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Czechoslovak Mathematical Journal, Vol. 31 (1981), No. 2, 306–313

Persistent URL: `http://dml.cz/dmlcz/101745`

Czechoslovak Mathematical Journal, 31 (106) 1981, Praha

ON VALUE SELECTORS AND TORSION CLASSES OF LATTICE ORDERED GROUPS

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(Received November 7, 1979)

In this note we will investigate a problem proposed by J. Martinez [7] on the relation between value selectors and torsion classes of lattice ordered groups.

1. PRELIMINARIES

We shall use the standard notation for lattice ordered groups (cf. Conrad [1] and Fuchs [2]). The group operation will be written additively.

The system of all convex l-subgroups of a lattice ordered group G will be denoted by $c(G)$; this system is partially ordered by inclusion. Then $c(G)$ is a complete lattice; the lattice operations in $c(G)$ are denoted by $\wedge$, $\vee$.

In what follows we shall consider objects belonging to some type of the following hierarchy:

1) lattice ordered groups and their elements;
2) classes of lattice ordered groups;
3) classes of classes of lattice ordered groups.

Let $\mathscr{G}$ be the class of all lattice ordered groups. Let A be a nonempty subclass of $\mathscr{G}$. Consider the following conditions for A:

(a) If $G \in \mathscr{G}$ and if $\{H_i\}_{i \in I} \subseteq A \cap c(G)$, then $\bigvee_{i \in I} H_i \in A$.
(b) *If* $G \in A$ and $H \in c(G)$, then $H \in A$.
(c) A is closed with respect to homomorphisms.

The class A is said to be a torsion class, if it satisfies (a), (b) and (c) (cf. Martinez [5], [6], [7]; a different terminology (using the term 'hereditary torsion class') has been applied in [4], [8]). Each variety of lattice ordered groups is a torsion class (Holland [3]).

Let T be the class of all torsion classes; T is partially ordered by inclusion. Then T is a complete lattice [5]. Several properties of the lattice T were established in [5], [9].

2. VALUE SELECTORS

The notion of a values selector was introduced in [7]. Let us recall some definitions and results concerning this notion.

Let $G \in \mathscr{G}$, $x \in G$. A convex l-subgroup of G maximal with respect to the property of noncontaining x is called a value of x. A convex l-subgroup of G is said to be a value if it is a value of an element of G. Let $M_0(G)$ be the set of all values of G.

A value selector is a function M assigning to each lattice ordered group G a subset $M(G)$ of $M_0(G)$ such that the following conditions are fulfilled:

(1) If $H \in c(G)$, then $M(H) = \{C \cap H : C \in M(G) \text{ and } C \not\supseteq H\}$.

(2) If K is an l-ideal of G, then $M(G/K) \supseteq \{C/K : C \in M(G) \text{ and } C \supseteq K\}$.

(Of course, we also assume that the mapping M is defined intrinsically, i.e., if φ is an isomorphism of a lattice ordered group G_1 onto a lattice ordered group G_2, then $M(G_2) = \{\varphi(C) : C \in M(G_1)\}$.)

Let M_1 and M_2 be value selectors. We put $M_1 \leqq M_2$ if $M_1(G) \subseteq M_2(G)$ for each lattice ordered group G. Let $\{M_i\}_{i\in I}$ be a family of value selectors; we define $M_1(G) = \bigcap_{i\in I} M_i(G)$ and $M_2(G) = \bigcup_{i\in I} M_i(G)$ for each $G \in \mathscr{G}$. Then M_1 and M_2 are value selectors, and $M_1 = \bigwedge_{i\in I} M_i$, $M_2 = \bigvee_{i\in I} M_i$.

Let M be a value selector. We denote by $T(M)$ the class of all latice ordered groups G such that $M(G) = M_0(G)$. For each torsion class A and each $G \in \mathscr{G}$ we put

$$A^\wedge(G) = \{H \in M_0(G) : A(G) \nsubseteq H\},$$

where $A(G)$ is the join of all convex l-subgroups of G belonging to A.

Then we have (cf. [7]; Lemmas 1.1 – 1.3):

2.1. Lemma. *For each value selector M, $T(M)$ is a torsion class.*

2.2. Lemma. *For each torsion class A, $A^\wedge$ is a value selector; moreover, for $G \in \mathscr{G}$ we have $G \in A$ if and only if $A^\wedge(G) = M_0(G)$.*

2.3. Lemma. *If A is a torsion class and M is a value selectcr, then $T(M)^\wedge \leqq M$ and $T(A^\wedge) = A$.*

The following problem has been proposed in [7]:

'The function $M \to T(M)$ preserves arbitrary intersections. But it is unknown whether it also preserves joins. It would be of interest to know it, for it would shed light on the following question: If A is a torsion class, is there a largest value selector M such that $T(M) = A$? There is always a smallest, namely $A^\wedge$. In view of the inequality in 1.3, the author doubts that it preserve joins.'

3. THE MAPPINGS s_1 AND s_2

Let $G \in \mathscr{G}$ and $X \subseteq G$. We denote $X^\delta = \{g \in G : |g| \wedge |x| = 0 \text{ for each } x \in X\}$. If we consider several lattice ordered groups then we sometimes write $X^{\delta(G)}$ rather than X^δ. It is well-known that X^δ is a convex l-subgroup of G.

The following lemma is easy to verify.

3.1. Lemma. *Let $0 < x \in G$ and suppose that the interval $[0, x]$ is a chain. Then $\{x\}^{\delta\delta}$ is a linearly ordered group.*

3.2. Lemma. *Let $0 < x \in G$ and suppose that the interval $[0, x]$ is a chain. Then x possesses a unique value $B + \{x\}^\delta$, where B is the value of x in $\{x\}^{\delta\delta}$.*

Proof. Put $\{x\}^{\delta\delta} = A$ and let $\{A_i\}_{i \in I}$ be the set of all convex l-subgroups of A such that $x \notin A_i$. Denote $B = \bigvee_{i \in I} A_i$. The fact that the system of all convex l-subgroups of a linearly ordered group is linearly ordered and 3.1 imply that B is the unique value of x in A.

We set $\{x\}^\delta = C$, $B + C = D$. Clearly $C = A^\delta$. Hence we obtain by a routine calculation that D is a convex l-subgroup of G. Moreover, D is a direct sum of its l-subgroups B and C, and $B \vee C = D$ is valid in the lattice $c(G)$. We also have $x \notin D$.

Let D_1 be a convex l-subgroup of G with $x \notin D_1$. Let $0 \leqq d_1 \in D_1$. Then $x \nleqq d_1$. Denote $x \wedge d_1 = y$, $-y + d_1 = z$, $-y + x = y_1$. We have $z \geqq 0$, $0 < y_1 \leqq x$ and $y_1 \wedge z = 0$. This and the fact that A is linearly ordered yields $a \wedge z = 0$ for each $0 \leqq a \in A$. Thus $z \in A^\delta$ and hence $d_1 \in D$. Therefore $D_1 \subseteq D$, which completes the proof.

If I is a linearly ordered set and if G_i is a linearly ordered group for each $i \in I$, then $\Gamma_{i \in I} G_i$ denotes the lexicographic product of the system $\{G_i\}$ $(i \in I)$ (cf., e.g., Fuchs [2]).

Let N be the set of all positive integers and let $P = \{p_n\}$ $(n \in N)$ be the set of all primes. Further, let R_0 be the set of all rational numbers (with the natural linear order).

Let f be a one-to-one mapping of the set R_0 onto N and let R_1, R_2 be infinite subsets of R_0 such that (i) $R_1 \cap R_2 = \emptyset$, $R_1 \cup R_2 = R_0$, and (ii) both R_1 and R_2 are dense subsets of R_0. For each $x \in R_0$ let K_x be the set of all rational numbers of the form lp_n^{-m}, where $n = f(x)$, $m \in N$ and l is any integer. We consider K_x as an additive group with the natural linear order. If $x, y \in R_0$ are distinct, then the linearly ordered groups K_x and K_y fail to be isomorphic. We denote by H_0 the class of all lattice ordered groups H that can be expressed as

$$H = \Gamma_{i \in I} H_i, \tag{3}$$

where

(i) I is a convex subset of R_0;

(ii) for each $i \in I$, H_i is isomorphic with K_i.

From the definition of H_0 it follows that if K is a homomorphic image of a lattice ordered group H belonging to H_0 then either K belongs to H_0 or $K = \{0\}$. The same is valid for each convex l-subgroup of H.

Let $H \in H_0$ be as in (3) and let $0 < g \in H$. Let us denote by i_0 the least $i \in I$ with $g(i) \neq 0$. If $i_0 \in R_i$ $(i \in \{1, 2\})$, then the element g will be said to be of type R_i. Let $R_i(H)$ be the set of all elements of H which are of type R_i $(i = 1, 2)$. We have $R_1(H) \cap R_2(H) = \emptyset$. If φ is an isomorphism of H onto a linearly ordered group $H' \in H_0$, then $\varphi(R_i(H)) = R_i(H')$ $(i = 1, 2)$.

An isomorphism φ of a lattice ordered group G_1 into a lattice ordered group G_2 is said to be convex if $\varphi(G_1)$ is a convex l-subgroup of G_2. Let G be a lattice ordered group and $0 < x \in G$. The element x will be called of type R_1 if there exist $H \in H_0$ and a convex isomorphism φ of H into G such that $x \in \varphi(H)$ and $\varphi^{-1}(x) \in R_1(H)$. Let $R_1(G)$ be the set of all elements of G which are of type R_1. The set $R_2(G)$ is defined analogously. Then $R_1(G) \cap R_2(G) = \emptyset$ is valid. Moreover, 3.2 implies that each element $x \in R_1(G) \cup R_2(G)$ possesses a unique value $v_G(x)$ in G. We put

$$s_1(G) = \{v_G(x) : x \in R_1(G)\}\,, \quad s_2(G) = \{v_G(x) : x \in R_2(G)\}\,.$$

3.3. Lemma. *The mappings s_1 and s_2 fulfil the condition* (1).

Proof. Let G be a lattice ordered group and let G_1 be a convex l-subgroup of G. We have to verify that $s_1(G_1) = \{C \cap G_1 : C \in s_1(G) \text{ and } C \not\supseteq G_1\}$.

Let $C_1 \in s_1(G_1)$. There is $x \in R_1(G_1)$ such that $C_1 = v_{G_1}(x)$. Let B be the convex l-subgroup of $\{x\}^{\delta(G_1)\delta(G_1)}$ that is maximal with respect to the property of non-containing x; i.e., B is the value of x in $\{x\}^{\delta(G_1)\delta(G_1)}$. Then B is also the value of x $\{x\}^{\delta\delta}$. From 3.2 it follows that

$$C_1 = v_{G_1}(x) = B + \{x\}^{\delta(G_1)}\,.$$

Further, we have $x \in R_1(G)$. Thus x has a unique value in G; let us denote this value by $C = v_G(x)$. Then $C \in s_1(G)$, $C \not\supseteq G_1$ and by using 3.2 again we obtain

$$C = B + \{x\}^{\delta}\,.$$

Since $\{x\}^{\delta(G_1)} = \{x\}^{\delta} \cap G_1$, we get $C_1 = C \cap G_1$. Thus $s_1(G_1) \subseteq \{C \cap G_1 : C \in$ $\in s_1(G)$ and $C \not\supseteq G_1\}$.

Now let $C \in s_1(G)$ such that $C \not\supseteq G_1$. There is $x \in R_1(G)$ with $C = v_G(x)$. Let B be the value of x in $\{x\}^{\delta\delta}$; then $C = B + \{x\}^{\delta}$. We shall show that $x \in G_1$.

By way of contradiction, assume that x does not belong to G_1. From $C \not\supseteq G_1$ it follows that there exists $0 < g_1 \in G_1$ such that $g_1 \notin C$. If $g_1 \geqq x$, then $x \in G_1$, which is a contradiction. If $0 < z \in G$ and $z \leqq x$, then the structure of lattice ordered groups belonging to H_0 yields that either $z \in B$ or the value of z in $\{x\}^{\delta\delta}$ coincides with B. If $g_1 < x$, then $g_1 \notin B$ (because $g_1 \notin C$) and thus the value of g_1 in $\{x\}^{\delta\delta}$ coincides with B; but in this case there is a positive integer n with $ng_1 > x$, implying $x \in G_1$.

Hence we can suppose that g_1 is incomparable with x. Put $y = x \wedge g_1$, $z = -y + g_1$. Then $y \in B$ and $z \in \{x\}^{\delta}$, hence $g_1 \in C$, which is a contradiction. Therefore $x \in G_1$ and so $B \subseteq G_1$.

The relation $x \in R_1(G) \cap G_1$ implies $x \in R_1(G_1)$. Thus

$$\begin{aligned} C \cap G_1 &= (B + \{x\}^{\delta}) \cap G_1 = (B \vee \{x\}^{\delta}) \wedge G_1 = \\ &= (B \wedge G_1) \vee (\{x\}^{\delta} \wedge G_1) = B \vee (\{x\}^{\delta} \wedge G_1) = \\ &= B \vee \{x\}^{\delta(G_1)} = B + \{x\}^{\delta(G_1)} = v_{G_1}(x) \in s_1(G) . \end{aligned}$$

We have proved that s_1 fulfils (1). The same proof can be applied to s_2.

3.4. Lemma. *The mappings s_1 and s_2 fulfil the condition* (2).

Proof. Let K be an l-ideal of a lattice ordered group G and let $C \in s_1(G)$, $C \supseteq K$. We have to verify that C/K belongs to $s_1(G/K)$.

According to the assumption there exists $x \in R_1(G)$ such that $C = v_G(x)$. As above, put $A = \{x\}^{\delta\delta}$, $B = v_A(x)$. For each $y \in G$, $Y \subseteq G$ put $\bar{y} = y + K$, $\bar{Y} = \{y + K\}_{y \in Y}$. The structure of A yields that the lattice ordered group $\bar{A}$ belongs to H_0 (the case $\bar{A} = \{0\}$ is impossible because $\bar{x} \in \bar{A}$ and $\bar{x} \neq K$); moreover $\bar{x} \in R_1(\bar{A})$ and $\bar{B} = v_{\bar{A}}(\bar{x})$. Thus $\bar{x} \in \bar{G}$).

Put $D = \{x\}^{\delta}$. From 3.2 it follows that $\bar{C} = \bar{B} + \bar{D}$. Hence in order to prove that $\bar{C} = v_{\bar{G}}(\bar{x})$ it suffices to verify that

$$\bar{D} = \{\bar{g} \in \bar{G} : |\bar{g}| \wedge \bar{x} = \bar{0}\} ,$$

the symbol $\bar{0}$ denoting the zero element in $\bar{G}$.

If $\bar{g} \in \bar{D}$, then there is $g_1 \in \bar{g} \cap D$, hence $|\bar{g}| \wedge \bar{x} = |\bar{g}_1| \wedge \bar{x} = \overline{|g_1| \wedge x} = \bar{0}$. Conversely, suppose that $\bar{g} \in \bar{G}$ and that $|\bar{g}| \wedge \bar{x} = \bar{0}$ is valid. There exists $0 \leqq g_2 \in |\bar{g}| = \overline{|g|}$. We have $\bar{g}_2 \wedge \bar{x} = \bar{0}$, hence $0 \leqq z = g_2 \wedge x \in K$. Put $g_3 = -z + g_2$, $x_1 = -z + x$. Clearly $x \notin K$, thus $0 < x_1 \leqq x$. Moreover, we have $g_3 \wedge x_1 = 0$. This and the fact that $[0, x]$ is a chain imply $g_3 \wedge x = 0$. Hence $g_3 \in D$ and therefore $|\bar{g}| = \bar{g}_3 \in \bar{D}$. Thus $\bar{g} \in \bar{D}$, which completes the proof for s_1. The proof for s_2 is analogous.

From 3.3 and 3.4 we obtain:

3.5. Lemma. *s_1 and s_2 are value selectors.*

4. THE MAPPINGS s_1' AND s_2'

In this paragraph we shall use the same notation as in § 3. Let R_{01}' be the class of all lattice ordered groups H such that H is isomorphic to some K_t, $t \in R_1$. The class R_{02}' is defined analogously. We put $R_0' = R_{01}' \cup R_{02}'$.

Let $G \in \mathscr{G}$, $0 < x \in G$. If there exists a convex l-subgroup H of G with $x \in H$ such that H belongs to R'_{01}, then the element x is said to be of type R_{01}. The elements of type R_{02} or R_0, respectively, are defined analogously. Let $R_{01}(G)$ be the set of all elements of G which are of type R_{01}. Similarly we define the sets $R_{02}(G)$ and $R_0(G)$. According to 3.2, each element $x \in R_0(G) = R_{01}(G) \cup R_{02}(G)$ possesses a unique value $v_G(x)$ in G. Put

$$s_{0i}(G) = \{v_G(x) : x \in R_{0i}(G)\} \; (i = 1, 2), \quad s_0(G) = \{v_G(x) : x \in R_0(G)\} .$$

4.1. Lemma. *s_{01}, s_{02} and s_0 are value selectors.*

The proof is analogous to that used in § 3 for s_1 and s_2, and therefore will be omitted.

Put $s'_i = s_i \vee s_0$ for $i = 1, 2$ (i.e., $s'_i(G) = s_i(G) \cup s_0(G)$ for each $G \in \mathscr{G}$). From 3.5 and 4.1 we obtain

4.2. Lemma. *s'_1 and s'_2 are value selectors.*

Let us denote by A_0 the class of all lattice ordered groups G such that either $G = \{0\}$ or G is a direct sum (= discrete direct product) of lattice ordered groups belonging to R'_0. Similarly we define the classes A_1 and A_2. It is easy to verify that all these classes are torsion classes (this follows also immediately from [9], Thm. 2.6).

Put $B_1 = T(s'_1)$. For each $K_t \in R'_0$ we have $s_0(K_t) = \{\{0\}\} = M_0(K_t)$, whence $K_t \in T(s_0) \subseteq T(s'_1)$. Because each lattice ordered group $G \in A_0$ is a join of lattice ordered groups belonging to R'_0 and since $T(s'_1)$ is a torsion class (cf. 2.1) we infer that

$$A_0 \subseteq T(s'_1) \tag{4}$$

is valid.

For each $G \in \mathscr{G}$ we denote by $A_0(G)$ the join of all convex l-subgroups of G which belong to A_0. Then $A_0(G)$ belongs to A_0 as well.

4.3. Lemma. $s_1(G) \cap s_2(G) = \emptyset$.

Proof. By way of contradiction, assume that $C \in s_1(G) \cap s_2(G)$. According to 3.2 there exists $0 < x \in R_1(G)$, $0 < y \in R_2(G)$, $B_1 \in c(G)$, $B_2 \in c(G)$ such that

$$C = B_1 + \{x\}^\delta , \quad C = B_2 + \{y\}^\delta ,$$

where B_1 is the value of x in $\{x\}^{\delta\delta}$ and B_2 is the value of y in $\{y\}^{\delta\delta}$. Since $R_1(G) \cap R_2(G) = \emptyset$ we have $x \neq y$. If $x < y$, then $x \in B_2 \subseteq C$, which is impossible; similarly, $y \nless x$. Hence x is incomparable with y; because $[0, x]$ and $[0, y]$ are chains, it follows that $x \wedge y = 0$, and thus $y \in \{x\}^\delta \subseteq C$, which is a contradiction.

4.4. Lemma. *Let $y \in R_2(G)$, $y \notin R_{02}(G)$. Then $v_G(y) \notin s_0(G)$.*

Proof. Clearly $s_{01} \leqq s_1$, hence 4.3 implies $s_{01}(G) \cap s_2(G) = \emptyset$. Because of $v_G(y) \in \in s_2(G)$ we have to verify that $v_G(y) \notin s_{02}(G)$.

By way of contradiction, assume that $v_G(y) \in s_{02}(G)$. Hence there exists $z \in R_{02}(G)$ such that $v_G(y) = v_G(z)$. From the structure of lattice ordered groups belonging to H_0 we infer that we have neither $z = y$ nor $z > y$. The cases (i) $y > z$ and (*ii*) y is incomparable with z lead to a contradiction in a similar way as in the proof of 4.3.

4.5. Lemma. *Let $G \in B_1$, $C \in M_0(C)$. Then there is $x \in R_0(G)$ such that $C = v_G(x)$.*

Proof. By way of contradiction, assume that $C \neq v_G(x)$ for each $x \in R_0(G)$. Then there is $x \in R_1(G) \setminus R_{01}(G)$ such that $C = v_G(x)$. Now the definition of H_0 implies that there is $y \in R_2(G) \setminus R_{02}(g)$ with $y < x$ (we use the density of R_2 in R_0). From 4.3 and 4.4 we obtain $v_G(y) \notin s_1'(G)$ implying $G \notin B_1$, which is a contradiction.

4.6. Lemma. *Let G belong to B_1. Then $G = A_0(G)$.*

Proof. Suppose that $G \neq A_0(G)$. Then there is $y \in G \setminus A_0(G)$. There exists a value C of y in G such that $A_0(G) \subseteq C$. In view of 4.5, there is $x \in R_0(G)$ with $C = v_G(x)$. The convex l-subgroup C_1 of G generated by x belongs to A_0, hence $x \in C_1 \subseteq A_0(G) \subseteq C$, which is a contradiction.

From (4) and 4.6 we conclude

4.7. Lemma. $T(s_1') = A_0$.

Analogously we obtain

4.8. Lemma. $T(s_2') = A_0$.

4.9. Lemma. *Let H be as in (3) with $I = R_0$. Then $H \in T(s_1' \vee s_2')$ and $H \notin A_0$.*

Proof. If C is a value in H, then there is $0 < x \in H$ such that C is a value of x. Since x belongs either to $R_1(H)$ or to $R_2(H)$, C belongs to $(s_1' \vee s_2')(H)$. Hence $H \in T(s_1' \vee s_2')$. Moreover, H is linearly ordered and thus H is directly indecomposable. Hence from $H \notin R_0'$ it follows that H does not belong to A_0.

4.10. Corollary. *There does not exist any largest value selector M with $T(M) = A_0$.*

Hence the above questions quoted from [7] are answered by the following

Proposition. *The function $M \to T(M)$ does not, in general, preserve joins. If A is a torsion class, then there need not exist a largest value selector M with $T(M) = A$; moreover, the class of all value selectors M_1 with $T(M_1) = A$ need not be directed.*

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