{A}$.

5. U^2 is a continuous function of U.

The reality of $\mathfrak{A}$ is expressed in II.2 and II.4.

Sherman [2] proved that II.4 is redundant as it is a consequence of the other postulates — he in fact showed that the sum of squares is a square and this, by Corollary 1 of [1] (II.4 is not needed for its proof), implies the desired result.

However, this can be also seen directly.

$$\|U^2\| = \|(U^2 + V^2) - V^2\| \leqq \text{Max}\,(\|U^2 + V^2\|, \|V^2\|)$$

as the sum of squares is a square and so II.2 can be used. If we suppose $\|U^2\| > \|V^2\|$, then $\|U^2\| \leqq \|U^2 + V^2\|$. If $\|U^2\| = \|V^2\|$ then we can write

$$\|U^2\| = \|(U^2 + tV^2) - tV^2\|\,, \quad 0 < t < 1\,,$$

and thus $\|U^2\| \leqq \|U^2 + tV^2\|$. As $\|U^2 + V^2 - (U^2 + tV^2)\| = (1 - t)\,\|V^2\| \to 0$ for $t \to 1$, we have $\|U^2\| \leqq \|U^2 + V^2\|$ as well.

Remark. When proving that the sum of squares is a square [2], the author uses the series for $\sqrt{(1 - t)}$. Now, we must know how the values of $\|U^n\|$ are distributed for using the series for $\sqrt{(I - U)}$.

If we have II.2, then we can calculate

$$U^{n+1} = \tfrac{1}{4}\{(U^n + U)^2 - (U^2 - U)^2\}$$

and consequently, if $\|U\| \leqq 1$, then, by induction, $\|U^{n+1}\| \leqq 1$.

If we have II.4, then we have to use the inequality $\|U^n\| \leqq 2\|U^{n-1}\| \cdot \|U\|$ (see below) for evaluating $\|U^n\|$ and so $\|U^{n+1}\| \leqq 2^n$ for $\|U\| \leqq 1$.

In [2] only a non-vanishing radius of convergence was used for the proof.

End of the remark.

We shall show now:

For a commutative system of observables $\mathfrak{A}$ the positivity of squares as expressed in II.4 is sufficient for the demonstration of Theorem 1 in [1] and so II.2 is a consequence of II.4 (for commuting observables).

The product in $\mathfrak{A}$ is $x \circ y = \tfrac{1}{4}\{(x + y)^2 - (x - y)^2\}$ and so $4\|x \circ y\| = \|(x + y)^2 - (x - y)^2\| \leqq \|(x + y)^2\| + \|(x - y)^2\| = \|x + y\|^2 = \|x - y\|^2$ by II.3.

Hence for $\|x\|, \|y\| \leqq 1$ we have $4\|x \circ y\| \leqq 8$ and thus $\|x \circ y\| \leqq 2\|x\| \,.\, \|y\|$ for all x, y.

If we set $|x| = 2\|x\|$ as a new norm, $\mathfrak{A}$ will be a real Banach algebra. In this new norm, we have

$$|x|^2 = 2|x^2|\,.$$

Let $\mathfrak{A}_C$ be the complexification of $\mathfrak{A}$:

$$\mathfrak{A} = \{z \mid z = x + \mathrm{i}y,\ x, y \in \mathfrak{A}\}\,.$$

For $z = x + \mathrm{i}y$ we set $|z| = |x| + |y|$, $z^* = x - \mathrm{i}y$. Then $|z + \zeta| \leqq |z| + |\zeta|$, $|az| = |a| \,.\, |z|$ for a real a,

$$|z\zeta| = |x\xi - y\eta| + \ldots \leqq |x| \,.\, |\xi| + \ldots = |z| \,.\, |\zeta|\,.$$

Finally,

$$|z|^2 = (|x| + |y|)^2 = |x|^2 + |y|^2 + 2|x| \,.\, |y| \leqq 2|x|^2 + 2|x|^2 =$$
$$= 4|x|^2 = 8|x^2| \leqq 8|x^2 + y^2| \quad \text{for} \quad |x| \geqq |y|\,.$$

On the other hand,

$$|zz^*| = |x^2 + y^2|\,, \quad \text{hence} \quad |z|^2 \leqq 8|zz^*|\,.$$

If we set

$$N(z) = \sup_{0 \in \vartheta \leqq 2\pi} |\exp(\mathrm{i}\vartheta)\, z|\,,$$

we have

$$N(z + \zeta) \leqq N(z) + N(\zeta)\,, \quad N(az) = |a|\, N(z)\,,$$
$$N(z\zeta) \leqq N(z)\, N(\zeta)\,,$$

and

$$N^2(z) \leqq 8\, N(zz^*)\,.$$

Now we shall apply the result of 26.E from [3]. $\mathfrak{A}_C$ with the norm N is $*$-algebraically isomorphic to $C(\mathfrak{M})$ – the space of continuous functions on a compact – and we have

$$|\hat{z}|_\infty \leqq N(z) \leqq 8|\hat{z}|_\infty\,, \quad z \in \mathfrak{A}_C\,,$$

where $\hat{z}$ is the corresponding function and $|z|_\infty$ is the norm in $C(\mathfrak{M})$.

Now $N(x) = \sqrt{(2)}\,\|x\|$ for $x \in \mathfrak{A}$ and hence

$$2^{-1/2}|\hat{x}|_\infty \leqq \|x\| \leqq 4\sqrt{(2)}\,|\hat{x}|_\infty\,.$$

By II.3, $\|x^{2^k}\| = \|x\|^{2^k}$ in $\mathfrak{A}$ and the same is true in $C(\mathfrak{M}) : (\hat{x}^{2^k}|_\infty = |\hat{x}|_\infty^{2^k}$.

Hence it must be $\|x\| = |\hat{x}|_\infty$ and this is the rest of Theorem 1.

Remark. The proof works well with the inequality

$$A\|U^2\| \leqq \|U\|^2 \leqq B\|U^2\| \quad \text{for all} \quad U \in \mathfrak{A}$$

and $\log(\|U\|^k/\|U^k\|)$ bounded for every U and a sequence $k \to \infty$ instee of II.3.

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References

[1] *Segal I. E.:* Postulates for general quantum mechanics, Ann. of Math., *48* (1947), 930—948.
[2] *Sherman S.:* On Segal's postulates for general quantum mechanics, Ann. of Math., *64* (1956), 593—601.
[3] *Loomis L. H.:* An Introduction to Abstract Harmonic Analysis, D. Van Nostrand Company, 1953.

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